10 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Figure 10.1 Hydroponic systems allow botanists to grow crops without land. (credit: "Izhamwong"/Wikimedia Commons)

Chapter Outline

10.1 Finding Composite and Inverse Functions
10.2 Evaluate and Graph Exponential Functions
10.3 Evaluate and Graph Logarithmic Functions
10.4 Use the Properties of Logarithms
10.5 Solve Exponential and Logarithmic Equations

Introduction

As the world population continues to grow, food supplies are becoming less able to meet the increasing demand. At the same time, available resources of fertile soil for growing plants is dwindling. One possible solution—grow plants without soil. Botanists around the world are expanding the potential of hydroponics, which is the process of growing plants without soil. To provide the plants with the nutrients they need, the botanists keep careful growth records. Some growth is described by the types of functions you will explore in this chapter—exponential and logarithmic. You will evaluate and graph these functions, and solve equations using them.

10.1 Finding Composite and Inverse Functions

Learning Objectives

By the end of this section, you will be able to:

- Find and evaluate composite functions
- Determine whether a function is one-to-one
- Find the inverse of a function

Be Prepared!

Before you get started, take this readiness quiz.

1. If $f(x) = 2x - 3$ and $g(x) = x^2 + 2x - 3$, find $f(4)$.
   If you missed this problem, review Example 3.48.
2. Solve for x, $3x + 2y = 12$.
   If you missed this problem, review Example 2.31.
3. Simplify: $5\frac{(x+4)}{5} - 4$.
   If you missed this problem, review Example 1.25.

In this chapter, we will introduce two new types of functions, exponential functions and logarithmic functions. These

functions are used extensively in business and the sciences as we will see.

Find and Evaluate Composite Functions

Before we introduce the functions, we need to look at another operation on functions called composition. In composition, the output of one function is the input of a second function. For functions f and g, the composition is written $f \circ g$ and is defined by $(f \circ g)(x) = f(g(x))$.

We read $f(g(x))$ as "f of g of x."

To do a composition, the output of the first function, $g(x)$, becomes the input of the second function, *f*, and so we must be sure that it is part of the domain of *f*.

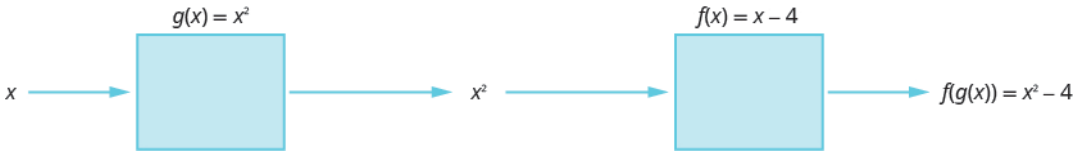
Composition of Functions

The composition of functions *f* and *g* is written $f \cdot g$ and is defined by

$$(f \circ g)(x) = f(g(x))$$

We read $f(g(x))$ as f of g of *x*.

We have actually used composition without using the notation many times before. When we graphed quadratic functions using translations, we were composing functions. For example, if we first graphed $g(x) = x^2$ as a parabola and then shifted it down vertically four units, we were using the composition defined by $(f \circ g)(x) = f(g(x))$ where $f(x) = x - 4$.



The next example will demonstrate that $(f \circ g)(x)$, $(g \circ f)(x)$ and $(f \cdot g)(x)$ usually result in different outputs.

EXAMPLE 10.1

For functions $f(x) = 4x - 5$ and $g(x) = 2x + 3$, find: ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

Solution

ⓐ

Use the definition of $(f \circ g)(x)$.	$(f \circ g)(x) = f(g(x))$
Substitute $2x + 3$ for $g(x)$.	$(f \circ g)(x) = f(2x + 3)$
Find $f(2x + 3)$ where $f(x) = 4x - 5$.	$(f \circ g)(x) = 4(2x + 3) - 5$
Distribute.	$(f \circ g)(x) = 8x + 12 - 5$
Simplify.	$(f \circ g)(x) = 8x + 7$

ⓑ

Use the definition of $(f \circ g)(x)$.	$(g \circ f)(x) = g(f(x))$
Substitute $4x - 5$ for $f(x)$.	$(g \circ f)(x) = g(4x - 5)$
Find $g(4x - 5)$ where $g(x) = 2x + 3$.	$(g \circ f)(x) = 2(4x - 5) + 3$
Distribute.	$(g \circ f)(x) = 8x - 10 + 3$
Simplify.	$(g \circ f)(x) = 8x - 7$

Notice the difference in the result in part ⓐ and part ⓑ.

ⓒ Notice that $(f \cdot g)(x)$ is different than $(f \circ g)(x)$. In part ⓐ we did the composition of the functions. Now in part ⓒ we are not composing them, we are multiplying them.

Use the definition o $(f \cdot g)(x)$.	$(f \cdot g)(x) = f(x) \cdot g(x)$
Substitute $f(x) = 4x - 5$ and $g(x) = 2x + 3$.	$(f \cdot g)(x) = (4x - 5) \cdot (2x + 3)$
Multiply.	$(f \cdot g)(x) = 8x^2 + 2x - 15$

TRY IT : : 10.1 For functions $f(x) = 3x - 2$ and $g(x) = 5x + 1$, find ⓐ $(f \circ g)(x)$ ⓑ $(g \circ f)(x)$ ⓒ $(f \cdot g)(x)$.

TRY IT : : 10.2

For functions $f(x) = 4x - 3$, and $g(x) = 6x - 5$, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

In the next example we will evaluate a composition for a specific value.

EXAMPLE 10.2

For functions $f(x) = x^2 - 4$, and $g(x) = 3x + 2$, find: ⓐ $(f \circ g)(-3)$, ⓑ $(g \circ f)(-1)$, and ⓒ $(f \circ f)(2)$.

Solution

ⓐ

Use the definition of $(f \circ g)(-3)$.	$(f \circ g)(-3) = f(g(-3))$
Find $g(-3)$ where $g(x) = 3x + 2$.	$(f \circ g)(-3) = f(3 \cdot (-3) + 2)$
Simplify.	$(f \circ g)(-3) = f(-7)$
Find $f(-7)$ where $f(x) = x^2 - 4$.	$(f \circ g)(-3) = (-7)^2 - 4$
Simplify.	$(f \circ g)(-3) = 45$

ⓑ

Use the definition of $(g \circ f)(-1)$.	$(g \circ f)(-1) = g(f(-1))$
Find $f(-1)$ where $f(x) = x^2 - 4$.	$(g \circ f)(-1) = g((-1)^2 - 4)$
Simplify.	$(g \circ f)(-1) = g(-3)$
Find $g(-3)$ where $g(x) = 3x + 2$.	$(g \circ f)(-1) = 3(-3) + 2$
Simplify.	$(g \circ f)(-1) = -7$

ⓒ

Use the definition of $(f \circ f)(2)$.	$(f \circ f)(2) = f(f(2))$
Find $f(2)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = f(2^2 - 4)$
Simplify.	$(f \circ f)(2) = f(0)$
Find $f(0)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = 0^2 - 4$
Simplify.	$(f \circ f)(2) = -4$

TRY IT : : 10.3

For functions $f(x) = x^2 - 9$, and $g(x) = 2x + 5$, find ⓐ $(f \circ g)(-2)$, ⓑ $(g \circ f)(-3)$, and ⓒ $(f \circ f)(4)$.

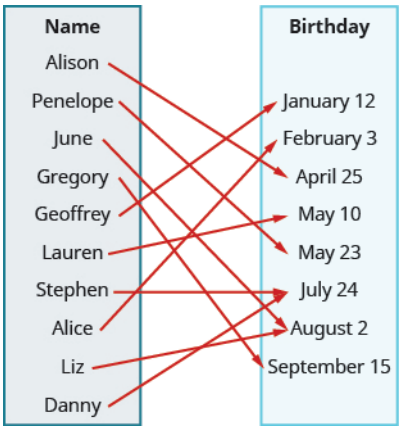
TRY IT : : 10.4

For functions $f(x) = x^2 + 1$, and $g(x) = 3x - 5$, find ⓐ $(f \circ g)(-1)$, ⓑ $(g \circ f)(2)$, and ⓒ $(f \circ f)(-1)$.

Determine Whether a Function is One-to-One

When we first introduced functions, we said a function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

We used the birthday example to help us understand the definition. Every person has a birthday, but no one has two birthdays and it is okay for two people to share a birthday. Since each person has exactly one birthday, that relation is a function.



A function is **one-to-one** if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

Our example of the birthday relation is not a one-to-one function. Two people can share the same birthday. The range value August 2 is the birthday of Liz and June, and so one range value has two domain values. Therefore, the function is

not one-to-one.

One-to-One Function

A function is **one-to-one** if each value in the range corresponds to one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value. There are no repeated *y*-values.

EXAMPLE 10.3

For each set of ordered pairs, determine if it represents a function and, if so, if the function is one-to-one.

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$ and ⓑ $\{(0, 0), (1, 1), (4, 2), (9, 3), (16, 4)\}$.

Solution

ⓐ

$$\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

But each *y*-value is not paired with only one *x*-value, $(-3, 27)$ and $(3, 27)$, for example. So this function is not one-to-one.

ⓑ

$$\{(0, 0), (1, 1), (4, 2), (9, 3), (16, 4)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

Since each *y*-value is paired with only one *x*-value, this function is one-to-one.

TRY IT : : 10.5

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$

ⓑ $\{(-4, 8), (-2, 4), (-1, 2), (0, 0), (1, 2), (2, 4), (4, 8)\}$

TRY IT : : 10.6

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(27, -3), (8, -2), (1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$

ⓑ $\{(7, -3), (-5, -4), (8, 0), (0, 0), (-6, 4), (-2, 2), (-1, 3)\}$

To help us determine whether a relation is a function, we use the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. Also, if any vertical line intersects the graph in more than one point, the graph does not represent a function.

The vertical line is representing an *x*-value and we check that it intersects the graph in only one *y*-value. Then it is a function.

To check if a function is one-to-one, we use a similar process. We use a horizontal line and check that each horizontal line intersects the graph in only one point. The horizontal line is representing a *y*-value and we check that it intersects the graph in only one *x*-value. If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function. This is the **horizontal line test**.

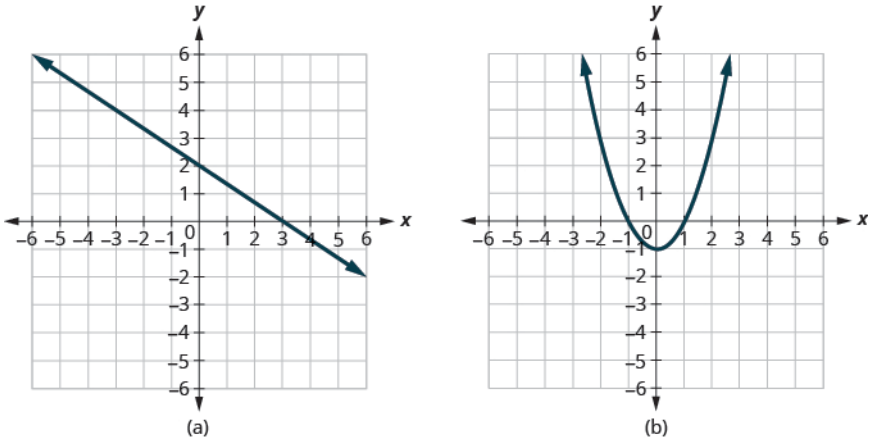
Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function.

We can test whether a graph of a relation is a function by using the vertical line test. We can then tell if the function is one-to-one by applying the horizontal line test.

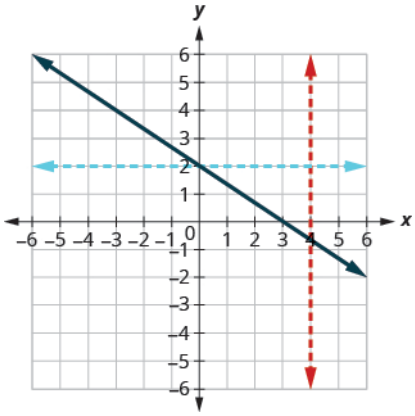
EXAMPLE 10.4

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



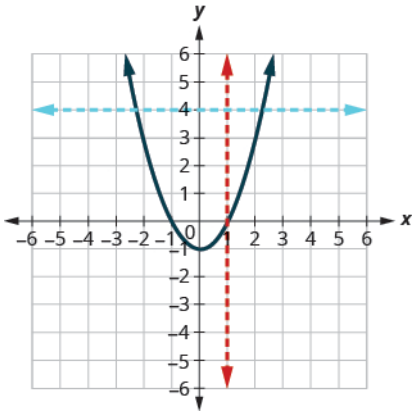
Solution

ⓐ



Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.

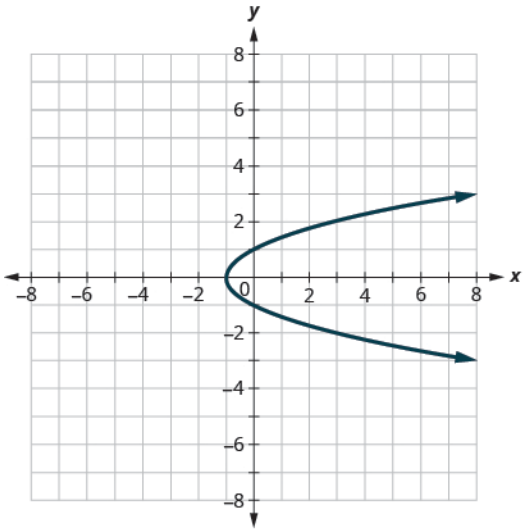
ⓑ



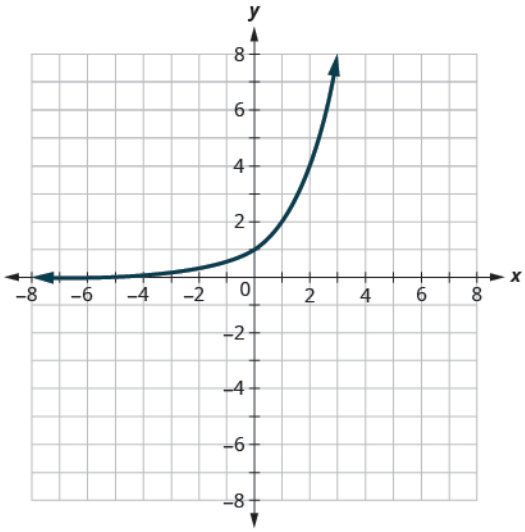
Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. The horizontal line shown on the graph intersects it in two points. This graph does not represent a one-to-one function.

TRY IT : : 10.7

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



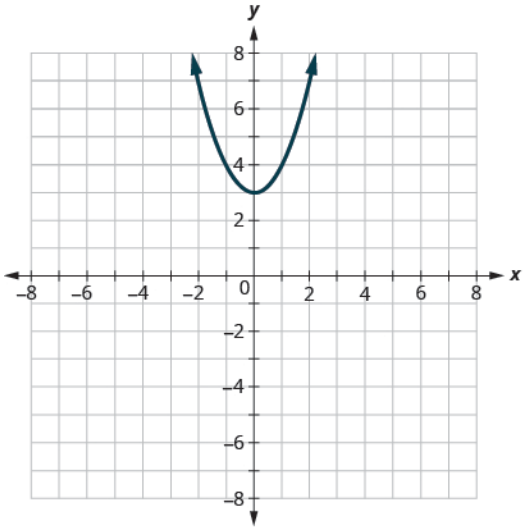
(a)



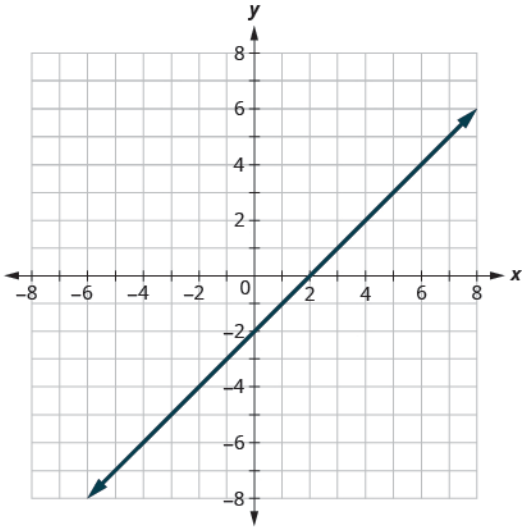
(b)

TRY IT : : 10.8

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



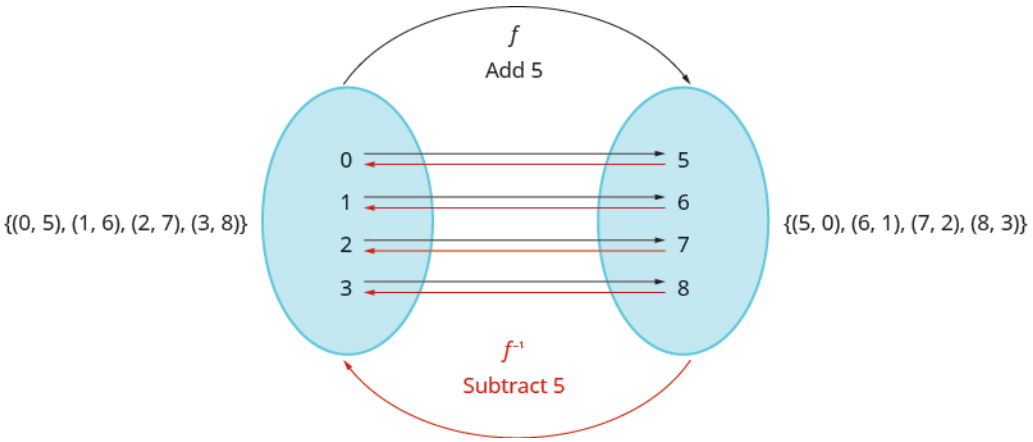
(a)



(b)

Find the Inverse of a Function

Let's look at a one-to one function, f, represented by the ordered pairs $\{(0, 5), (1, 6), (2, 7), (3, 8)\}$. For each x-value, f adds 5 to get the y-value. To 'undo' the addition of 5, we subtract 5 from each y-value and get back to the original x-value. We can call this "taking the inverse of f" and name the function f^{-1}.



Notice that that the ordered pairs of f and f^{-1} have their x-values and y-values reversed. The domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f.

Inverse of a Function Defined by Ordered Pairs

If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).

In the next example we will find the inverse of a function defined by ordered pairs.

EXAMPLE 10.5

Find the inverse of the function $\{(0, 3), (1, 5), (2, 7), (3, 9)\}$. Determine the domain and range of the inverse function.

Solution

This function is one-to-one since every x-value is paired with exactly one y-value.

To find the inverse we reverse the x-values and y-values in the ordered pairs of the function.

Function	$\{(0, 3), (1, 5), (2, 7), (3, 9)\}$
Inverse Function	$\{(3, 0), (5, 1), (7, 2), (9, 3)\}$
Domain of Inverse Function	$\{3, 5, 7, 9\}$
Range of Inverse Function	$\{0, 1, 2, 3\}$

TRY IT : : 10.9

Find the inverse of $\{(0, 4), (1, 7), (2, 10), (3, 13)\}$. Determine the domain and range of the inverse function.

TRY IT : : 10.10

Find the inverse of $\{(-1, 4), (-2, 1), (-3, 0), (-4, 2)\}$. Determine the domain and range of the inverse function.

We just noted that if $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).

So if a point (a, b) is on the graph of a function $f(x)$, then the ordered pair (b, a) is on the graph of $f^{-1}(x)$. See Figure 10.2.

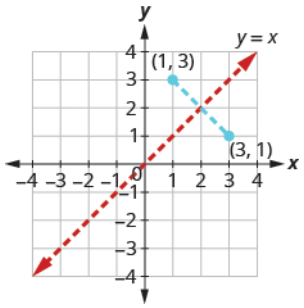


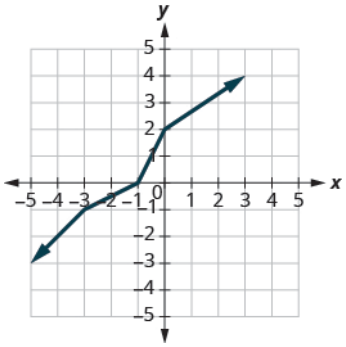
Figure 10.2

The distance between any two pairs (a, b) and (b, a) is cut in half by the line $y = x$. So we say the points are mirror images of each other through the line $y = x$.

Since every point on the graph of a function $f(x)$ is a mirror image of a point on the graph of $f^{-1}(x)$, we say the graphs are mirror images of each other through the line $y = x$. We will use this concept to graph the inverse of a function in the next example.

EXAMPLE 10.6

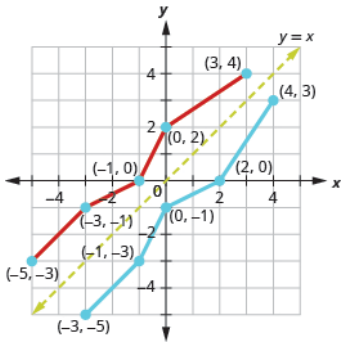
Graph, on the same coordinate system, the inverse of the one-to one function shown.



Solution

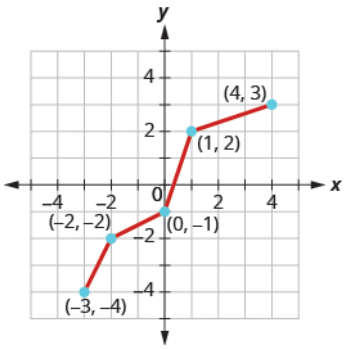
We can use points on the graph to find points on the inverse graph. Some points on the graph are: $(-5, -3), (-3, -1), (-1, 0), (0, 2), (3, 4)$.

So, the inverse function will contain the points: $(-3, -5), (-1, -3), (0, -1), (2, 0), (4, 3)$.

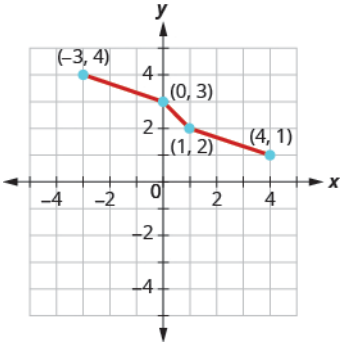


Notice how the graph of the original function and the graph of the inverse functions are mirror images through the line $y = x$.

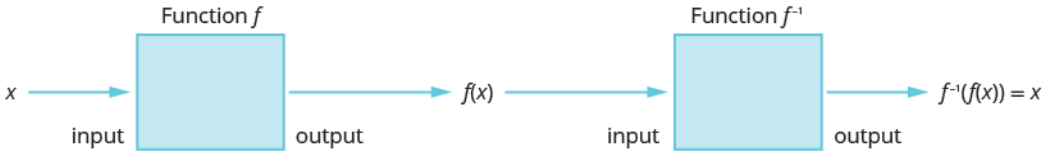
TRY IT : : 10.11 Graph, on the same coordinate system, the inverse of the one-to one function.



TRY IT : : 10.12 Graph, on the same coordinate system, the inverse of the one-to one function.



When we began our discussion of an inverse function, we talked about how the inverse function 'undoes' what the original function did to a value in its domain in order to get back to the original *x*-value.



Inverse Functions

$$f^{-1}(f(x)) = x, \text{ for all } x \text{ in the domain of } f$$
$$f\left(f^{-1}(x)\right) = x, \text{ for all } x \text{ in the domain of } f^{-1}$$

We can use this property to verify that two functions are inverses of each other.

EXAMPLE 10.7

Verify that $f(x) = 5x - 1$ and $g(x) = \frac{x+1}{5}$ are inverse functions.

Solution

The functions are inverses of each other if $g(f(x)) = x$ and $f(g(x)) = x$.

	$g(f(x)) \stackrel{?}{=} x$
Substitute $5x-1$ for $f(x)$.	$g(5x-1) \stackrel{?}{=} x$
Find $g(5x-1)$ where $g(x)=\frac{x+1}{5}$.	$\frac{(5x-1)+1}{5} \stackrel{?}{=} x$
Simplify.	$\frac{5x}{5} \stackrel{?}{=} x$
Simplify.	$x = x$ ✓

	$f(g(x)) \stackrel{?}{=} x$
Substitute $\frac{x+1}{5}$ for $g(x)$.	$f\left(\frac{x+1}{5}\right) \stackrel{?}{=} x$
Find $f\left(\frac{x+1}{5}\right)$ where $f(x)=5x-1$.	$5\left(\frac{x+1}{5}\right)-1 \stackrel{?}{=} x$
Simplify.	$x+1-1 \stackrel{?}{=} x$
Simplify.	$x = x$ ✓

Since both $g(f(x))=x$ and $f(g(x))=x$ are true, the functions $f(x)=5x-1$ and $g(x)=\frac{x+1}{5}$ are inverse functions. That is, they are inverses of each other.

TRY IT : : 10.13 Verify that the functions are inverse functions.

$f(x)=4x-3$ and $g(x)=\frac{x+3}{4}$.

TRY IT : : 10.14 Verify that the functions are inverse functions.

$f(x)=2x+6$ and $g(x)=\frac{x-6}{2}$.

We have found inverses of function defined by ordered pairs and from a graph. We will now look at how to find an inverse using an algebraic equation. The method uses the idea that if $f(x)$ is a one-to-one function with ordered pairs $(x,\ y)$, then its inverse function $f^{-1}(x)$ is the set of ordered pairs $(y,\ x)$.

If we reverse the *x* and *y* in the function and then solve for *y*, we get our inverse function.

EXAMPLE 10.8 HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

Find the inverse of $f(x)=4x+7$.

Solution

Step 1. Substitute *y* for *f*(*x*).	Replace *f*(*x*) with *y*.	$f(x)=4x+7$ $y=4x+7$
Step 2. Interchange the variables *x* and *y*.	Replace *x* with *y* and then *y* with *x*.	$x=4y+7$

Step 3. Solve for y.	Subtract 7 from each side. Divide by 4.	$x-7=4y$ $\frac{x-7}{4}=y$
Step 4. Substitute $f^{-1}(x)$ for y.	Replace y with $f^{-1}(x)$.	$\frac{x-7}{4}=f^{-1}(x)$
Step 5. Verify that the functions are inverses.	Show $f^{-1}(f(x))=x$ and $f(f^{-1}(x))=x$	$f^{-1}(f(x)) \stackrel{?}{=} x$ $f^{-1}(4x+7) \stackrel{?}{=} x$ $\frac{(4x+7)-7}{4} \stackrel{?}{=} x$ $\frac{4x}{4} \stackrel{?}{=} x$ $x=x$ ✓ $f(f^{-1}(x)) \stackrel{?}{=} x$ $f\left(\frac{x-7}{4}\right) \stackrel{?}{=} x$ $4\left(\frac{x-7}{4}\right)+7 \stackrel{?}{=} x$ $x-7+7 \stackrel{?}{=} x$ $x=x$ ✓

TRY IT : : 10.15 Find the inverse of the function $f(x)=5x-3$.

TRY IT : : 10.16 Find the inverse of the function $f(x)=8x+5$.

We summarize the steps below.

HOW TO : : HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

Step 1. Substitute y for $f(x)$.

Step 2. Interchange the variables x and y.

Step 3. Solve for y.

Step 4. Substitute $f^{-1}(x)$ for y.

Step 5. Verify that the functions are inverses.

EXAMPLE 10.9 HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

Find the inverse of $f(x)=\sqrt[5]{2x-3}$.

Solution

$$f(x) = \sqrt[5]{2x-3}$$

Substitute y for $f(x)$. $\quad y = \sqrt[5]{2x-3}$

Interchange the variables x and y. $\quad x = \sqrt[5]{2y-3}$

Solve for y. $\quad (x)^5 = \left(\sqrt[5]{2y-3}\right)^5$

$$x^5 = 2y-3$$

$$x^5+3 = 2y$$

$$\frac{x^5+3}{2} = y$$

Substitute $f^{-1}(x)$ for y. $\quad f^{-1}(x) = \frac{x^5+3}{2}$

Verify that the functions are inverses.

$$f^{-1}(f(x)) \stackrel{?}{=} x \qquad f\left(f^{-1}(x)\right) \stackrel{?}{=} x$$

$$f^{-1}\left(\sqrt[5]{2x-3}\right) \stackrel{?}{=} x \qquad f\left(\frac{x^5+3}{2}\right) \stackrel{?}{=} x$$

$$\frac{\left(\sqrt[5]{2x-3}\right)^5+3}{2} \stackrel{?}{=} x \qquad \sqrt[5]{2\left(\frac{x^5+3}{2}\right)-3} \stackrel{?}{=} x$$

$$\frac{2x-3+3}{2} \stackrel{?}{=} x \qquad \sqrt[5]{x^5+3-3} \stackrel{?}{=} x$$

$$\frac{2x}{2} \stackrel{?}{=} x \qquad \sqrt[5]{x^5} \stackrel{?}{=} x$$

$$x = x \checkmark \qquad x = x \checkmark$$

TRY IT : : 10.17 Find the inverse of the function $f(x) = \sqrt[5]{3x-2}$.

TRY IT : : 10.18 Find the inverse of the function $f(x) = \sqrt[4]{6x-7}$.

10.1 EXERCISES

Practice Makes Perfect

Find and Evaluate Composite Functions

In the following exercises, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

1. $f(x) = 4x + 3$ and $g(x) = 2x + 5$

2. $f(x) = 3x - 1$ and $g(x) = 5x - 3$

3. $f(x) = 6x - 5$ and $g(x) = 4x + 1$

4. $f(x) = 2x + 7$ and $g(x) = 3x - 4$

5. $f(x) = 3x$ and $g(x) = 2x^2 - 3x$

6. $f(x) = 2x$ and $g(x) = 3x^2 - 1$

7. $f(x) = 2x - 1$ and $g(x) = x^2 + 2$

8. $f(x) = 4x + 3$ and $g(x) = x^2 - 4$

In the following exercises, find the values described.

9. For functions $f(x) = 2x^2 + 3$ and $g(x) = 5x - 1$, find

ⓐ $(f \circ g)(-2)$

ⓑ $(g \circ f)(-3)$

ⓒ $(f \circ f)(-1)$

10. For functions $f(x) = 5x^2 - 1$ and $g(x) = 4x - 1$, find

ⓐ $(f \circ g)(1)$

ⓑ $(g \circ f)(-1)$

ⓒ $(f \circ f)(2)$

11. For functions $f(x) = 2x^3$ and $g(x) = 3x^2 + 2$, find

ⓐ $(f \circ g)(-1)$

ⓑ $(g \circ f)(1)$

ⓒ $(g \circ g)(1)$

12. For functions $f(x) = 3x^3 + 1$ and $g(x) = 2x^2 - 3$, find

ⓐ $(f \circ g)(-2)$

ⓑ $(g \circ f)(-1)$

ⓒ $(g \circ g)(1)$

Determine Whether a Function is One-to-One

In the following exercises, determine if the set of ordered pairs represents a function and if so, is the function one-to-one.

13. $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$

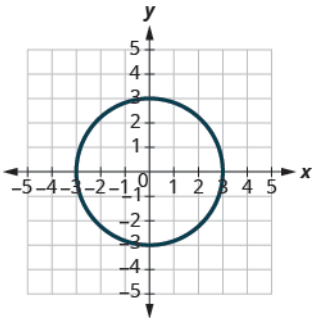
14. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

15. $\{(-3, -5), (-2, -3), (-1, -1), (0, 1), (1, 3), (2, 5), (3, 7)\}$

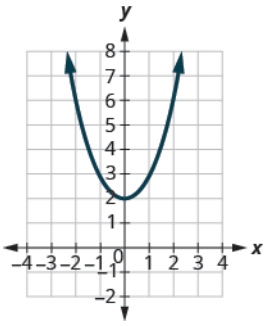
16. $\{(5, 3), (4, 2), (3, 1), (2, 0), (1, -1), (0, -2), (-1, -3)\}$

In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

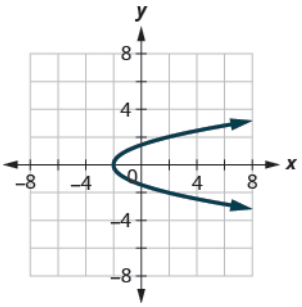
17. ⓐ



ⓑ



18. ⓐ



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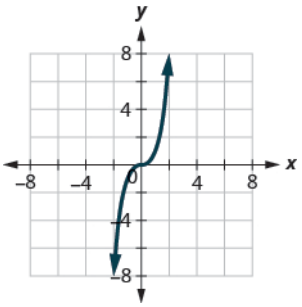
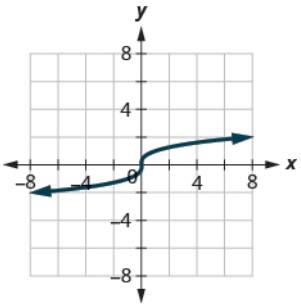


19. ⓐ



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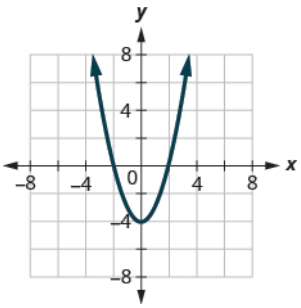
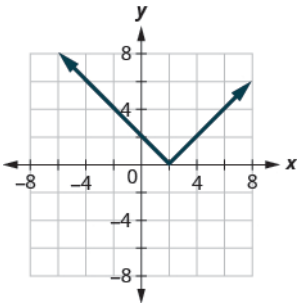
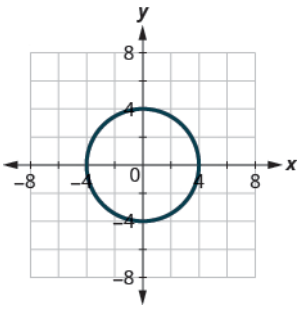


20. ⓐ



ⓑ



In the following exercises, find the inverse of each function. Determine the domain and range of the inverse function.

21. {(2, 1), (4, 2), (6, 3), (8, 4)}

22. {(6, 2), (9, 5), (12, 8), (15, 11)}

23. {(0, −2), (1, 3), (2, 7), (3, 12)}

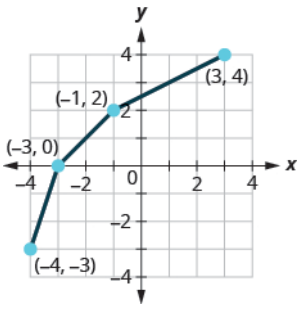
24. {(0, 0), (1, 1), (2, 4), (3, 9)}

25. {(−2, −3), (−1, −1), (0, 1), (1, 3)}

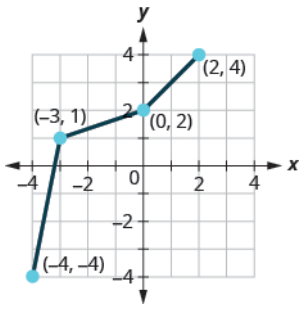
26. {(5, 3), (4, 2), (3, 1), (2, 0)}

In the following exercises, graph, on the same coordinate system, the inverse of the one-to-one function shown.

27.



28.



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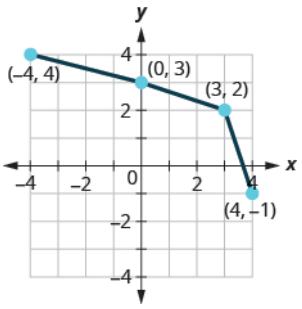
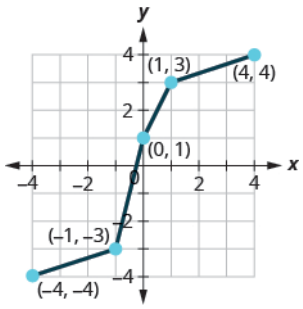


30.



In the following exercises, determine whether or not the given functions are inverses.

31. $f(x) = x + 8$ and $g(x) = x - 8$

32. $f(x) = x - 9$ and $g(x) = x + 9$

33. $f(x) = 7x$ and $g(x) = \frac{x}{7}$

34. $f(x) = \frac{x}{11}$ and $g(x) = 11x$

35. $f(x) = 7x + 3$ and $g(x) = \frac{x-3}{7}$

36. $f(x) = 5x - 4$ and $g(x) = \frac{x-4}{5}$

37. $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 2$

38. $f(x) = \sqrt[3]{x-4}$ and $g(x) = x^3 + 4$

In the following exercises, find the inverse of each function.

39. $f(x) = x - 12$

40. $f(x) = x + 17$

41. $f(x) = 9x$

42. $f(x) = 8x$

43. $f(x) = \frac{x}{6}$

44. $f(x) = \frac{x}{4}$

45. $f(x) = 6x - 7$

46. $f(x) = 7x - 1$

47. $f(x) = -2x + 5$

48. $f(x) = -5x - 4$

49. $f(x) = x^2 + 6, \quad x \geq 0$

50. $f(x) = x^2 - 9, \quad x \geq 0$

51. $f(x) = x^3 - 4$

52. $f(x) = x^3 + 6$

53. $f(x) = \frac{1}{x+2}$

54. $f(x) = \frac{1}{x-6}$

55. $f(x) = \sqrt{x-2}, \quad x \geq 2$

56. $f(x) = \sqrt{x+8}, \quad x \geq -8$

57. $f(x) = \sqrt[3]{x-3}$

58. $f(x) = \sqrt[3]{x+5}$

59. $f(x) = \sqrt[4]{9x-5}, \quad x \geq \frac{5}{9}$

60. $f(x) = \sqrt[4]{8x-3}, \quad x \geq \frac{3}{8}$

61. $f(x) = \sqrt[5]{-3x+5}$

62. $f(x) = \sqrt[5]{-4x-3}$

Writing Exercises

63. Explain how the graph of the inverse of a function is related to the graph of the function.

64. Explain how to find the inverse of a function from its equation. Use an example to demonstrate the steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find and evaluate composite functions.			
determine whether a function is one-to-one.			
find the inverse of a function.			

ⓑ *If most of your checks were:*

...confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

...with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

...no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

10.2 Evaluate and Graph Exponential Functions

Learning Objectives

By the end of this section, you will be able to:

- Graph exponential functions
- Solve Exponential equations
- Use exponential models in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\left(\frac{x^3}{x^2}\right)$.
   If you missed this problem, review Example 5.13.
2. Evaluate: ⓐ 2^0 ⓑ $\left(\frac{1}{3}\right)^0$.
   If you missed this problem, review Example 5.14.
3. Evaluate: ⓐ 2^{-1} ⓑ $\left(\frac{1}{3}\right)^{-1}$.
   If you missed this problem, review Example 5.15.

Graph Exponential Functions

The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

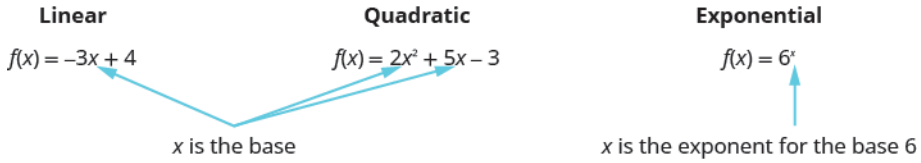
An **exponential function** is a function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$.

Exponential Function

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form

$$f(x) = a^x$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.



Our definition says $a \neq 1$. If we let $a = 1$, then $f(x) = a^x$ becomes $f(x) = 1^x$. Since $1^x = 1$ for all real numbers, $f(x) = 1$. This is the constant function.

Our definition also says $a > 0$. If we let a base be negative, say -4, then $f(x) = (-4)^x$ is not a real number when $x = \frac{1}{2}$.

$$\begin{aligned} f(x) &= (-4)^x \\ f\left(\tfrac{1}{2}\right) &= (-4)^{\frac{1}{2}} \\ f\left(\tfrac{1}{2}\right) &= \sqrt{-4} \quad \text{not a real number} \end{aligned}$$

In fact, $f(x) = (-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a > 0$.

By graphing a few exponential functions, we will be able to see their unique properties.

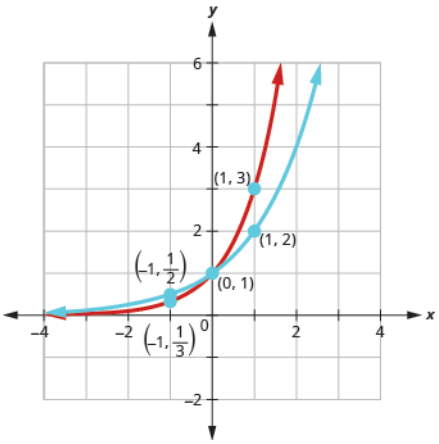
EXAMPLE 10.10

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

Solution

We will use point plotting to graph the functions.

x	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 3^x$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	$(0, 1)$	$3^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$	$3^1 = 3$	$(1, 3)$
2	$2^2 = 4$	$(2, 4)$	$3^2 = 9$	$(2, 9)$
3	$2^3 = 8$	$(3, 8)$	$3^3 = 27$	$(3, 27)$



TRY IT : : 10.19 Graph: $f(x) = 4^x$.

TRY IT : : 10.20 Graph: $g(x) = 5^x$.

If we look at the graphs from the previous Example and Try Its, we can identify some of the properties of exponential functions.

The graphs of $f(x) = 2^x$ and $g(x) = 3^x$, as well as the graphs of $f(x) = 4^x$ and $g(x) = 5^x$, all have the same basic shape. This is the shape we expect from an exponential function where $a > 1$.

We notice, that for each function, the graph contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any a.

The graph of each function, $f(x) = a^x$ also contains the point $(1, a)$. The graph of $f(x) = 2^x$ contained $(1, 2)$ and the graph of $g(x) = 3^x$ contained $(1, 3)$. This makes sense as $a^1 = a$.

Notice too, the graph of each function $f(x) = a^x$ also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x) = 2^x$ contained $\left(-1, \frac{1}{2}\right)$ and the graph of $g(x) = 3^x$ contained $\left(-1, \frac{1}{3}\right)$. This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty, \infty)$.

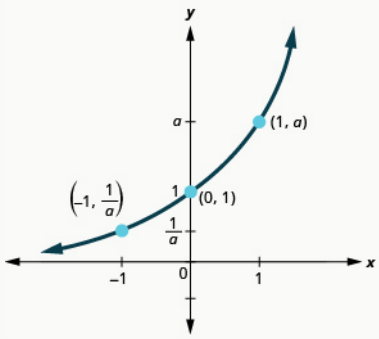
Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers.

We write the range in interval notation as $(0, \infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y = 0$, the *x*-axis, a horizontal asymptote.

Properties of the Graph of $f(x) = a^x$ when $a > 1$

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
x-intercept	None
y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$



Our definition of an exponential function $f(x) = a^x$ says $a > 0$, but the examples and discussion so far has been about functions where $a > 1$. What happens when $0 < a < 1$? The next example will explore this possibility.

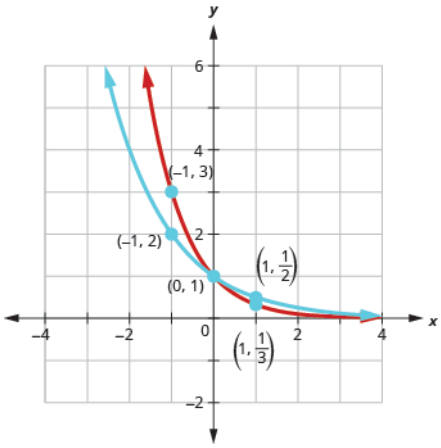
EXAMPLE 10.11

On the same coordinate system, graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

Solution

We will use point plotting to graph the functions.

x	$f(x)=\left(\frac{1}{2}\right)^x$	$(x, f(x))$	$g(x)=\left(\frac{1}{3}\right)^x$	$(x, g(x))$
–2	$\left(\frac{1}{2}\right)^{-2}=2^2=4$	(–2, 4)	$\left(\frac{1}{3}\right)^{-2}=3^2=9$	(–2, 9)
–1	$\left(\frac{1}{2}\right)^{-1}=2^1=2$	(–1, 2)	$\left(\frac{1}{3}\right)^{-1}=3^1=3$	(–1, 3)
0	$\left(\frac{1}{2}\right)^0=1$	(0, 1)	$\left(\frac{1}{3}\right)^0=1$	(0, 1)
1	$\left(\frac{1}{2}\right)^1=\frac{1}{2}$	$\left(1,\frac{1}{2}\right)$	$\left(\frac{1}{3}\right)^1=\frac{1}{3}$	$\left(1,\frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$	$\left(2,\frac{1}{4}\right)$	$\left(\frac{1}{3}\right)^2=\frac{1}{9}$	$\left(2,\frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^3=\frac{1}{8}$	$\left(3,\frac{1}{8}\right)$	$\left(\frac{1}{3}\right)^3=\frac{1}{27}$	$\left(3,\frac{1}{27}\right)$



TRY IT : : 10.21 Graph: $f(x)=\left(\frac{1}{4}\right)^x$.

TRY IT : : 10.22 Graph: $g(x)=\left(\frac{1}{5}\right)^x$.

Now let's look at the graphs from the previous Example and Try Its so we can now identify some of the properties of exponential functions where $0<a<1$.

The graphs of $f(x)=\left(\frac{1}{2}\right)^x$ and $g(x)=\left(\frac{1}{3}\right)^x$ as well as the graphs of $f(x)=\left(\frac{1}{4}\right)^x$ and $g(x)=\left(\frac{1}{5}\right)^x$ all have the same basic shape. While this is the shape we expect from an exponential function where $0<a<1$, the graphs go down from left to right while the previous graphs, when $a>1$, went from up from left to right.

We notice that for each function, the graph still contains the point (0, 1). This make sense because $a^0=1$ for any *a*.

As before, the graph of each function, $f(x)=a^x$, also contains the point $(1, a)$. The graph of $f(x)=\left(\frac{1}{2}\right)^x$ contained $\left(1, \frac{1}{2}\right)$ and the graph of $g(x)=\left(\frac{1}{3}\right)^x$ contained $\left(1, \frac{1}{3}\right)$. This makes sense as $a^1=a$.

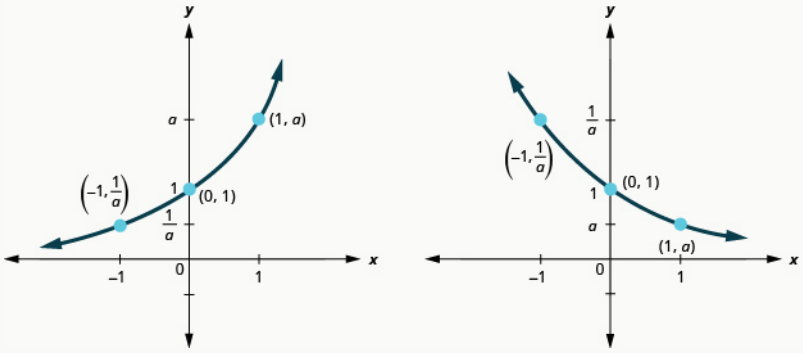
Notice too that the graph of each function, $f(x)=a^x$, also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x)=\left(\frac{1}{2}\right)^x$ contained $(-1, 2)$ and the graph of $g(x)=\left(\frac{1}{3}\right)^x$ contained $(-1, 3)$. This makes sense as $a^{-1}=\frac{1}{a}$.

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $(-\infty, \infty)$. Again, the graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Properties of the Graph of $f(x) = a^x$

when $a > 1$		when $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

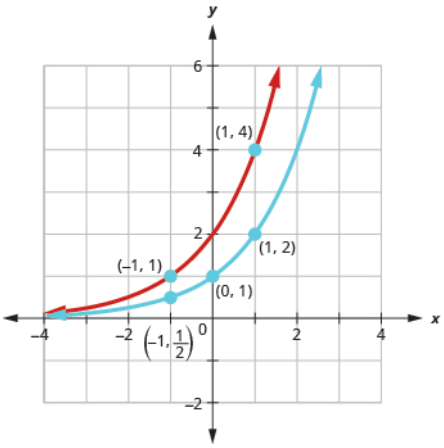
EXAMPLE 10.12

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 2^{x+1}$.

Solution

We will use point plotting to graph the functions.

x	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 2^{x+1}$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$2^{-2+1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-2, \frac{1}{2}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$2^{-1+1} = 2^0 = 1$	$(-1, 1)$
0	$2^0 = 1$	$(0, 1)$	$2^{0+1} = 2^1 = 2$	$(0, 2)$
1	$2^1 = 2$	$(1, 2)$	$2^{1+1} = 2^2 = 4$	$(1, 4)$
2	$2^2 = 4$	$(2, 4)$	$2^{2+1} = 2^3 = 8$	$(2, 8)$
3	$2^3 = 8$	$(3, 8)$	$2^{3+1} = 2^4 = 16$	$(3, 16)$



TRY IT : : 10.23 On the same coordinate system, graph: $f(x) = 2^x$ and $g(x) = 2^{x-1}$.

TRY IT : : 10.24 On the same coordinate system, graph: $f(x) = 3^x$ and $g(x) = 3^{x+1}$.

Looking at the graphs of the functions $f(x) = 2^x$ and $g(x) = 2^{x+1}$ in the last example, we see that adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

Let's now consider another situation that might be graphed more easily by translation, once we recognize the pattern.

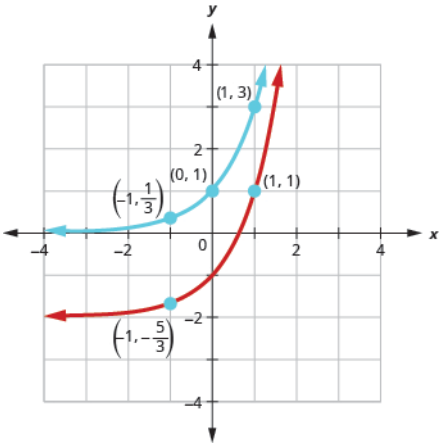
EXAMPLE 10.13

On the same coordinate system graph $f(x) = 3^x$ and $g(x) = 3^x - 2$.

Solution

We will use point plotting to graph the functions.

x	$f(x) = 3^x$	$(x, g(x))$	$g(x) = 3^x - 2$	$(x, g(x))$
-2	$3^{-2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$	$3^{-2} - 2 = \frac{1}{9} - 2 = -\frac{17}{9}$	$\left(-2, -\frac{17}{9}\right)$
-1	$3^{-1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$	$3^{-1} - 2 = \frac{1}{3} - 2 = -\frac{5}{3}$	$\left(-1, -\frac{5}{3}\right)$
0	$3^0 = 1$	$(0, 1)$	$3^0 - 2 = 1 - 2 = -1$	$(0, -1)$
1	$3^1 = 3$	$(1, 3)$	$3^1 - 2 = 3 - 2 = 1$	$(1, 1)$
2	$3^2 = 9$	$(2, 9)$	$3^2 - 2 = 9 - 2 = 7$	$(2, 8)$



TRY IT : : 10.25 On the same coordinate system, graph: $f(x) = 3^x$ and $g(x) = 3^x + 2$.

TRY IT : : 10.26 On the same coordinate system, graph: $f(x) = 4^x$ and $g(x) = 4^x - 2$.

Looking at the graphs of the functions $f(x) = 3^x$ and $g(x) = 3^x - 2$ in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, *e*. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $\left(1+\frac{1}{n}\right)^n$ as *n* gets larger and larger. We say, as *n* approaches infinity, or increases without bound. The table shows the value of $\left(1+\frac{1}{n}\right)^n$ for several values of n.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829...
1,000	2.716923932...
10,000	2.718145927...
100,000	2.718268237...
1,000,000	2.718280469...
1,000,000,000	2.718281827...

Table 10.7

$$e \approx 2.718281827$$

The number *e* is like the number π in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number *e* is called the **natural base**.

Natural Base e

The number *e* is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as *n* increases without bound. We say, as *n* approaches infinity,

$$e \approx 2.718281827...$$

The exponential function whose base is e, $f(x)=e^x$ is called the **natural exponential function**.

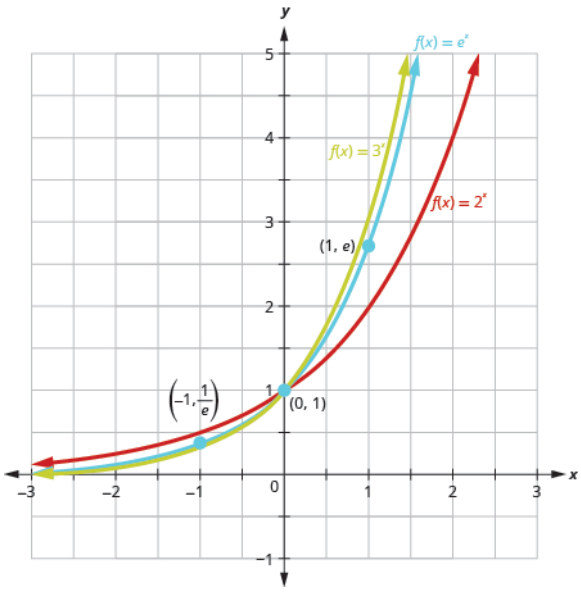
Natural Exponential Function

The natural exponential function is an exponential function whose base is e

$$f(x)=e^x$$

The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Let's graph the function $f(x)=e^x$ on the same coordinate system as $g(x)=2^x$ and $h(x)=3^x$.



Notice that the graph of $f(x)=e^x$ is "between" the graphs of $g(x)=2^x$ and $h(x)=3^x$. Does this make sense as $2<e<3$?

Solve Exponential Equations

Equations that include an exponential expression a^x are called exponential equations. To solve them we use a property that says as long as $a>0$ and $a\neq 1$, if $a^x=a^y$ then it is true that $x=y$. In other words, in an exponential equation, if the bases are equal then the exponents are equal.

One-to-One Property of Exponential Equations

For $a>0$ and $a\neq 1$,

$$\text{If } a^x=a^y, \text{ then } x=y.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

EXAMPLE 10.14 HOW TO SOLVE AN EXPONENTIAL EQUATION

Solve: $3^{2x-5}=27$.

Solution

Step 1. Write both sides of the equation with the same base.	Since the left side has base 3, we write the right side with base 3. $27 = 3^3$	$3^{2x-5} = 27$ $3^{2x-5} = 3^3$
Step 2. Write a new equation by setting the exponents equal.	Since the bases are the same, the exponents must be equal.	$2x - 5 = 3$
Step 3. Solve the equation.	Add 5 to each side. Divide by 2.	$2x = 8$ $x = 4$
Step 4. Check the solution.	Substitute $x = 4$ into the original equation.	$3^{2x-5} = 27$ $3^{2\cdot 4-5} \stackrel{?}{=} 27$ $3^3 \stackrel{?}{=} 27$ $27 = 27$ ✓

TRY IT : : 10.27 Solve: $3^{3x-2} = 81$.

TRY IT : : 10.28 Solve: $7^{x-3} = 7$.

The steps are summarized below.

HOW TO : : HOW TO SOLVE AN EXPONENTIAL EQUATION

Step 1. Write both sides of the equation with the same base, if possible.
Step 2. Write a new equation by setting the exponents equal.
Step 3. Solve the equation.
Step 4. Check the solution.

In the next example, we will use our properties on exponents.

EXAMPLE 10.15

Solve $\frac{e^{x^2}}{e^3} = e^{2x}$.

Solution

	$\frac{e^{x^2}}{e^3} = e^{2x}$
Use the Property of Exponents: $\frac{a^m}{a^n} = a^{m-n}$.	$e^{x^2-3} = e^{2x}$
Write a new equation by setting the exponents equal.	$x^2 - 3 = 2x$
Solve the equation.	$x^2 - 2x - 3 = 0$

$$(x-3)(x+1)=0$$

$$x=3,\quad x=-1$$

Check the solutions.

$x=3$	$x=-1$
$\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$	$\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$
$\frac{e^{3^2}}{e^3} \stackrel{?}{=} e^{2\cdot 3}$	$\frac{e^{(-1)^2}}{e^3} \stackrel{?}{=} e^{2\cdot(-1)}$
$\frac{e^9}{e^3} \stackrel{?}{=} e^6$	$\frac{e^1}{e^3} \stackrel{?}{=} e^{-2}$
$e^6 = e^6$ ✓	$e^{-2} = e^{-2}$ ✓

TRY IT : : 10.29 Solve: $\frac{e^{x^2}}{e^x}=e^2$.

TRY IT : : 10.30 Solve: $\frac{e^{x^2}}{e^x}=e^6$.

Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, *P*, is invested at an interest rate, *r*, for *t* years, the new balance, *A*, will depend on how often the interest is compounded. If the interest is compounded *n* times a year we use the formula $A=P\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A=Pe^{rt}$. These are the formulas for **compound interest**.

Compound Interest

For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A*, is:

$A=P\left(1+\frac{r}{n}\right)^{nt}$ when compounded n times a year.

$A=Pe^{rt}$ when compounded continuously.

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

EXAMPLE 10.16

A total of $\$10{,}000$ was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$\begin{aligned} A &= ? \\ P &= \$10{,}000 \\ r &= 0.05 \\ t &= 18 \text{ years} \end{aligned}$$

ⓐ

For quarterly compounding, $n = 4$. There are 4 quarters in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10{,}000\left(1 + \frac{0.05}{4}\right)^{4 \cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24{,}459.20$$

ⓑ

For monthly compounding, $n = 12$. There are 12 months in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10{,}000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 18}$$

Compute the amount.

$$A = \$24{,}550.08$$

ⓒ

For compounding continuously,

$$A = Pe^{rt}$$

Substitute the values in the formula.

$$A = 10{,}000e^{0.05 \cdot 18}$$

Compute the amount.

$$A = \$24{,}596.03$$

TRY IT : : 10.31

Angela invested $\$15{,}000$ in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

TRY IT : : 10.32

Allan invested $10,000 in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0, as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, *r*, for a certain time, *t*, the final amount, *A*, is:

$$A = A_0 e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

EXAMPLE 10.17

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution

Identify the values of each variable in the formulas.	$A = ?$
Be sure to put the percent in decimal form.	$A_0 = 100$
Be sure the units match—the rate is per hour and the time is in hours.	$r = 0.25/\text{hour}$ $t = 24 \text{ hours}$
Substitute the values in the formula: $A = A_0 e^{rt}$.	$A = 100e^{0.25 \cdot 24}$
Compute the amount.	$A = 40{,}342.88$
Round to the nearest whole virus.	$A = 40{,}343$ The researcher will find 40,34 viruses.

TRY IT : : 10.33

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

TRY IT : : 10.34

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

MEDIA : :

Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

- **Graphing Exponential Functions (https://openstax.org/l/37Graphexponent)**
- **Solving Exponential Equations (https://openstax.org/l/37Solvelikebase)**
- **Applications of Exponential Functions (https://openstax.org/l/37Exponentapp)**
- **Continuously Compound Interest (https://openstax.org/l/37Compoundint)**
- **Radioactive Decay and Exponential Growth (https://openstax.org/l/37Exponentdecay)**

10.2 EXERCISES

Practice Makes Perfect

Graph Exponential Functions

In the following exercises, graph each exponential function.

65. $f(x) = 2^x$

66. $g(x) = 3^x$

67. $f(x) = 6^x$

68. $g(x) = 7^x$

69. $f(x) = (1.5)^x$

70. $g(x) = (2.5)^x$

71. $f(x) = \left(\frac{1}{2}\right)^x$

72. $g(x) = \left(\frac{1}{3}\right)^x$

73. $f(x) = \left(\frac{1}{6}\right)^x$

74. $g(x) = \left(\frac{1}{7}\right)^x$

75. $f(x) = (0.4)^x$

76. $g(x) = (0.6)^x$

In the following exercises, graph each function in the same coordinate system.

77. $f(x) = 4^x,\ g(x) = 4^{x-1}$

78. $f(x) = 3^x,\ g(x) = 3^{x-1}$

79. $f(x) = 2^x,\ g(x) = 2^{x-2}$

80. $f(x) = 2^x,\ g(x) = 2^{x+2}$

81. $f(x) = 3^x,\ g(x) = 3^x + 2$

82. $f(x) = 4^x,\ g(x) = 4^x + 2$

83. $f(x) = 2^x,\ g(x) = 2^x + 1$

84. $f(x) = 2^x,\ g(x) = 2^x - 1$

In the following exercises, graph each exponential function.

85. $f(x) = 3^{x+2}$

86. $f(x) = 3^{x-2}$

87. $f(x) = 2^x + 3$

88. $f(x) = 2^x - 3$

89. $f(x) = \left(\frac{1}{2}\right)^{x-4}$

90. $f(x) = \left(\frac{1}{2}\right)^x - 3$

91. $f(x) = e^x + 1$

92. $f(x) = e^{x-2}$

93. $f(x) = -2^x$

94. $f(x) = 3^x$

Solve Exponential Equations

In the following exercises, solve each equation.

95. $2^{3x-8} = 16$

96. $2^{2x-3} = 32$

97. $3^{x+3} = 9$

98. $3^{x^2} = 81$

99. $4^{x^2} = 4$

100. $4^x = 32$

101. $4^{x+2} = 64$

102. $4^{x+3} = 16$

103. $2^{x^2+2x} = \frac{1}{2}$

104. $3^{x^2-2x} = \frac{1}{3}$

105. $e^{3x} \cdot e^4 = e^{10}$

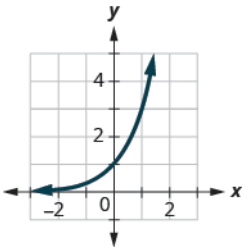
106. $e^{2x} \cdot e^3 = e^9$

107. $\frac{e^{x^2}}{e^2} = e^x$

108. $\frac{e^{x^2}}{e^3} = e^{2x}$

In the following exercises, match the graphs to one of the following functions: ⓐ 2^x ⓑ 2^{x+1} ⓒ 2^{x-1} ⓓ $2^x + 2$ ⓔ $2^x - 2$ ⓕ 3^x

109.



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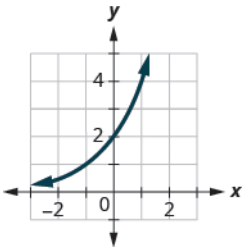

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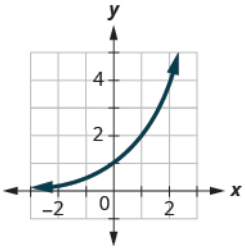
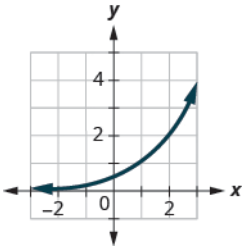
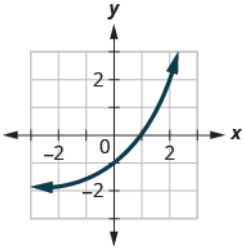


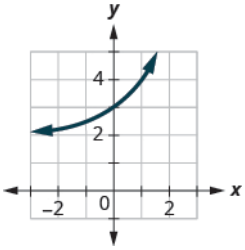
112.



113.



114.



Use exponential models in applications

In the following exercises, use an exponential model to solve.

115. Edgar accumulated $\$5{,}000$ in credit card debt. If the interest rate is 20% per year, and he does not make any payments for 2 years, how much will he owe on this debt in 2 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

116. Cynthia invested $\$12{,}000$ in a savings account. If the interest rate is 6%, how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

117. Rochelle deposits $\$5{,}000$ in an IRA. What will be the value of her investment in 25 years if the investment is earning 8% per year and is compounded continuously?

118. Nazerhy deposits $\$8{,}000$ in a certificate of deposit. The annual interest rate is 6% and the interest will be compounded quarterly. How much will the certificate be worth in 10 years?

119. A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. He starts his experiment with 100 of the bacteria that grows at a rate of 6% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

120. A biologist is observing the growth pattern of a virus. She starts with 50 of the virus that grows at a rate of 20% per hour. She will check on the virus in 24 hours. How many viruses will she find?

121. In the last ten years the population of Indonesia has grown at a rate of 1.12% per year to 258,316,051. If this rate continues, what will be the population in 10 more years?

122. In the last ten years the population of Brazil has grown at a rate of 0.9% per year to 205,823,665. If this rate continues, what will be the population in 10 more years?

Writing Exercises

123. Explain how you can distinguish between exponential functions and polynomial functions.

124. Compare and contrast the graphs of $y = x^2$ and $y = 2^x$.

125. What happens to an exponential function as the values of x decreases? Will the graph ever cross the y-axis? Explain.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
graph exponential eunctions.			
solve exponential equations.			
use exponential models in applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

10.3 Evaluate and Graph Logarithmic Functions

Learning Objectives

By the end of this section, you will be able to:

- Convert between exponential and logarithmic form
- Evaluate logarithmic functions
- Graph Logarithmic functions
- Solve logarithmic equations
- Use logarithmic models in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 81$.
   If you missed this problem, review **Example 6.46**.
2. Evaluate: 3^{-2}.
   If you missed this problem, review **Example 5.15**.
3. Solve: $2^4 = 3x - 5$.
   If you missed this problem, review **Example 2.2**.

We have spent some time finding the inverse of many functions. It works well to 'undo' an operation with another operation. Subtracting 'undoes' addition, multiplication 'undoes' division, taking the square root 'undoes' squaring.

As we studied the exponential function, we saw that it is one-to-one as its graphs pass the horizontal line test. This means an exponential function does have an inverse. If we try our algebraic method for finding an inverse, we run into a problem.

	$f(x) = a^x$
Rewrite with $y = f(x)$.	$y = a^x$
Interchange the variables x and y.	$x = a^y$
Solve for y.	Oops! We have no way to solve for y!

To deal with this we define the logarithm function with base a to be the inverse of the exponential function $f(x) = a^x$.

We use the notation $f^{-1}(x) = \log_a x$ and say the inverse function of the exponential function is the logarithmic function.

Logarithmic Function

The function $f(x) = \log_a x$ is the **logarithmic function** with base a, where $a > 0$, $x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

Convert Between Exponential and Logarithmic Form

Since the equations $y = \log_a x$ and $x = a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See **Figure 10.3**. Notice the positions of the exponent and base.

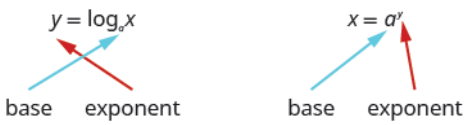


Figure 10.3

If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, "base to the exponent give us the number."

EXAMPLE 10.18

Convert to logarithmic form: ⓐ $2^3 = 8$, ⓑ $5^{\frac{1}{2}} = \sqrt{5}$, and ⓒ $\left(\frac{1}{2}\right)^x = \frac{1}{16}$.

Solution

Identify the base and the exponent.

(a)	(b)	(c)
$2^3 = 8$	$5^{\frac{1}{2}} = \sqrt{5}$	$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$
$y = \log_a x$	$y = \log_a x$	$y = \log_a x$
$3 = \log_2 8$	$\frac{1}{2} = \log_5 \sqrt{5}$	$4 = \log_{\frac{1}{2}} \frac{1}{16}$
If $2^3 = 8$, then $3 = \log_2 8$.	If $5^{\frac{1}{2}} = \sqrt{5}$, then $\frac{1}{2} = \log_5 \sqrt{5}$.	If $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, then $4 = \log_{\frac{1}{2}} \frac{1}{16}$.

TRY IT : : 10.35 Convert to logarithmic form: ⓐ $3^2 = 9$ ⓑ $7^{\frac{1}{2}} = \sqrt{7}$ ⓒ $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

TRY IT : : 10.36 Convert to logarithmic form: ⓐ $4^3 = 64$ ⓑ $4^{\frac{1}{3}} = \sqrt[3]{4}$ ⓒ $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

In the next example we do the reverse—convert logarithmic form to exponential form.

EXAMPLE 10.19

Convert to exponential form: ⓐ $2 = \log_8 64$, ⓑ $0 = \log_4 1$, and ⓒ $-3 = \log_{10} \frac{1}{1000}$.

Solution

Identify the base and the exponent.

(a)	(b)	(c)
$2 = \log_8 64$	$0 = \log_4 1$	$-3 = \log_{10} \frac{1}{1000}$
$x = a^y$	$x = a^y$	$x = a^y$
$64 = 8^2$	$1 = 4^0$	$\frac{1}{1000} = \log^{-3}$
If $2 = \log_8 64$, then $64 = 8^2$.	If $0 = \log_4 1$, then $1 = 4^0$.	If $-3 = \log_{10} \frac{1}{1000}$, then $\frac{1}{1000} = 10^{-3}$.

TRY IT : : 10.37 Convert to exponential form: ⓐ $3 = \log_4 64$ ⓑ $0 = \log_x 1$ ⓒ $-2 = \log_{10} \frac{1}{100}$

TRY IT : : 10.38 Convert to exponential form: ⓐ $3 = \log_3 27$ ⓑ $0 = \log_x 1$ ⓒ $-1 = \log_{10} \frac{1}{10}$

Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

EXAMPLE 10.20

Find the value of x: ⓐ $\log_x 36 = 2$, ⓑ $\log_4 x = 3$, and ⓒ $\log_{\frac{1}{2}} \frac{1}{8} = x$.

Solution

ⓐ

	$\log_x 36 = 2$
Convert to exponential form.	$x^2 = 36$
Solve the quadratic.	$x = 6, \quad \cancel{x = -6}$
The base of a logarithmic function must be positive, so we eliminate $x = -6$.	$x = 6$ Therefore, $\log_6 36 = 2$.

ⓑ

	$\log_4 x = 3$
Convert to exponential form.	$4^3 = x$
Simplify.	$x = 64$ Therefore, $\log_4 64 = 3$.

ⓒ

	$\log_{\frac{1}{2}} \frac{1}{8} = x$
Convert to exponential form.	$\left(\frac{1}{2}\right)^x = \frac{1}{8}$
Rewrite $\frac{1}{8}$ as $\left(\frac{1}{2}\right)^3$.	$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$
With the same base, the exponents must be equal.	$x = 3$ Therefore, $\log_{\frac{1}{2}} \frac{1}{8} = 3$

TRY IT : : 10.39 Find the value of x: ⓐ $\log_x 64 = 2$ ⓑ $\log_5 x = 3$ ⓒ $\log_{\frac{1}{2}} \frac{1}{4} = x$

TRY IT : : 10.40 Find the value of x: ⓐ $\log_x 81 = 2$ ⓑ $\log_3 x = 5$ ⓒ $\log_{\frac{1}{3}} \frac{1}{27} = x$

When see an expression such as $\log_3 27$, we can find its exact value two ways. By inspection we realize it means "3 to what power will be 27"? Since $3^3 = 27$, we know $\log_3 27 = 3$. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

EXAMPLE 10.21

Find the exact value of each logarithm without using a calculator:

ⓐ $\log_5 25$,

ⓑ $\log_9 3$, and ⓒ $\log_2 \frac{1}{16}$.

Solution

ⓐ

$\log_5 25$

5 to what power will be 25?

$\log_5 25 = 2$

Or

Set the expression equal to x. $\log_5 25 = x$

Change to exponential form. $5^x = 25$

Rewrite 25 as 5^2. $5^x = 5^2$

With the same base the exponents must be equal. $x = 2$ Therefore, $\log_5 25 = 2$.

ⓑ

$\log_9 3$

Set the expression equal to x. $\log_9 3 = x$

Change to exponential form. $9^x = 3$

Rewrite 9 as 3^2. $\left(3^2\right)^x = 3^1$

Simplify the exponents. $3^{2x} = 3^1$

With the same base the exponents must be equal. $2x = 1$

Solve the equation. $x = \frac{1}{2}$ Therefore, $\log_9 3 = \frac{1}{2}$.

ⓒ

$\log_2 \frac{1}{16}$

Set the expression equal to x. $\log_2 \frac{1}{16} = x$

Change to exponential form. $2^x = \frac{1}{16}$

Rewrite 16 as 2^4. $2^x = \frac{1}{2^4}$

$2^x = 2^{-4}$

With the same base the exponents must be equal. $x = -4$ Therefore, $\log_2 \frac{1}{16} = -4$.

TRY IT : : 10.41 Find the exact value of each logarithm without using a calculator:

ⓐ $\log_{12} 144$

ⓑ $\log_4 2$

ⓒ $\log_2 \frac{1}{32}$

TRY IT : : 10.42 Find the exact value of each logarithm without using a calculator:

ⓐ $\log_9 81$

ⓑ $\log_8 2$

ⓒ $\log_3 \frac{1}{9}$

Graph Logarithmic Functions

To graph a logarithmic function $y = \log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$. Generally, when we look for ordered pairs for the graph of a function, we usually choose an x-value and then determine its corresponding y-value. In this case you may find it easier to choose y-values and then determine its corresponding x-value.

EXAMPLE 10.22

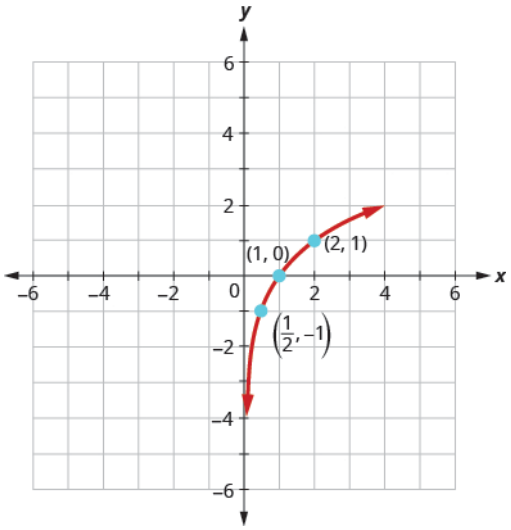
Graph $y = \log_2 x$.

Solution

To graph the function, we will first rewrite the logarithmic equation, $y = \log_2 x$, in exponential form, $2^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

y	$2^y = x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(\frac{1}{4}, 2\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(\frac{1}{2}, -1\right)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$



TRY IT : : 10.43 Graph: $y = \log_3 x$.

TRY IT : : 10.44 Graph: $y = \log_5 x$.

The graphs of $y = \log_2 x$, $y = \log_3 x$, and $y = \log_5 x$ are the shape we expect from a logarithmic function where $a > 1$.

We notice that for each function the graph contains the point $(1, 0)$. This make sense because $0 = \log_a 1$ means $a^0 = 1$ which is true for any *a*.

The graph of each function, also contains the point $(a, 1)$. This makes sense as $1 = \log_a a$ means $a^1 = a$. which is true for any *a*.

Notice too, the graph of each function $y = \log_a x$ also contains the point $\left(\frac{1}{a}, -1\right)$. This makes sense as $-1 = \log_a \frac{1}{a}$ means $a^{-1} = \frac{1}{a}$, which is true for any a.

Look at each graph again. Now we will see that many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

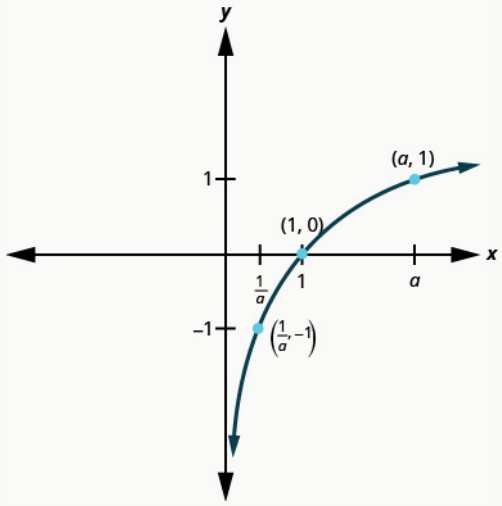
What is the domain of the function? The graph never hits the y-axis. The domain is all positive numbers. We write the domain in interval notation as $(0, \infty)$.

What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as $(-\infty, \infty)$.

When the graph approaches the y-axis so very closely but will never cross it, we call the line $x = 0$, the y-axis, a vertical asymptote.

Properties of the Graph of $y = \log_a x$ when $a > 1$

Domain	$(0, \infty)$
Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$
y-intercept	None
Contains	$(a, 1)$, $\left(\frac{1}{a}, -1\right)$
Asymptote	y-axis



Our next example looks at the graph of $y = \log_a x$ when $0 < a < 1$.

EXAMPLE 10.23

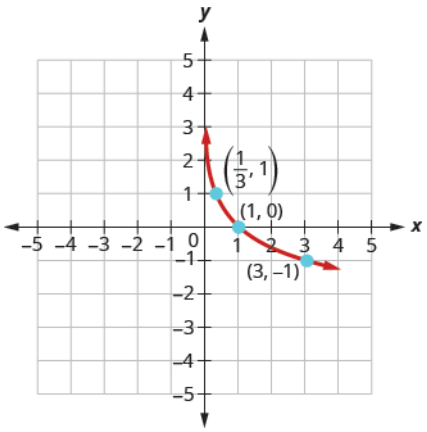
Graph $y = \log_{\frac{1}{3}} x$.

Solution

To graph the function, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}} x$, in exponential form, $\left(\frac{1}{3}\right)^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of y and then get x.

y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(9, -2)$
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(3, -1)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(1, 0)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(\frac{1}{9}, 2\right)$
3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(\frac{1}{27}, 3\right)$



TRY IT : : 10.45 Graph: $y = \log_{\frac{1}{2}} x$.

TRY IT : : 10.46 Graph: $y = \log_{\frac{1}{4}} x$.

Now, let's look at the graphs $y = \log_{\frac{1}{2}} x$, $y = \log_{\frac{1}{3}} x$ and $y = \log_{\frac{1}{4}} x$, so we can identify some of the properties of logarithmic functions where $0 < a < 1$.

The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where $0 < a < 1$.

We notice, that for each function again, the graph contains the points, $(1, 0)$, $(a, 1)$, $\left(\frac{1}{a}, -1\right)$. This make sense for the

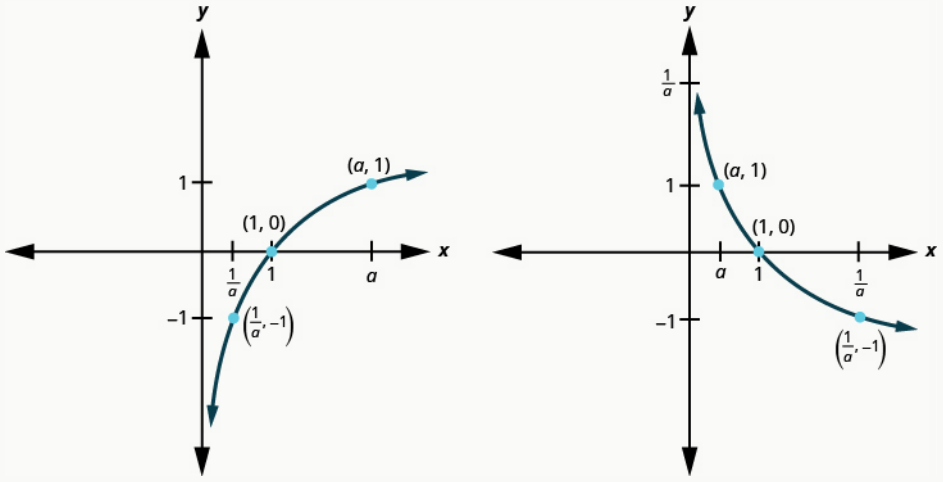
same reasons we argued above.

We notice the domain and range are also the same—the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. The y-axis is again the vertical asymptote.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Properties of the Graph of $y = \log_a x$

when $a > 1$		when $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
y-intercept	none	y-intercept	None
Contains	$(a, 1)$, $\left(\frac{1}{a}, -1\right)$	Contains	$(a, 1)$, $\left(\frac{1}{a}, -1\right)$
Asymptote	y-axis	Asymptote	y-axis
Basic shape	increasing	Basic shape	Decreasing



We talked earlier about how the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$. The graphs in Figure 10.4 show both the exponential (blue) and logarithmic (red) functions on the same graph for both $a > 1$ and $0 < a < 1$.

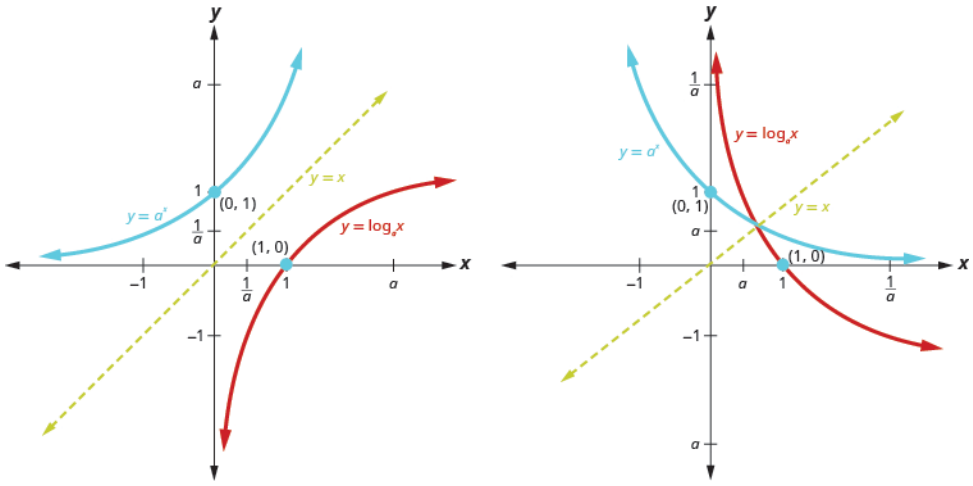


Figure 10.4

Notice how the graphs are reflections of each other through the line $y = x$. We know this is true of inverse functions.

Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the *x*-axis is the horizontal asymptote for the exponential functions and the *y*-axis is the vertical asymptote for the logarithmic functions.

Solve Logarithmic Equations

When we talked about exponential functions, we introduced the number *e*. Just as *e* was a base for an exponential function, it can be used a base for logarithmic functions too. The logarithmic function with base *e* is called the **natural logarithmic function**. The function $f(x) = \log_e x$ is generally written $f(x) = \ln x$ and we read it as "el en of x."

Natural Logarithmic Function

The function $f(x) = \ln x$ is the **natural logarithmic function** with base e, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

When the base of the logarithm function is 10, we call it the **common logarithmic function** and the base is not shown. If the base *a* of a logarithm is not shown, we assume it is 10.

Common Logarithmic Function

The function $f(x) = \log x$ is the **common logarithmic function** with base 10, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

It will be important for you to use your calculator to evaluate both common and natural logarithms.

Look for the log and ln keys on your calculator.

To solve logarithmic equations, one strategy is to change the equation to exponential form and then solve the exponential equation as we did before. As we solve logarithmic equations, $y = \log_a x$, we need to remember that for the base *a*, $a > 0$ and $a \neq 1$. Also, the domain is $x > 0$. Just as with radical equations, we must check our solutions to eliminate any extraneous solutions.

EXAMPLE 10.24

Solve: ⓐ $\log_a 49 = 2$ and ⓑ $\ln x = 3$.

Solution

ⓐ

$$\log_a 49 = 2$$

Rewrite in exponential form. $a^2 = 49$

Solve the equation using the square root property. $a = \pm 7$

The base cannot be negative, so we eliminate $a = -7$. $a = 7, \cancel{a = -7}$

Check.

$a = 7$ $\log_a 49 = 2$

$\log_7 49 \stackrel{?}{=} 2$

$7^2 \stackrel{?}{=} 49$

$49 = 49$ ✓

ⓑ

$$\ln x = 3$$

Rewrite in exponential form. $e^3 = x$

Check.

$x = e^3$ $\ln x = 3$

$\ln e^3 \stackrel{?}{=} 3$

$e^3 = e^3$ ✓

TRY IT : : 10.47 Solve: ⓐ $\log_a 121 = 2$ ⓑ $\ln x = 7$

TRY IT : : 10.48 Solve: ⓐ $\log_a 64 = 3$ ⓑ $\ln x = 9$

EXAMPLE 10.25

Solve: ⓐ $\log_2(3x-5) = 4$ and ⓑ $\ln e^{2x} = 4$.

Solution

ⓐ

$$\log_2(3x-5) = 4$$

Rewrite in exponential form. $2^4 = 3x-5$

Simplify. $16 = 3x-5$

Solve the equation. $21 = 3x$

$7 = x$

Check.

$x = 7$ $\log_2(3x-5) = 4$

$\log_2(3\cdot 7-5) \stackrel{?}{=} 4$

$\log_2(16) \stackrel{?}{=} 4$

$2^4 \stackrel{?}{=} 16$

$16 = 16$ ✓

ⓑ

	$\ln e^{2x} = 4$
Rewrite in exponential form.	$e^4 = e^{2x}$
Since the bases are the same the exponents are equal.	$4 = 2x$
Solve the equation.	$2 = x$

Check.

$x = 2$

$$\ln e^{2x} = 4$$
$$\ln e^{2\cdot 2} \stackrel{?}{=} 4$$
$$\ln e^4 \stackrel{?}{=} 4$$
$$e^4 = e^4 \checkmark$$

TRY IT : : 10.49 Solve: ⓐ $\log_2(5x-1)=6$ ⓑ $\ln e^{3x}=6$

TRY IT : : 10.50 Solve: ⓐ $\log_3(4x+3)=3$ ⓑ $\ln e^{4x}=4$

Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The 10^{-12} in the formula represents the intensity of sound that is barely audible.

Decibel Level of Sound

The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is

$$D = 10\log\left(\frac{I}{10^{-12}}\right)$$

EXAMPLE 10.26

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity 10^{-2} watts per square inch?

Solution

	$D = 10\log\left(\frac{I}{10^{-12}}\right)$
Substitute in the intensity level, I.	$D = 10\log\left(\frac{10^{-2}}{10^{-12}}\right)$
Simplify.	$D = 10\log(10^{10})$
Since $\log 10^{10} = 10$.	$D = 10 \cdot 10$
Multiply.	$D = 100$
	The decibel level of music coming through earphones is 100 dB.

TRY IT : : 10.51

What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

TRY IT : : 10.52 What is the decibel level heavy city traffic with intensity 10^{-3} watts per square inch?

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where I is the intensity of the shock wave. This model provides a way to measure earthquake intensity.

Earthquake Intensity

The magnitude *R* of an earthquake is measured by $R = \log I$, where *I* is the intensity of its shock wave.

EXAMPLE 10.27

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $108 million dollars of damage. Compare the intensities of the two earthquakes.

Solution

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

Convert the magnitudes to intensities.	$R = \log I$
1906 earthquake	$7.8 = \log I$
Convert to exponential form.	$I = 10^{7.8}$
2014 earthquake	$5.1 = \log I$
Convert to exponential form.	$I = 10^{5.1}$
Form a ratio of the intensities.	$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$
Substitute in the values.	$\frac{10^{7.8}}{10^{5.1}}$
Divide by subtracting the exponents.	$10^{2.7}$
Evaluate.	501

The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake.

TRY IT : : 10.53

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

TRY IT : : 10.54

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

MEDIA : :

Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

- **Re-writing logarithmic equations in exponential form (https://openstax.org/l/37logasexponent)**
- **Simplifying Logarithmic Expressions (https://openstax.org/l/37Simplifylog)**
- **Graphing logarithmic functions (https://openstax.org/l/37Graphlog)**
- **Using logarithms to calculate decibel levels (https://openstax.org/l/37Finddecibel)**

10.3 EXERCISES

Practice Makes Perfect

Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

126. $4^2 = 16$

127. $2^5 = 32$

128. $3^3 = 27$

129. $5^3 = 125$

130. $10^3 = 1000$

131. $10^{-2} = \frac{1}{100}$

132. $x^{\frac{1}{2}} = \sqrt{3}$

133. $x^{\frac{1}{3}} = \sqrt[3]{6}$

134. $32^x = \sqrt[4]{32}$

135. $17^x = \sqrt[5]{17}$

136. $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

137. $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

138. $3^{-2} = \frac{1}{9}$

139. $4^{-3} = \frac{1}{64}$

140. $e^x = 6$

141. $e^3 = x$

In the following exercises, convert each logarithmic equation to exponential form.

142. $3 = \log_4 64$

143. $6 = \log_2 64$

144. $4 = \log_x 81$

145. $5 = \log_x 32$

146. $0 = \log_{12} 1$

147. $0 = \log_7 1$

148. $1 = \log_3 3$

149. $1 = \log_9 9$

150. $-4 = \log_{10} \frac{1}{10{,}000}$

151. $3 = \log_{10} 1{,}000$

152. $5 = \log_e x$

153. $x = \log_e 43$

Evaluate Logarithmic Functions

In the following exercises, find the value of x in each logarithmic equation.

154. $\log_x 49 = 2$

155. $\log_x 121 = 2$

156. $\log_x 27 = 3$

157. $\log_x 64 = 3$

158. $\log_3 x = 4$

159. $\log_5 x = 3$

160. $\log_2 x = -6$

161. $\log_3 x = -5$

162. $\log_{\frac{1}{4}} \frac{1}{16} = x$

163. $\log_{\frac{1}{3}} \frac{1}{9} = x$

164. $\log_{\frac{1}{4}} 64 = x$

165. $\log_{\frac{1}{9}} 81 = x$

In the following exercises, find the exact value of each logarithm without using a calculator.

166. $\log_7 49$

167. $\log_6 36$

168. $\log_4 1$

169. $\log_5 1$

170. $\log_{16} 4$

171. $\log_{27} 3$

172. $\log_{\frac{1}{2}} 2$

173. $\log_{\frac{1}{2}} 4$

174. $\log_2 \frac{1}{16}$

175. $\log_3 \frac{1}{27}$

176. $\log_4 \frac{1}{16}$

177. $\log_9 \frac{1}{81}$

Graph Logarithmic Functions

In the following exercises, graph each logarithmic function.

178. $y = \log_2 x$

179. $y = \log_4 x$

180. $y = \log_6 x$

181. $y = \log_7 x$

182. $y = \log_{1.5} x$

183. $y = \log_{2.5} x$

184. $y = \log_{\frac{1}{3}} x$

185. $y = \log_{\frac{1}{5}} x$

186. $y = \log_{0.4} x$

187. $y = \log_{0.6} x$

Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

188. $\log_a 16 = 2$

189. $\log_a 81 = 2$

190. $\log_a 8 = 3$

191. $\log_a 27 = 3$

192. $\log_a 32 = 2$

193. $\log_a 24 = 3$

194. $\ln x = 5$

195. $\ln x = 4$

196. $\log_2 (5x + 1) = 4$

197. $\log_2 (6x + 2) = 5$

198. $\log_3 (4x - 3) = 2$

199. $\log_3 (5x - 4) = 4$

200. $\log_4 (5x + 6) = 3$

201. $\log_4 (3x - 2) = 2$

202. $\ln e^{4x} = 8$

203. $\ln e^{2x} = 6$

204. $\log x^2 = 2$

205. $\log(x^2 - 25) = 2$

206. $\log_2\left(x^2 - 4\right) = 5$

207. $\log_3\left(x^2 + 2\right) = 3$

Use Logarithmic Models in Applications

In the following exercises, use a logarithmic model to solve.

208. What is the decibel level of normal conversation with intensity 10^{-6} watts per square inch?

209. What is the decibel level of a whisper with intensity 10^{-10} watts per square inch?

210. What is the decibel level of the noise from a motorcycle with intensity 10^{-2} watts per square inch?

211. What is the decibel level of the sound of a garbage disposal with intensity 10^{-2} watts per square inch?

212. In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2010, Haiti also experienced an intense earthquake which measured 7.0 on the Richter scale. Compare the intensities of the two earthquakes.

213. The Los Angeles area experiences many earthquakes. In 1994, the Northridge earthquake measured magnitude of 6.7 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Writing Exercises

214. Explain how to change an equation from logarithmic form to exponential form.

215. Explain the difference between common logarithms and natural logarithms.

216. Explain why $\log_a a^x = x$.

217. Explain how to find the $\log_7 32$ on your calculator.

Self Check

ⓐ

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
convert between exponential and logarithmic form.			
evaluate logarithmic functions.			
graph logarithmic functions.			
solve logarithmic functions.			
use logarithmic models in applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

10.4 Use the Properties of Logarithms

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate: ⓐ a^0 ⓑ a^1.
   If you missed this problem, review **Example 5.14**.
2. Write with a rational exponent: $\sqrt[3]{x^2 y}$.
   If you missed this problem, review **Example 8.27**.
3. Round to three decimal places: 2.5646415.
   If you missed this problem, review **Example 1.34**.

Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0 = 1,$ we can convert this to logarithmic form and get $\log_a 1 = 0.$ Also, since $a^1 = a,$ we get $\log_a a = 1.$

Properties of Logarithms

$$\log_a 1 = 0 \qquad \log_a a = 1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

EXAMPLE 10.28

Evaluate using the properties of logarithms: ⓐ $\log_8 1$ and ⓑ $\log_6 6.$

Solution

ⓐ

	$\log_8 1$	
Use the property, $\log_a 1 = 0.$	0	$\log_8 1 = 0$

ⓑ

	$\log_6 6$	
Use the property, $\log_a a = 1.$	1	$\log_6 6 = 1$

TRY IT : : 10.55 Evaluate using the properties of logarithms: ⓐ $\log_{13} 1$ ⓑ $\log_9 9.$

TRY IT : : 10.56 Evaluate using the properties of logarithms: ⓐ $\log_5 1$ ⓑ $\log_7 7.$

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $a^{\log_a x} = x$ converts to the logarithmic equation $\log_a x = \log_a x,$ which is a true statement

for positive values for *x* only.

The logarithmic equation $\log_a a^x = x$ converts to the exponential equation $a^x = a^x$, which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power. These two properties show the composition of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

Inverse Properties of Logarithms

For $a > 0$, $x > 0$ and $a \neq 1$,

$$a^{\log_a x} = x \qquad \log_a a^x = x$$

In the next example, apply the inverse properties of logarithms.

EXAMPLE 10.29

Evaluate using the properties of logarithms: ⓐ $4^{\log_4 9}$ and ⓑ $\log_3 3^5$.

Solution

ⓐ

Use the property, $a^{\log_a x} = x$.

$4^{\log_4 9}$

9

$4^{\log_4 9} = 9$

ⓑ

Use the property, $a^{\log_a x} = x$.

$\log_3 3^5$

5

$\log_3 3^5 = 5$

TRY IT : : 10.57 Evaluate using the properties of logarithms: ⓐ $5^{\log_5 15}$ ⓑ $\log_7 7^4$.

TRY IT : : 10.58 Evaluate using the properties of logarithms: ⓐ $2^{\log_2 8}$ ⓑ $\log_2 2^{15}$.

There are three more properties of logarithms that will be useful in our work. We know exponential functions and logarithmic function are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $a^m \cdot a^n = a^{m+n}$, we see that to multiply the same base, we add the exponents. The **Product Property of Logarithms**, $\log_a M \cdot N = \log_a M + \log_a N$ tells us to take the log of a product, we add the log of the factors.

Product Property of Logarithms

If $M > 0$, $N > 0$, $a > 0$ and $a \neq 1$, then,

$$\log_a (M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

EXAMPLE 10.30

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible: ⓐ $\log_3 7x$ and ⓑ $\log_4 64xy$.

Solution

ⓐ

	$\log_3 7x$
Use the Product Property, $\log_a(M \cdot N) = \log_a M + \log_a N$.	$\log_3 7 + \log_3 x$
	$\log_3 7x = \log_3 7 + \log_3 x$

ⓑ

	$\log_4 64xy$
Use the Product Property, $\log_a(M \cdot N) = \log_a M + \log_a N$.	$\log_4 64 + \log_4 x + \log_4 y$
Simplify by evaluating $\log_4 64$.	$3 + \log_4 x + \log_4 y$
	$\log_4 64xy = 3 + \log_4 x + \log_4 y$

TRY IT : : 10.59

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

ⓐ $\log_3 3x$ ⓑ $\log_2 8xy$

TRY IT : : 10.60

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

ⓐ $\log_9 9x$ ⓑ $\log_3 27xy$

Similarly, in the Quotient Property of Exponents, $\frac{a^m}{a^n} = a^{m-n}$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $\log_a \frac{M}{N} = \log_a M - \log_a N$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Quotient Property of Logarithms

If $M > 0$, $N > 0$,a > 0 and a $\neq 1$, then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_a M - \log_a N \neq \log_a(M - N)$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

EXAMPLE 10.31

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_5 \frac{5}{7}$ and ⓑ $\log \frac{x}{100}$

Solution

ⓐ

	$\log_5 \frac{5}{7}$
Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.	$\log_5 5 - \log_5 7$
Simplify.	$1 - \log_5 7$
	$\log_5 \frac{5}{7} = 1 - \log_5 7$

ⓑ

	$\log \frac{x}{100}$
Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.	$\log x - \log 100$
Simplify.	$\log x - 2$
	$\log \frac{x}{100} = \log x - 2$

TRY IT : : 10.61

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_4 \frac{3}{4}$ ⓑ $\log \frac{x}{1000}$

TRY IT : : 10.62

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_2 \frac{5}{4}$ ⓑ $\log \frac{10}{y}$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_a M^p = p \log_a M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Power Property of Logarithms

If $M > 0$, $a > 0$, $a \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power as the product product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

EXAMPLE 10.32

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_5 4^3$ and ⓑ $\log x^{10}$

Solution

ⓐ

	$\log_5 4^3$
Use the Power Property, $\log_a M^p = p \log_a M$.	$3\log_5 4$
	$\log_5 4^3 = 3\log_5 4$

ⓑ

	$\log x^{10}$
Use the Power Property, $\log_a M^p = p \log_a M$.	$10 \log x$
	$\log x^{10} = 10\log x$

TRY IT : : 10.63

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_7 5^4$ ⓑ $\log x^{100}$

TRY IT : : 10.64

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_2 3^7$ ⓑ $\log x^{20}$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Properties of Logarithms

If $M > 0$, $a > 0$, $a \neq 1$ and p is any real number then,

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

Now that we have the properties we can use them to "expand" a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

EXAMPLE 10.33

Use the Properties of Logarithms to expand the logarithm $\log_4\left(2x^3 y^2\right)$. Simplify, if possible.

Solution

	$\log_4\left(2x^3y^2\right)$
Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$.	$\log_4 2 + \log_4 x^3 + \log_4 y^2$
Use the Power Property, $\log_a M^p = p\log_a M$, on the last two terms.	$\log_4 2 + 3\log_4 x + 2\log_4 y$
Simplify.	$\frac{1}{2} + 3\log_4 x + 2\log_4 y$

$$\log_4\left(2x^3y^2\right) = \frac{1}{2} + 3\log_4 x + 2\log_4 y$$

TRY IT : : 10.65

Use the Properties of Logarithms to expand the logarithm $\log_2\left(5x^4y^2\right)$. Simplify, if possible.

TRY IT : : 10.66

Use the Properties of Logarithms to expand the logarithm $\log_3\left(7x^5y^3\right)$. Simplify, if possible.

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

EXAMPLE 10.34

Use the Properties of Logarithms to expand the logarithm $\log_2\sqrt[4]{\frac{x^3}{3y^2z}}$. Simplify, if possible.

Solution

	$\log_2\sqrt[4]{\frac{x^3}{3y^2z}}$
Rewrite the radical with a rational exponent.	$\log_2\left(\frac{x^3}{3y^2z}\right)^{\frac{1}{4}}$
Use the Power Property, $\log_a M^p = p\log_a M$.	$\frac{1}{4}\log_2\left(\frac{x^3}{3y^2z}\right)$
Use the Quotient Property, $\log_a M \cdot N = \log_a M - \log_a N$.	$\frac{1}{4}\left(\log_2\left(x^3\right) - \log_2\left(3y^2z\right)\right)$
Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$, in the second term.	$\frac{1}{4}\left(\log_2\left(x^3\right) - \left(\log_2 3 + \log_2 y^2 + \log_2 z\right)\right)$
Use the Power Property, $\log_a M^p = p\log_a M$, inside the parentheses.	$\frac{1}{4}\left(3\log_2 x - \left(\log_2 3 + 2\log_2 y + \log_2 z\right)\right)$
Simplify by distributing.	$\frac{1}{4}\left(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z\right)$

$$\log_2\sqrt[4]{\frac{x^3}{3y^2z}} = \frac{1}{4}\left(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z\right)$$

TRY IT : : 10.67

Use the Properties of Logarithms to expand the logarithm $\log_4 \sqrt[5]{\frac{x^4}{2y^3z^2}}$. Simplify, if possible.

TRY IT : : 10.68

Use the Properties of Logarithms to expand the logarithm $\log_3 \sqrt[3]{\frac{x^2}{5yz}}$. Simplify, if possible.

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

EXAMPLE 10.35

Use the Properties of Logarithms to condense the logarithm $\log_4 3 + \log_4 x - \log_4 y$. Simplify, if possible.

Solution

The log expressions all have the same base, 4.

The fir t two terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.

Since the logs are subtracted, we use the Quotient Property, $\log_a M - \log_a N = \log_a \frac{M}{N}$.

$$\log_4 3 + \log_4 x - \log_4 y$$

$$\log_4 3x - \log_4 y$$

$$\log_4 \frac{3x}{y}$$

$$\log_4 3 + \log_4 x - \log_4 y = \log_4 \frac{3x}{y}$$

TRY IT : : 10.69

Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

TRY IT : : 10.70

Use the Properties of Logarithms to condense the logarithm $\log_3 6 - \log_3 x - \log_3 y$. Simplify, if possible.

EXAMPLE 10.36

Use the Properties of Logarithms to condense the logarithm $2\log_3 x + 4\log_3 (x+1)$. Simplify, if possible.

Solution

The log expressions have the same base, 3.

Use the Power Property, $\log_a M + \log_a N = \log_a M \cdot N$.

The terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.

$$2\log_3 x + 4\log_3 (x+1)$$

$$\log_3 x^2 + \log_3 (x+1)^4$$

$$\log_3 x^2 (x+1)^4$$

$$2\log_3 x + 4\log_3 (x+1) = \log_3 x^2 (x+1)^4$$

TRY IT : : 10.71

Use the Properties of Logarithms to condense the logarithm $3\log_2 x + 2\log_2 (x-1)$. Simplify, if possible.

TRY IT : : 10.72

Use the Properties of Logarithms to condense the logarithm $2\log x + 2\log(x+1)$. Simplify, if possible.

Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the Change-of-Base Formula. We will show how this is derived.

Suppose we want to evaluate $\log_a M$.	$\log_a M$
Let $y = \log_a M$.	$y = \log_a M$
Rewrite the expression in exponential form.	$a^y = M$
Take the $\log_b$ of each side.	$\log_b a^y = \log_b M$
Use the Power Property.	$y\log_b a = \log_b M$
Solve for y.	$y = \frac{\log_b M}{\log_b a}$
Substitute $y = \log_a M$.	$\log_a M = \frac{\log_b M}{\log_b a}$

The Change-of-Base Formula introduces a new base b. This can be any base *b* we want where $b > 0$, $b \neq 1$. Because our calculators have keys for logarithms base 10 and base *e*, we will rewrite the Change-of-Base Formula with the new base as 10 or *e*.

Change-of-Base Formula

For any logarithmic bases a, b and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$

new base b | new base 10 | new base e

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol $(\approx)$.

EXAMPLE 10.37

Rounding to three decimal places, approximate $\log_4 35$.

Solution

	$\log_4 35$
Use the Change-of-Base Formula.	$\log_a M = \frac{\log_b M}{\log_b a}$
Identify *a* and *M*. Choose 10 for *b*.	$\log_4 35 = \frac{\log 35}{\log 4}$
Enter the expression $\frac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places.	$\log_4 35 \approx 2.565$

TRY IT : : 10.73 Rounding to three decimal places, approximate $\log_3 42$.

TRY IT : : 10.74 Rounding to three decimal places, approximate $\log_5 46$.

MEDIA : :

Access these online resources for additional instruction and practice with using the properties of logarithms.

- **Using Properties of Logarithms to Expand Logs (https://openstax.org/l/37Logproperties)**
- **Using Properties of Logarithms to Condense Logs (https://openstax.org/l/37Condenselogs)**
- **Change of Base (https://openstax.org/l/37Changeofbase)**

10.4 EXERCISES

Practice Makes Perfect

Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

218. ⓐ $\log_4 1$ ⓑ $\log_8 8$

219. ⓐ $\log_{12} 1$ ⓑ $\ln e$

220. ⓐ $3^{\log_3 6}$ ⓑ $\log_2 2^7$

221. ⓐ $5^{\log_5 10}$ ⓑ $\log_4 4^{10}$

222. ⓐ $8^{\log_8 7}$ ⓑ $\log_6 6^{-2}$

223. ⓐ $6^{\log_6 15}$ ⓑ $\log_8 8^{-4}$

224. ⓐ $10^{\log\sqrt{5}}$ ⓑ $\log 10^{-2}$

225. ⓐ $10^{\log\sqrt{3}}$ ⓑ $\log 10^{-1}$

226. ⓐ $e^{\ln 4}$ ⓑ $\ln e^2$

227. ⓐ $e^{\ln 3}$ ⓑ $\ln e^7$

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

228. $\log_4 6x$

229. $\log_5 8y$

230. $\log_2 32xy$

231. $\log_3 81xy$

232. $\log 100x$

233. $\log 1000y$

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

234. $\log_3 \frac{3}{8}$

235. $\log_6 \frac{5}{6}$

236. $\log_4 \frac{16}{y}$

237. $\log_5 \frac{125}{x}$

238. $\log \frac{x}{10}$

239. $\log \frac{10{,}000}{y}$

240. $\ln \frac{e^3}{3}$

241. $\ln \frac{e^4}{16}$

In the following exercises, use the Power Property of Logarithms to expand each. Simplify if possible.

242. $\log_3 x^2$

243. $\log_2 x^5$

244. $\log x^{-2}$

245. $\log x^{-3}$

246. $\log_4 \sqrt{x}$

247. $\log_5 \sqrt[3]{x}$

248. $\ln x^{\sqrt{3}}$

249. $\ln x^{\sqrt[3]{4}}$

In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.

250. $\log_5\left(4x^6y^4\right)$

251. $\log_2\left(3x^5y^3\right)$

252. $\log_3\left(\sqrt{2}x^2\right)$

253. $\log_5\left(\sqrt[4]{21}y^3\right)$

254. $\log_3\dfrac{xy^2}{z^2}$

255. $\log_5\dfrac{4ab^3c^4}{d^2}$

256. $\log_4\dfrac{\sqrt{x}}{16y^4}$

257. $\log_3\dfrac{\sqrt[3]{x^2}}{27y^4}$

258. $\log_2\dfrac{\sqrt{2x+y^2}}{z^2}$

259. $\log_3\dfrac{\sqrt{3x+2y^2}}{5z^2}$

260. $\log_2\sqrt[4]{\dfrac{5x^3}{2y^2z^4}}$

261. $\log_5\sqrt[3]{\dfrac{3x^2}{4y^3z}}$

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.

262. $\log_6 4+\log_6 9$

263. $\log 4+\log 25$

264. $\log_2 80-\log_2 5$

265. $\log_3 36-\log_3 4$

266. $\log_3 4+\log_3(x+1)$

267. $\log_2 5-\log_2(x-1)$

268. $\log_7 3+\log_7 x-\log_7 y$

269. $\log_5 2-\log_5 x-\log_5 y$

270. $4\log_2 x+6\log_2 y$

271. $6\log_3 x+9\log_3 y$

272. $\log_3\left(x^2-1\right)-2\log_3(x-1)$

273. $\log\left(x^2+2x+1\right)-2\log(x+1)$

274. $4\log x-2\log y-3\log z$

275. $3\ln x+4\ln y-2\ln z$

276. $\frac{1}{3}\log x-3\log(x+1)$

277. $2\log(2x+3)+\frac{1}{2}\log(x+1)$

Use the Change-of-Base Formula

In the following exercises, use the Change-of-Base Formula, rounding to three decimal places, to approximate each logarithm.

278. $\log_3 42$

279. $\log_5 46$

280. $\log_{12} 87$

281. $\log_{15} 93$

282. $\log_{\sqrt{2}} 17$

283. $\log_{\sqrt{3}} 21$

Writing Exercises

284. Write the Product Property in your own words. Does it apply to each of the following? $\log_a 5x$, $\log_a(5+x)$. Why or why not?

285. Write the Power Property in your own words. Does it apply to each of the following? $\log_a x^p$, $(\log_a x)^r$. Why or why not?

286. Use an example to show that $\log(a+b) \neq \log a + \log b$?

287. Explain how to find the value of $\log_7 15$ using your calculator.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
use the properties of logarithms.			
use the Change of Base Formula.			

ⓑ *On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

10.5 Solve Exponential and Logarithmic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve logarithmic equations using the properties of logarithms
- Solve exponential equations using logarithms
- Use exponential models in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 16$.
   If you missed this problem, review Example 6.46.
2. Solve: $x^2 - 5x + 6 = 0$.
   If you missed this problem, review Example 6.45.
3. Solve: $x(x + 6) = 2x + 5$.
   If you missed this problem, review Example 6.47.

Solve Logarithmic Equations Using the Properties of Logarithms

In the section on logarithmic functions, we solved some equations by rewriting the equation in exponential form. Now that we have the properties of logarithms, we have additional methods we can use to solve logarithmic equations.

If our equation has two logarithms we can use a property that says that if $\log_a M = \log_a N$ then it is true that $M = N$. This is the **One-to-One Property of Logarithmic Equations**.

One-to-One Property of Logarithmic Equations

For $M > 0$, $N > 0$, $a>0$, and $a \neq 1$ is any real number:

$$\text{If } \log_a M = \log_a N, \text{ then } M = N.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Remember that logarithms are defined only for positive real numbers. Check your results in the original equation. You may have obtained a result that gives a logarithm of zero or a negative number.

EXAMPLE 10.38

Solve: $2\log_5 x = \log_5 81$.

Solution

	$2\log_5 x = \log_5 81$
Use the Power Property.	$\log_5 x^2 = \log_5 81$
Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.	$x^2 = 81$
Solve using the Square Root Property.	$x = \pm 9$
We eliminate $x = -9$ as we cannot take the logarithm of a negative number.	$x = 9, \cancel{x = -9}$

Check.

$x = 9$

$$\begin{aligned} 2\log_5 x &= \log_5 81 \\ 2\log_5 9 &\stackrel{?}{=} \log_5 81 \\ \log_5 9^2 &\stackrel{?}{=} \log_5 81 \\ \log_5 81 &= \log_5 81 \checkmark \end{aligned}$$

TRY IT : : 10.75 Solve: $2\log_3 x = \log_3 36$

TRY IT : : 10.76 Solve: $3\log x = \log 64$

Another strategy to use to solve logarithmic equations is to condense sums or differences into a single logarithm.

EXAMPLE 10.39

Solve: $\log_3 x + \log_3 (x-8) = 2$.

Solution

	$\log_3 x + \log_3 (x-8) = 2$
Use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.	$\log_3 x(x-8) = 2$
Rewrite in exponential form.	$3^2 = x(x-8)$
Simplify.	$9 = x^2 - 8x$
Subtract 9 from each side.	$0 = x^2 - 8x - 9$
Factor.	$0 = (x-9)(x+1)$
Use the Zero-Product Property.	$x-9 = 0, \quad x+1 = 0$
Solve each equation.	$x = 9, \quad \cancel{x = -1}$

Check.

$x = -1$

$$\log_3 x + \log_3 (x-8) = 2$$
$$\log_3 (-1) + \log_3 (-1-8) \stackrel{?}{=} 2$$

We cannot take the log of a negative number.

$x = 9$

$$\begin{aligned} \log_3 x + \log_3 (x-8) &= 2 \\ \log_3 9 + \log_3 (9-8) &\stackrel{?}{=} 2 \\ 2 + 0 &\stackrel{?}{=} 2 \\ 2 &= 2 \checkmark \end{aligned}$$

TRY IT : : 10.77 Solve: $\log_2 x + \log_2 (x-2) = 3$

TRY IT : : 10.78 Solve: $\log_2 x + \log_2 (x-6) = 4$

When there are logarithms on both sides, we condense each side into a single logarithm. Remember to use the Power Property as needed.

EXAMPLE 10.40

Solve: $\log_4 (x+6) - \log_4 (2x+5) = -\log_4 x$.

Solution

	$\log_4 (x+6) - \log_4 (2x+5) = -\log_4 x$
Use the Quotient Property on the left side and the Power Property on the right.	$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 x^{-1}$
Rewrite $x^{-1} = \frac{1}{x}$.	$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 \frac{1}{x}$
Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.	$\frac{x+6}{2x+5} = \frac{1}{x}$
Solve the rational equation.	$x(x+6) = 2x+5$
Distribute.	$x^2 + 6x = 2x+5$
Write in standard form.	$x^2 + 4x - 5 = 0$
Factor.	$(x+5)(x-1) = 0$
Use the Zero-Product Property.	$x+5=0, \quad x-1=0$
Solve each equation.	$\cancel{x=-5}, \quad x=1$
Check. We leave the check for you.	

TRY IT : : 10.79 Solve: $\log(x+2) - \log(4x+3) = -\log x$.

TRY IT : : 10.80 Solve: $\log(x-2) - \log(4x+16) = \log\frac{1}{x}$.

Solve Exponential Equations Using Logarithms

In the section on exponential functions, we solved some equations by writing both sides of the equation with the same base. Next we wrote a new equation by setting the exponents equal.

It is not always possible or convenient to write the expressions with the same base. In that case we often take the common logarithm or natural logarithm of both sides once the exponential is isolated.

EXAMPLE 10.41

Solve $5^x = 11$. Find the exact answer and then approximate it to three decimal places.

Solution

	$5^x = 11$
Since the exponential is isolated, take the logarithm of both sides.	$\log 5^x = \log 11$
Use the Power Property to get the x as a factor, not an exponent.	$x\log 5 = \log 11$
Solve for x. Find the exact answer.	$x = \frac{\log 11}{\log 5}$
Approximate the answer.	$x \approx 1.490$
Since $5^1 = 5$ and $5^2 = 25$, does it makes sense that $5^{1.490} \approx 11$?	

TRY IT : : 10.81 Solve $7^x = 43$. Find the exact answer and then approximate it to three decimal places.

TRY IT : : 10.82 Solve $8^x = 98$. Find the exact answer and then approximate it to three decimal places.

When we take the logarithm of both sides we will get the same result whether we use the common or the natural logarithm (try using the natural log in the last example. Did you get the same result?) When the exponential has base *e*, we use the natural logarithm.

EXAMPLE 10.42

Solve $3e^{x+2} = 24$. Find the exact answer and then approximate it to three decimal places.

Solution

	$3e^{x+2} = 24$
Isolate the exponential by dividing both sides by 3.	$e^{x+2} = 8$
Take the natural logarithm of both sides.	$\ln e^{x+2} = \ln 8$
Use the Power Property to get the x as a factor, not an exponent.	$(x+2)\ln e = \ln 8$
Use the property $\ln e = 1$ to simplify.	$x+2 = \ln 8$
Solve the equation. Find the exact answer.	$x = \ln 8 - 2$
Approximate the answer.	$x \approx 0.079$

TRY IT : : 10.83 Solve $2e^{x-2} = 18$. Find the exact answer and then approximate it to three decimal places.

TRY IT : : 10.84 Solve $5e^{2x} = 25$. Find the exact answer and then approximate it to three decimal places.

Use Exponential Models in Applications

In previous sections we were able to solve some applications that were modeled with exponential equations. Now that we have so many more options to solve these equations, we are able to solve more applications.

We will again use the Compound Interest Formulas and so we list them here for reference.

Compound Interest

For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A* is:

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ when compounded n times a year.

$A = Pe^{rt}$ when compounded continuously.

EXAMPLE 10.43

Jermael's parents put $10,000 in investments for his college expenses on his first birthday. They hope the investments will be worth $50,000 when he turns 18. If the interest compounds continuously, approximately what rate of growth will they need to achieve their goal?

Solution

	A	$=$	$\$50{,}000$
	P	$=$	$\$10{,}000$
Identify the variables in the formula.	r	$=$	?
	t	$=$	17 years
	A	$=$	Pe^{rt}
Substitute the values into the formula.	$50{,}000$	$=$	$10{,}000e^{r \cdot 17}$
Solve for r.Divide each side by 10,000.	5	$=$	e^{17r}
Take the natural log of each side.	$\ln 5$	$=$	$\ln e^{17r}$
Use the Power Property.	$\ln 5$	$=$	$17r \ln e$
Simplify.	$\ln 5$	$=$	$17r$
Divide each side by 17.	$\frac{\ln 5}{17}$	$=$	r
Approximate the answer.	r	$\approx$	0.095
Convert to a percentage.	r	$\approx$	9.5%

They need the rate of growth to be approximately 9.5%.

TRY IT : : 10.85

Hector invests $\$10{,}000$ at age 21. He hopes the investments will be worth $\$150{,}000$ when he turns 50. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal?

TRY IT : : 10.86

Rachel invests $\$15{,}000$ at age 25. She hopes the investments will be worth $\$90{,}000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

We have seen that growth and decay are modeled by exponential functions. For growth and decay we use the formula $A = A_0 e^{kt}$. Exponential growth has a positive rate of growth or growth constant, k, and exponential decay has a negative rate of growth or decay constant, *k*.

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, *k*, for a certain time, *t*, the final amount, *A*, is:

$$A = A_0 e^{kt}$$

We can now solve applications that give us enough information to determine the rate of growth. We can then use that rate of growth to predict other situations.

EXAMPLE 10.44

Researchers recorded that a certain bacteria population grew from 100 to 300 in 3 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

Solution

This problem requires two main steps. First we must find the unknown rate, *k*. Then we use that value of *k* to help us find the unknown number of bacteria.

Identify the variables in the formula.

$$\begin{aligned} A &= 300 \\ A_0 &= 100 \\ k &= ? \\ t &= 3 \text{ hours} \\ A &= A_0 e^{kt} \end{aligned}$$

Substitute the values in the formula. $300 = 100e^{k \cdot 3}$

Solve for k. Divide each side by 100. $3 = e^{3k}$

Take the natural log of each side. $\ln 3 = \ln e^{3k}$

Use the Power Property. $\ln 3 = 3k \ln e$

Simplify. $\ln 3 = 3k$

Divide each side by 3. $\frac{\ln 3}{3} = k$

Approximate the answer. $k \approx 0.366$

We use this rate of growth to predict the number of bacteria there will be in 24 hours.

$$\begin{aligned} A &= ? \\ A_0 &= 100 \\ k &= \frac{\ln 3}{3} \\ t &= 24 \text{ hours} \\ A &= A_0 e^{kt} \end{aligned}$$

Substitute in the values. $A = 100e^{\frac{\ln 3}{3} \cdot 24}$

Evaluate. $A \approx 656{,}100$

At this rate of growth, they can expect 656,100 bacteria.

TRY IT : : 10.87

Researchers recorded that a certain bacteria population grew from 100 to 500 in 6 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

TRY IT : : 10.88

Researchers recorded that a certain bacteria population declined from 700,000 to 400,000 in 5 hours after the administration of medication. At this rate of decay, how many bacteria will there be 24 hours from the start of the experiment?

Radioactive substances decay or decompose according to the exponential decay formula. The amount of time it takes for the substance to decay to half of its original amount is called the half-life of the substance.

Similar to the previous example, we can use the given information to determine the constant of decay, and then use that constant to answer other questions.

EXAMPLE 10.45

The half-life of radium-226 is 1,590 years. How much of a 100 mg sample will be left in 500 years?

Solution

This problem requires two main steps. First we must find the decay constant k. If we start with 100-mg, at the half-life there will be 50-mg remaining. We will use this information to find k. Then we use that value of k to help us find the amount of sample that will be left in 500 years.

Identify the variables in the formula.

$$\begin{aligned} A &= 50 \\ A_0 &= 100 \\ k &= ? \\ t &= 1590\text{years} \end{aligned}$$

$$A = A_0 e^{kt}$$

Substitute the values in the formula. $50 = 100e^{k \cdot 1590}$

Solve for k. Divide each side by 100. $0.5 = e^{1590k}$

Take the natural log of each side. $\ln 0.5 = \ln e^{1590k}$

Use the Power Property. $\ln 0.5 = 1590k \ln e$

Simplify. $\ln 0.5 = 1590k$

Divide each side by 1590. $\frac{\ln 0.5}{1590} = k$ exact answer

We use this rate of growth to predict the amount that will be left in 500 years.

$$\begin{aligned} A &= ? \\ A_0 &= 100 \\ k &= \frac{\ln 0.5}{1590} \\ t &= 500\text{years} \end{aligned}$$

$$A = A_0 e^{kt}$$

Substitute in the values. $A = 100e^{\frac{\ln 0.5}{1590} \cdot 500}$

Evaluate. $A \approx 80.4$ mg

In 500 years there would be approximately 80.4 mg remaining.

TRY IT : : 10.89

The half-life of magnesium-27 is 9.45 minutes. How much of a 10-mg sample will be left in 6 minutes?

TRY IT : : 10.90

The half-life of radioactive iodine is 60 days. How much of a 50-mg sample will be left in 40 days?

MEDIA : :

Access these online resources for additional instruction and practice with solving exponential and logarithmic equations.

- **Solving Logarithmic Equations (https://openstax.org/l/37Solvelog)**
- **Solving Logarithm Equations (https://openstax.org/l/37sSolvelogeqs2)**
- **Finding the rate or time in a word problem on exponential growth or decay (https://openstax.org/l/37Solveforrate)**
- **Finding the rate or time in a word problem on exponential growth or decay (https://openstax.org/l/37Solveforrate2)**

10.5 EXERCISES

Practice Makes Perfect

Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for x.

288. $\log_4 64 = 2\log_4 x$

289. $\log 49 = 2\log x$

290. $3\log_3 x = \log_3 27$

291. $3\log_6 x = \log_6 64$

292. $\log_5(4x-2) = \log_5 10$

293. $\log_3\left(x^2+3\right) = \log_3 4x$

294. $\log_3 x + \log_3 x = 2$

295. $\log_4 x + \log_4 x = 3$

296. $\log_2 x + \log_2(x-3) = 2$

297. $\log_3 x + \log_3(x+6) = 3$

298. $\log x + \log(x+3) = 1$

299. $\log x + \log(x-15) = 2$

300. $\log(x+4) - \log(5x+12) = -\log x$

301. $\log(x-1) - \log(x+3) = \log\frac{1}{x}$

302. $\log_5(x+3) + \log_5(x-6) = \log_5 10$

303. $\log_5(x+1) + \log_5(x-5) = \log_5 7$

304. $\log_3(2x-1) = \log_3(x+3) + \log_3 3$

305. $\log(5x+1) = \log(x+3) + \log 2$

Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

306. $3^x = 89$

307. $2^x = 74$

308. $5^x = 110$

309. $4^x = 112$

310. $e^x = 16$

311. $e^x = 8$

312. $\left(\frac{1}{2}\right)^x = 6$

313. $\left(\frac{1}{3}\right)^x = 8$

314. $4e^{x+1} = 16$

315. $3e^{x+2} = 9$

316. $6e^{2x} = 24$

317. $2e^{3x} = 32$

318. $\frac{1}{4}e^x = 3$

319. $\frac{1}{3}e^x = 2$

320. $e^{x+1} + 2 = 16$

321. $e^{x-1} + 4 = 12$

In the following exercises, solve each equation.

322. $3^{3x+1} = 81$

323. $6^{4x-17} = 216$

324. $\frac{e^{x^2}}{e^{14}} = e^{5x}$

325. $\frac{e^{x^2}}{e^{x}} = e^{20}$

326. $\log_a 64 = 2$

327. $\log_a 81 = 4$

328. $\ln x = -8$

329. $\ln x = 9$

330. $\log_5(3x - 8) = 2$

331. $\log_4(7x + 15) = 3$

332. $\ln e^{5x} = 30$

333. $\ln e^{6x} = 18$

334. $3\log x = \log 125$

335. $7\log_3 x = \log_3 128$

336. $\log_6 x + \log_6(x - 5) = 24$

337. $\log_9 x + \log_9(x - 4) = 12$

338. $\log_2(x + 2) - \log_2(2x + 9) = -\log_2 x$

339. $\log_6(x + 1) - \log_6(4x + 10) = \log_6 \frac{1}{x}$

In the following exercises, solve for x, giving an exact answer as well as an approximation to three decimal places.

340. $6^x = 91$

341. $\left(\frac{1}{2}\right)^x = 10$

342. $7e^{x-3} = 35$

343. $8e^{x+5} = 56$

Use Exponential Models in Applications

In the following exercises, solve.

344. Sung Lee invests $\$5{,}000$ at age 18. He hopes the investments will be worth $\$10{,}000$ when he turns 25. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?

345. Alice invests $\$15{,}000$ at age 30 from the signing bonus of her new job. She hopes the investments will be worth $\$30{,}000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

346. Coralee invests $\$5{,}000$ in an account that compounds interest monthly and earns 7%. How long will it take for her money to double?

347. Simone invests $\$8{,}000$ in an account that compounds interest quarterly and earns 5%. How long will it take for his money to double?

348. Researchers recorded that a certain bacteria population declined from 100,000 to 100 in 24 hours. At this rate of decay, how many bacteria will there be in 16 hours?

349. Researchers recorded that a certain bacteria population declined from 800,000 to 500,000 in 6 hours after the administration of medication. At this rate of decay, how many bacteria will there be in 24 hours?

350. A virus takes 6 days to double its original population $(A = 2A_0)$. How long will it take to triple its population?

351. A bacteria doubles its original population in 24 hours $(A = 2A_0)$. How big will its population be in 72 hours?

352. Carbon-14 is used for archeological carbon dating. Its half-life is 5,730 years. How much of a 100-gram sample of Carbon-14 will be left in 1000 years?

353. Radioactive technetium-99m is often used in diagnostic medicine as it has a relatively short half-life but lasts long enough to get the needed testing done on the patient. If its half-life is 6 hours, how much of the radioactive material form a 0.5 ml injection will be in the body in 24 hours?

Writing Exercises

354. Explain the method you would use to solve these equations: $3^{x+1} = 81$, $3^{x+1} = 75$. Does your method require logarithms for both equations? Why or why not?

355. What is the difference between the equation for exponential growth versus the equation for exponential decay?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve logarithmic equations using the properties of logarithms.			
solve exponential equations using logarithms.			
use exponential models in applications.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

CHAPTER 10 REVIEW

KEY TERMS

asymptote A line which a graph of a function approaches closely but never touches.

common logarithmic function The function $f(x) = \log x$ is the common logarithmic function with base 10, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

exponential function An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form $f(x) = a^x$.

logarithmic function The function $f(x) = \log_a x$ is the logarithmic function with base a, where $a > 0$, $x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

natural base The number *e* is defined as the value of $\left(1 + \frac{1}{n}\right)^n$, as *n* gets larger and larger. We say, as *n* increases without bound, $e \approx 2.718281827\ldots$

natural exponential function The natural exponential function is an exponential function whose base is *e*: $f(x) = e^x$. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

natural logarithmic function The function $f(x) = \ln x$ is the natural logarithmic function with base e, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

one-to-one function A function is one-to-one if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

KEY CONCEPTS

10.1 Finding Composite and Inverse Functions

- **Composition of Functions:** The composition of functions f and g, is written $f \circ g$ and is defined by

$$(f \circ g)(x) = f(g(x))$$

  We read $f(g(x))$ as f of g of x.

- **Horizontal Line Test:** If every horizontal line, intersects the graph of a function in at most one point, it is a one-to-one function.
- **Inverse of a Function Defined by Ordered Pairs:** If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).
- **Inverse Functions:** For every x in the domain of one-to-one function f and f^{-1},

$$\begin{aligned} f^{-1}(f(x)) &= x \\ f\left(f^{-1}(x)\right) &= x \end{aligned}$$

- **How to Find the Inverse of a One-to-One Function:**

  Step 1. Substitute *y* for $f(x)$.

  Step 2. Interchange the variables *x* and *y*.

  Step 3. Solve for *y*.

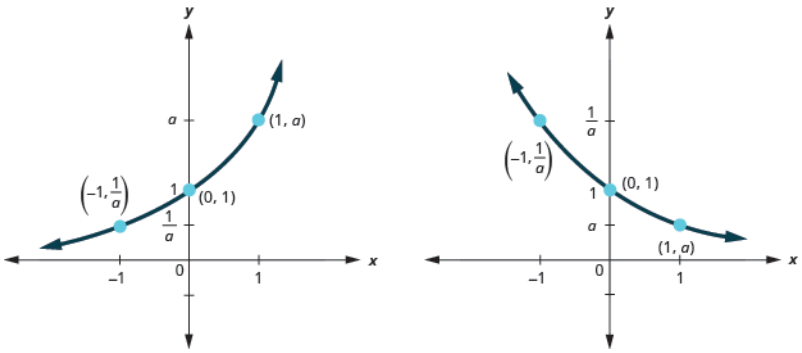
  Step 4. Substitute $f^{-1}(x)$ for y.

  Step 5. Verify that the functions are inverses.

10.2 Evaluate and Graph Exponential Functions

- **Properties of the Graph of** $f(x) = a^x$:

when $a > 1$		when $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



- **One-to-One Property of Exponential Equations:**
  For $a > 0$ and $a \neq 1$,

$$A = A_0 e^{rt}$$

- **How to Solve an Exponential Equation**
  Step 1. Write both sides of the equation with the same base, if possible.
  Step 2. Write a new equation by setting the exponents equal.
  Step 3. Solve the equation.
  Step 4. Check the solution.
- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is

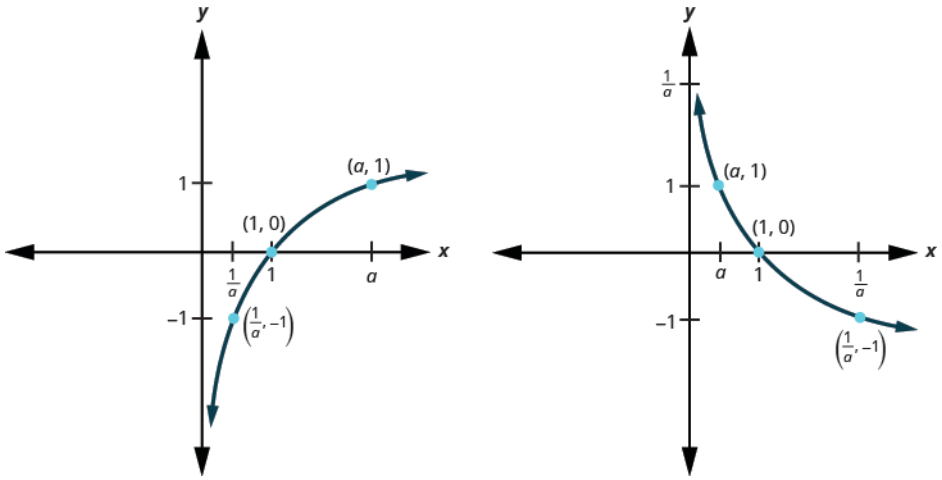
  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ when compounded n times a year.

  $A = Pe^{rt}$ when compounded continuously.
- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount, A, is $A = A_0 e^{rt}$.

10.3 Evaluate and Graph Logarithmic Functions

- **Properties of the Graph of** $y = \log_a x$:

when $a > 1$		when $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	*x*-intercept	$(1, 0)$
y-intercept	none	*y*-intercept	none
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	*y*-axis	Asymptote	*y*-axis
Basic shape	increasing	Basic shape	decreasing



- **Decibel Level of Sound:** The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is $D = 10\log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

10.4 Use the Properties of Logarithms

- **Properties of Logarithms**

$$\log_a 1 = 0 \qquad \log_a a = 1$$

- **Inverse Properties of Logarithms**
  - For $a > 0$, $x > 0$ and $a \neq 1$

$$a^{\log_a x} = x \qquad \log_a a^x = x$$

- **Product Property of Logarithms**
  - If $M > 0$, $N > 0$, $a > 0$ and $a \neq 1$, then,

$$\log_a M \cdot N = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

- **Quotient Property of Logarithms**

- If $M > 0,\ N > 0, \text{a} > 0$ and $\text{a} \neq 1,$ then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**
  - If $M > 0,\ \text{a} > 0,\ \text{a} \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

- **Properties of Logarithms Summary**
If $M > 0,\ \text{a} > 0,\ \text{a} \neq 1$ and p is any real number then,

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

- **Change-of-Base Formula**
For any logarithmic bases a and b, and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$

new base b new base 10 new base e

10.5 Solve Exponential and Logarithmic Equations

- **One-to-One Property of Logarithmic Equations:** For $M > 0,\ N > 0,\ \text{a} > 0,$ and $\text{a} \neq 1$ is any real number:

$$\text{If } \log_a M = \log_a N, \text{ then } M = N.$$

- **Compound Interest:**
For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{when compounded } n \text{ times a year.}$$

$$A = Pe^{rt} \qquad \text{when compounded continuously.}$$

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount, A, is $A = A_0 e^{rt}$.

REVIEW EXERCISES

10.1 Section 10.1 Finding Composite and Inverse Functions

Find and Evaluate Composite Functions

In the following exercises, for each pair of functions, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

356. $f(x) = 7x - 2$ and $g(x) = 5x + 1$

357. $f(x) = 4x$ and $g(x) = x^2 + 3x$

In the following exercises, evaluate the composition.

358. For functions $f(x) = 3x^2 + 2$ and $g(x) = 4x - 3$, find

ⓐ $(f \circ g)(-3)$

ⓑ $(g \circ f)(-2)$

ⓒ $(f \circ f)(-1)$

359. For functions $f(x) = 2x^3 + 5$ and $g(x) = 3x^2 - 7$, find

ⓐ $(f \circ g)(-1)$

ⓑ $(g \circ f)(-2)$

ⓒ $(g \circ g)(1)$

Determine Whether a Function is One-to-One

In the following exercises, for each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

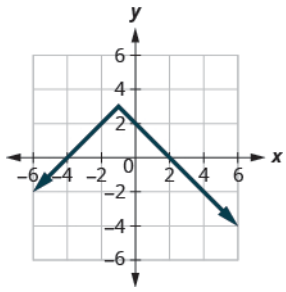
360. $\{(-3, -5), (-2, -4), (-1, -3), (0, -2), (-1, -1), (-2, 0), (-3, 1)\}$

361. $\{(-3, 0), (-2, -2), (-1, 0), (0, 1), (1, 2), (2, 1), (3, -1)\}$

362. $\{(-3, 3), (-2, 1), (-1, -1), (0, -3), (1, -5), (2, -4), (3, -2)\}$

In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

363. ⓐ



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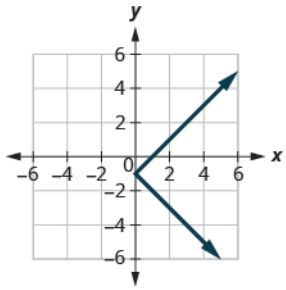
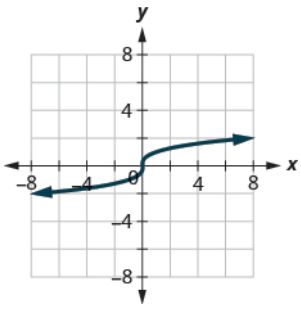


364. ⓐ



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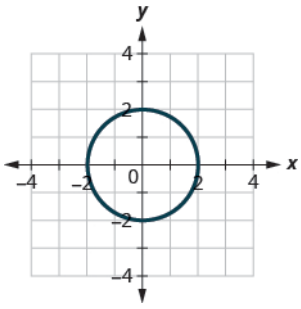


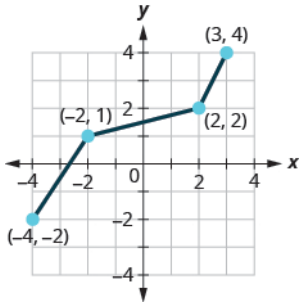
Find the Inverse of a Function

In the following exercise, find the inverse of the function. Determine the domain and range of the inverse function.

365. $\{(-3, 10), (-2, 5), (-1, 2), (0, 1)\}$

In the following exercise, graph the inverse of the one-to-one function shown.

366.



In the following exercises, verify that the functions are inverse functions.

367. $f(x) = 3x + 7$ and $g(x) = \frac{x-7}{3}$

368. $f(x) = 2x + 9$ and $g(x) = \frac{x+9}{2}$

In the following exercises, find the inverse of each function.

369. $f(x) = 6x - 11$

370. $f(x) = x^3 + 13$

371. $f(x) = \frac{1}{x+5}$

372. $f(x) = \sqrt[5]{x-1}$

10.2 Section 10.2 Evaluate and Graph Exponential Functions

Graph Exponential Functions

In the following exercises, graph each of the following functions.

373. $f(x) = 4^x$

374. $f(x) = \left(\frac{1}{5}\right)^x$

375. $g(x) = (0.75)^x$

376. $g(x) = 3^{x+2}$

377. $f(x) = (2.3)^x - 3$

378. $f(x) = e^x + 5$

379. $f(x) = -e^x$

Solve Exponential Equations

In the following exercises, solve each equation.

380. $3^{5x-6} = 81$

381. $2^{x^2} = 16$

382. $9^x = 27$

383. $5^{x^2+2x} = \frac{1}{5}$

384. $e^{4x} \cdot e^7 = e^{19}$

385. $\frac{e^{x^2}}{e^{15}} = e^{2x}$

Use Exponential Models in Applications

In the following exercises, solve.

386. Felix invested $\$12{,}000$ in a savings account. If the interest rate is 4% how much will be in the account in 12 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously.

387. Sayed deposits $\$20{,}000$ in an investment account. What will be the value of his investment in 30 years if the investment is earning 7% per year and is compounded continuously?

388. A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. She starts her experiment with 150 of the bacteria that grows at a rate of 15% per hour. She will check on the bacteria every 24 hours. How many bacteria will he find in 24 hours?

389. In the last five years the population of the United States has grown at a rate of 0.7% per year to about 318,900,000. If this rate continues, what will be the population in 5 more years?

10.3 Section 10.3 Evaluate and Graph Logarithmic Functions

Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

390. $5^4 = 625$

391. $10^{-3} = \frac{1}{1{,}000}$

392. $63^{\frac{1}{5}} = \sqrt[5]{63}$

393. $e^y = 16$

In the following exercises, convert each logarithmic equation to exponential form.

394. $7 = \log_2 128$

395. $5 = \log 100{,}000$

396. $4 = \ln x$

Evaluate Logarithmic Functions

In the following exercises, solve for x.

397. $\log_x 125 = 3$

398. $\log_7 x = -2$

399. $\log_{\frac{1}{2}} \frac{1}{16} = x$

In the following exercises, find the exact value of each logarithm without using a calculator.

400. $\log_2 32$

401. $\log_8 1$

402. $\log_3 \frac{1}{9}$

Graph Logarithmic Functions

In the following exercises, graph each logarithmic function.

403. $y = \log_5 x$

404. $y = \log_{\frac{1}{4}} x$

405. $y = \log_{0.8} x$

Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

406. $\log_a 36 = 5$

407. $\ln x = -3$

408. $\log_2 (5x - 7) = 3$

409. $\ln e^{3x} = 24$

410. $\log(x^2 - 21) = 2$

Use Logarithmic Models in Applications

411. What is the decibel level of a train whistle with intensity 10^{-3} watts per square inch?

10.4 Section 10.4 Use the Properties of Logarithms

Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

412. ⓐ $\log_7 1$ ⓑ $\log_{12} 12$

413. ⓐ $5^{\log_5 13}$ ⓑ $\log_3 3^{-9}$

414. ⓐ $10^{\log\sqrt{5}}$ ⓑ $\log 10^{-3}$

415. ⓐ $e^{\ln 8}$ ⓑ $\ln e^5$

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

416. $\log_4(64xy)$

417. $\log 10{,}000m$

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

418. $\log_7 \frac{49}{y}$

419. $\ln \frac{e^5}{2}$

In the following exercises, use the Power Property of Logarithms to expand each logarithm. Simplify, if possible.

420. $\log x^{-9}$

421. $\log_4 \sqrt[7]{z}$

In the following exercises, use properties of logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

422. $\log_3\left(\sqrt{4}x^7 y^8\right)$

423. $\log_5 \frac{8a^2b^6c}{d^3}$

424. $\ln \frac{\sqrt{3x^2 - y^2}}{z^4}$

425. $\log_6 \sqrt[3]{\frac{7x^2}{6y^3z^5}}$

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.

426. $\log_2 56 - \log_2 7$

427. $3\log_3 x + 7\log_3 y$

428. $\log_5\left(x^2 - 16\right) - 2\log_5(x+4)$

429. $\frac{1}{4}\log y - 2\log(y-3)$

Use the Change-of-Base Formula

In the following exercises, rounding to three decimal places, approximate each logarithm.

430. $\log_5 97$

431. $\log_{\sqrt{3}} 16$

10.5 Section 10.5 Solve Exponential and Logarithmic Equations

Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for x.

432. $3\log_5 x = \log_5 216$

433. $\log_2 x + \log_2(x-2) = 3$

434. $\log(x-1) - \log(3x+5) = -\log x$

435. $\log_4(x-2) + \log_4(x+5) = \log_4 8$

436. $\ln(3x-2) = \ln(x+4) + \ln 2$

Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

437. $2^x = 101$

438. $e^x = 23$

439. $\left(\frac{1}{3}\right)^x = 7$

440. $7e^{x+3} = 28$

441. $e^{x-4} + 8 = 23$

Use Exponential Models in Applications

442. Jerome invests $\$18{,}000$ at age 17. He hopes the investments will be worth $\$30{,}000$ when he turns 26. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?

443. Elise invests $\$4500$ in an account that compounds interest monthly and earns 6%. How long will it take for her money to double?

444. Researchers recorded that a certain bacteria population grew from 100 to 300 in 8 hours. At this rate of growth, how many bacteria will there be in 24 hours?

445. Mouse populations can double in 8 months $(A = 2A_0)$. How long will it take for a mouse population to triple?

446. The half-life of radioactive iodine is 60 days. How much of a 50 mg sample will be left in 40 days?

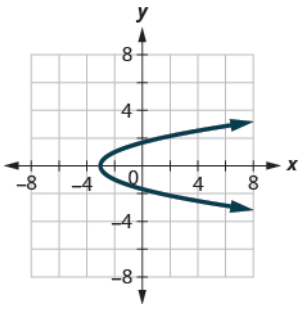
PRACTICE TEST

447. For the functions, $f(x) = 6x + 1$ and $g(x) = 8x - 3$, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

448. Determine if the following set of ordered pairs represents a function and if so, is the function one-to-one. $\{(-2, 2), (-1, -3), (0, 1), (1, -2), (2, -3)\}$

449. Determine whether each graph is the graph of a function and if so, is it one-to-one.

ⓐ



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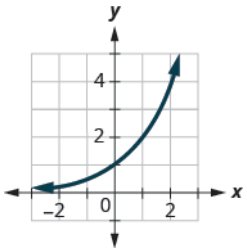
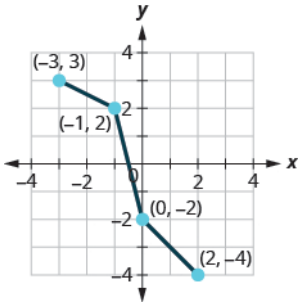


450. Graph, on the same coordinate system, the inverse of the one-to-one function shown.



451. Find the inverse of the function $f(x) = x^5 - 9$.

452. Graph the function $g(x) = 2^{x-3}$.

453. Solve the equation $2^{2x-4} = 64$.

454. Solve the equation $\frac{e^{x^2}}{e^4} = e^{3x}$.

455. Megan invested $\$21{,}000$ in a savings account. If the interest rate is 5%, how much will be in the account in 8 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously.

456. Convert the equation from exponential to logarithmic form: $10^{-2} = \frac{1}{100}$.

457. Convert the equation from logarithmic equation to exponential form: $3 = \log_7 343$

458. Solve for x: $\log_5 x = -3$

459. Evaluate $\log_{11} 1$.

460. Evaluate $\log_4 \frac{1}{64}$.

461. Graph the function $y = \log_3 x$.

462. Solve for x: $\log(x^2 - 39) = 1$

463. What is the decibel level of a small fan with intensity 10^{-8} watts per square inch?

464. Evaluate each. ⓐ $6^{\log_6 17}$ ⓑ $\log_9 9^{-3}$

In the following exercises, use properties of logarithms to write each expression as a sum of logarithms, simplifying if possible.

465. $\log_5 25ab$

466. $\ln\frac{e^{12}}{8}$

467. $\log_2 \sqrt[4]{\frac{5x^3}{16y^2z^7}}$

In the following exercises, use the Properties of Logarithms to condense the logarithm, simplifying if possible.

468. $5\log_4 x + 3\log_4 y$

469. $\frac{1}{6}\log x - 3\log(x+5)$

470. Rounding to three decimal places, approximate $\log_4 73$.

471. Solve for x: $\log_7(x+2) + \log_7(x-3) = \log_7 24$

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

472. $\left(\frac{1}{5}\right)^x = 9$

473. $5e^{x-4} = 40$

474. Jacob invests $14,000 in an account that compounds interest quarterly and earns 4%. How long will it take for his money to double?

475. Researchers recorded that a certain bacteria population grew from 500 to 700 in 5 hours. At this rate of growth, how many bacteria will there be in 20 hours?

476. A certain beetle population can double in 3 months $(A = 2A_0)$. How long will it take for that beetle population to triple?