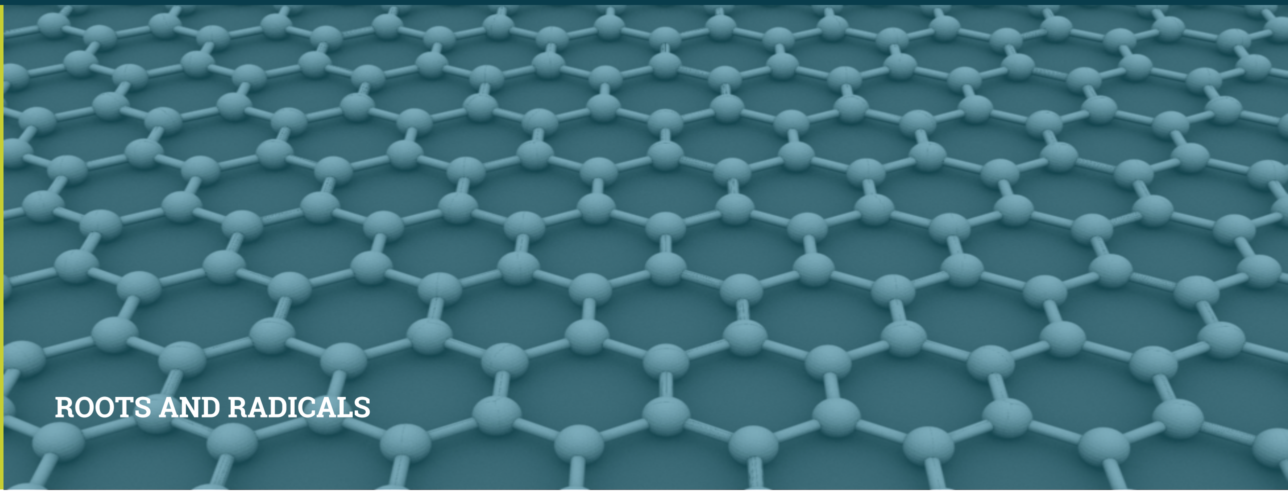


8 ROOTS AND RADICALS

Figure 8.1 Graphene is an incredibly strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: “AlexanderAIUS” / Wikimedia Commons)

Chapter Outline

Introduction

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

8.1 Simplify Expressions with Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: ⓐ $(-9)^2$ ⓑ -9^2 ⓒ $(-9)^3$.
   If you missed this problem, review Example 2.21.

2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review **Example 1.34**.
3. Simplify: ⓐ $x^3 \cdot x^3$ ⓑ $y^2 \cdot y^2 \cdot y^2$ ⓒ $z^3 \cdot z^3 \cdot z^3 \cdot z^3$.
   If you missed this problem, review **Example 5.12**.

Simplify Expressions with Roots

In **Foundations**, we briefly looked at square roots. Remember that when a real number *n* is multiplied by itself, we write n^2 and read it '*n* squared'. This number is called the **square** of *n*, and *n* is called the **square root**. For example,

$$13^2 \text{ is read "13 squared"}$$
$$169 \text{ is called the } \textit{square} \text{ of 13, since } 13^2 = 169$$
$$13 \text{ is a } \textit{square root} \text{ of } 169$$

Square and Square Root of a number

Square

$$\text{If } n^2 = m, \quad \text{then } m \text{ is the } \textbf{square} \text{ of } n.$$

Square Root

$$\text{If } n^2 = m, \quad \text{then } n \text{ is a } \textbf{square root} \text{ of } m.$$

Notice $(-13)^2 = 169$ also, so −13 is also a square root of 169. Therefore, both 13 and −13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of *m*. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Square Root Notation

$$\sqrt{m} \text{ is read "the square root of } m\text{".}$$
$$\text{If } n^2 = m, \quad \text{then } n = \sqrt{m}, \quad \text{for } n \geq 0.$$

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169} = 13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169} = -13$.

EXAMPLE 8.1

Simplify: ⓐ $\sqrt{144}$ ⓑ $-\sqrt{289}$.

Solution

ⓐ

	$\sqrt{144}$
Since $12^2 = 144$.	12

ⓑ

	$-\sqrt{289}$
Since $17^2 = 289$ and the negative is in front of the radical sign.	-17

TRY IT : : 8.1 Simplify: ⓐ $-\sqrt{64}$ ⓑ $\sqrt{225}$.

TRY IT : : 8.2 Simplify: ⓐ $\sqrt{100}$ ⓑ $-\sqrt{121}$.

Can we simplify $\sqrt{-49}$? Is there a number whose square is -49?

$$(\ )^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

EXAMPLE 8.2

Simplify: ⓐ $\sqrt{-196}$ ⓑ $-\sqrt{64}$.

Solution

ⓐ

	$\sqrt{-196}$
There is no real number whose square is -196.	$\sqrt{-196}$ is not a real number.

ⓑ

	$-\sqrt{64}$
The negative is in front of the radical.	-8

TRY IT : : 8.3 Simplify: ⓐ $\sqrt{-169}$ ⓑ $-\sqrt{81}$.

TRY IT : : 8.4 Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{-121}$.

So far we have only talked about squares and square roots. Let's now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth power
n^5	n to the fi th power

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from –5 to 5. See **Figure 8.2**.

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Figure 8.2

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of –2 to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Even power **Positive** result **Odd** power **Negative** result

We will now extend the square root definition to higher roots.

n^{th} Root of a Number

$$\text{If } b^n = a, \text{ then } b \text{ is an } n^{th} \text{ root of } a.$$
$$\text{The principal } n^{th} \text{ root of } a \text{ is written } \sqrt[n]{a}.$$
$$n \text{ is called the } \textbf{index} \text{ of the radical.}$$

Just like we use the word ‘cubed’ for b^3, we use the term ‘cube root’ for $\sqrt[3]{a}$.

We can refer to **Figure 8.2** to help find higher roots.

$$\begin{aligned} 4^3 &= 64 \\ 3^4 &= 81 \\ (-2)^5 &= -32 \end{aligned} \qquad \begin{aligned} \sqrt[3]{64} &= 4 \\ \sqrt[4]{81} &= 3 \\ \sqrt[5]{-32} &= -2 \end{aligned}$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Properties of $\sqrt[n]{a}$

When n is an even number and

- $a \geq 0,$ then $\sqrt[n]{a}$ is a real number.
- $a < 0,$ then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

We will apply these properties in the next two examples.

EXAMPLE 8.3

Simplify: ⓐ $\sqrt[3]{64}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{32}$.

Solution

ⓐ

	$\sqrt[3]{64}$
Since $4^3 = 64$.	4

ⓑ

	$\sqrt[4]{81}$
Since $(3)^4 = 81$.	3

ⓒ

	$\sqrt[5]{32}$
Since $(2)^5 = 32$.	2

TRY IT : : 8.5 Simplify: ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{256}$ ⓒ $\sqrt[5]{243}$.

TRY IT : : 8.6 Simplify: ⓐ $\sqrt[3]{1000}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$.

In this example be alert for the negative signs as well as even and odd powers.

EXAMPLE 8.4

Simplify: ⓐ $\sqrt[3]{-125}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{-243}$.

Solution

ⓐ

	$\sqrt[3]{-125}$
Since $(-5)^3 = -125$.	-5

ⓑ

	$\sqrt[4]{-16}$
Think, $(\,?\,)^4 = -16$. No real number raised to the fourth power is negative.	Not a real number.

ⓒ

	$\sqrt[5]{-243}$
Since $(-3)^5 = -243$.	-3

TRY IT : : 8.7 Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-256}$ ⓒ $\sqrt[5]{-32}$.

TRY IT : : 8.8 Simplify: ⓐ $\sqrt[3]{-216}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1024}$.

Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4} = 2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for perfect square numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

Number	Square Root
4	2
9	3
16	4
25	5

11 $\sqrt{11}$

$9 < 11 < 16$

$3 < \sqrt{11} < 4$

Number	Cube Root
8	2
27	3
64	4
125	5

91 $\sqrt[3]{91}$

$64 < 91 < 125$

$4 < \sqrt[3]{91} < 5$

Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

EXAMPLE 8.5

Estimate each root between two consecutive whole numbers: ⓐ $\sqrt{105}$ ⓑ $\sqrt[3]{43}$.

Solution

ⓐ Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

$\sqrt{105}$

Number	Square Root
81	9
100	10
121	11
144	12

105 $\sqrt{105}$

Locate 105 between two consecutive perfect squares.	$100 < 105 < 121$
$\sqrt{105}$ is between their square roots.	$10 < \sqrt{105} < 11$

ⓑ Similarly we locate 43 between two perfect cube numbers.

$\sqrt[3]{43}$

Number	Cube Root
8	2
27	3
64	4
125	5

43 $\sqrt[3]{43}$

Locate 43 between two consecutive perfect cubes.	$27 < 43 < 64$
$\sqrt[3]{43}$ is between their cube roots.	$3 < \sqrt[3]{43} < 4$

TRY IT : : 8.9 Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{38}$ ⓑ $\sqrt[3]{93}$

TRY IT : : 8.10 Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{84}$ ⓑ $\sqrt[3]{152}$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{ rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{ rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$\begin{aligned} (2.236067978)^2 &= 5.000000002 \qquad & (3.105422799)^4 &= 92.999999991 \\ (2.24)^2 &= 5.0176 & (3.11)^4 &= 93.54951841 \end{aligned}$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

EXAMPLE 8.6

Round to two decimal places: ⓐ $\sqrt{17}$ ⓑ $\sqrt[3]{49}$ ⓒ $\sqrt[4]{51}$.

Solution

ⓐ

	$\sqrt{17}$
Use the calculator square root key.	$4.123105626\ldots$
Round to two decimal places.	4.12
	$\sqrt{17} \approx 4.12$

ⓑ

	$\sqrt[3]{49}$
Use the calculator $\sqrt[y]{x}$ key.	$3.659305710\ldots$
Round to two decimal places.	3.66
	$\sqrt[3]{49} \approx 3.66$

ⓒ

	$\sqrt[4]{51}$
Use the calculator $\sqrt[y]{x}$ key.	$2.6723451177\ldots$
Round to two decimal places.	2.67
	$\sqrt[4]{51} \approx 2.67$

TRY IT : : 8.11 Round to two decimal places:

ⓐ $\sqrt{11}$ ⓑ $\sqrt[3]{71}$ ⓒ $\sqrt[4]{127}$.

TRY IT : : 8.12 Round to two decimal places:

ⓐ $\sqrt{13}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{98}$.

Simplify Variable Expressions with Roots

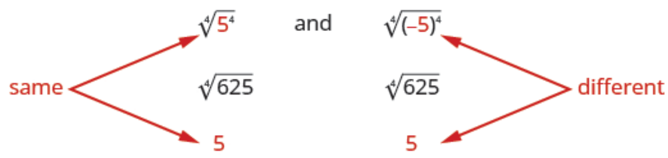
The odd root of a number can be either positive or negative. For example,

	$\sqrt[3]{4^3}$	and	$\sqrt[3]{(-4)^3}$	
same	$\sqrt[3]{64}$		$\sqrt[3]{-64}$	same
	4		-4	

In either case, when n is odd, $\sqrt[n]{a^n} = a$.

But what about an even root? We want the principal root, so $\sqrt[4]{625} = 5$.

But notice,



Here we see that sometimes when n is even, $\sqrt[n]{a^n} \neq a$.

How can we make sure the fourth root of −5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even $\sqrt[n]{a^n} = |a|$. This guarantees the principal root is positive.

Simplifying Odd and Even Roots

For any integer $n \geq 2$,

when the index n is odd	$\sqrt[n]{a^n} = a$
when the index n is even	$\sqrt[n]{a^n} = \|a\|$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

EXAMPLE 8.7

Simplify: ⓐ $\sqrt{x^2}$ ⓑ $\sqrt[3]{n^3}$ ⓒ $\sqrt[4]{p^4}$ ⓓ $\sqrt[5]{y^5}$.

Solution

ⓐ We use the absolute value to be sure to get the positive root.

	$\sqrt{x^2}$
Since the index n is even, $\sqrt[n]{a^n} = \|a\|$.	$\|x\|$

ⓑ This is an odd indexed root so there is no need for an absolute value sign.

	$\sqrt[3]{m^3}$
Since the index n is odd, $\sqrt[n]{a^n} = a$.	m

ⓒ

	$\sqrt[4]{p^4}$
Since the index n is even $\sqrt[n]{a^n} = \|a\|$.	$\|p\|$

ⓓ

	$\sqrt[5]{y^5}$
Since the index n is odd, $\sqrt[n]{a^n} = a$.	y

TRY IT : : 8.13 Simplify: ⓐ $\sqrt{b^2}$ ⓑ $\sqrt[3]{w^3}$ ⓒ $\sqrt[4]{m^4}$ ⓓ $\sqrt[5]{q^5}$.

TRY IT : : 8.14 Simplify: ⓐ $\sqrt{y^2}$ ⓑ $\sqrt[3]{p^3}$ ⓒ $\sqrt[4]{z^4}$ ⓓ $\sqrt[5]{q^5}$.

What about square roots of higher powers of variables? The Power Property of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root,

	$\sqrt{a^{2m}}$
Since $(a^m)^2 = a^{2m}$.	$\sqrt{(a^m)^2}$
Since n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	$\lvert a^m \rvert$

$$\text{So } \sqrt{a^{2m}} = \lvert a^m \rvert.$$

We apply this concept in the next example.

EXAMPLE 8.8

Simplify: ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$.

Solution

ⓐ

	$\sqrt{x^6}$
Since $\left(x^3\right)^2 = x^6$.	$\sqrt{\left(x^3\right)^2}$
Since the index n is even $\sqrt{a^n} = \lvert a \rvert$.	$\lvert x^3 \rvert$

ⓑ

	$\sqrt{y^{16}}$
Since $\left(y^8\right)^2 = y^{16}$.	$\sqrt{\left(y^8\right)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	y^8
In this case the absolute value sign is not needed as y^8 is positive.	

TRY IT : : 8.15 Simplify: ⓐ $\sqrt{y^{18}}$ ⓑ $\sqrt{z^{12}}$.

TRY IT : : 8.16 Simplify: ⓐ $\sqrt{m^4}$ ⓑ $\sqrt{b^{10}}$.

The next example uses the same idea for highter roots.

EXAMPLE 8.9

Simplify: ⓐ $\sqrt[3]{y^{18}}$ ⓑ $\sqrt[4]{z^8}$.

Solution

ⓐ

	$\sqrt[3]{y^{18}}$
Since $\left(y^6\right)^3 = y^{18}$.	$\sqrt[3]{\left(y^6\right)^3}$
Since n is odd, $\sqrt[n]{a^n} = a$.	y^6

ⓑ

	$\sqrt[4]{z^8}$
Since $\left(z^2\right)^4 = z^8$.	$\sqrt[4]{\left(z^2\right)^4}$
Since z^2 is positive, we do not need an absolute value sign.	z^2

TRY IT : : 8.17 Simplify: ⓐ $\sqrt[4]{u^{12}}$ ⓑ $\sqrt[3]{v^{15}}$.

TRY IT : : 8.18 Simplify: ⓐ $\sqrt[5]{c^{20}}$ ⓑ $\sqrt[6]{d^{24}}$

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = \left|a^m\right|$ works in much the same way.

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as u^4 is always positive.

EXAMPLE 8.10

Simplify: ⓐ $\sqrt{16n^2}$ ⓑ $-\sqrt{81c^2}$.

Solution

ⓐ

	$\sqrt{16n^2}$
Since $(4n)^2 = 16n^2$.	$\sqrt{(4n)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	$4\lvert n \rvert$

ⓑ

	$-\sqrt{81c^2}$
Since $(9c)^2 = 81c^2$.	$-\sqrt{(9c)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	$-9\lvert c \rvert$

TRY IT : : 8.19 Simplify: ⓐ $\sqrt{64x^2}$ ⓑ $-\sqrt{100p^2}$.

TRY IT : : 8.20 Simplify: ⓐ $\sqrt{169y^2}$ ⓑ $-\sqrt{121y^2}$.

This example just takes the idea farther as it has roots of higher index.

EXAMPLE 8.11

Simplify: ⓐ $\sqrt[3]{64p^6}$ ⓑ $\sqrt[4]{16q^{12}}$.

Solution

ⓐ

	$\sqrt[3]{64p^6}$
Rewrite $64p^6$ as $\left(4p^2\right)^3$.	$\sqrt[3]{\left(4p^2\right)^3}$
Take the cube root.	$4p^2$

ⓑ

	$\sqrt[4]{16q^{12}}$
Rewrite the radicand as a fourth power.	$\sqrt[4]{\left(2q^3\right)^4}$
Take the fourth root.	$2\left\|q^3\right\|$

> **TRY IT : : 8.21** Simplify: ⓐ $\sqrt[3]{27x^{27}}$ ⓑ $\sqrt[4]{81q^{28}}$.

> **TRY IT : : 8.22** Simplify: ⓐ $\sqrt[3]{125q^9}$ ⓑ $\sqrt[5]{243q^{25}}$.

The next examples have two variables.

EXAMPLE 8.12

Simplify: ⓐ $\sqrt{36x^2y^2}$ ⓑ $\sqrt{121a^6b^8}$ ⓒ $\sqrt[3]{64p^{63}q^9}$.

Solution

ⓐ

	$\sqrt{36x^2y^2}$
Since $(6xy)^2 = 36x^2y^2$	$\sqrt{(6xy)^2}$
Take the square root.	$6\|xy\|$

ⓑ

	$\sqrt{121a^6b^8}$
Since $\left(11a^3b^4\right)^2 = 121a^6b^8$	$\sqrt{\left(11a^3b^4\right)^2}$
Take the square root.	$11\left\|a^3\right\|b^4$

ⓒ

	$\sqrt[3]{64p^{63}q^9}$
Since $\left(4p^{21}q^3\right)^3 = 64p^{63}q^9$	$\sqrt[3]{\left(4p^{21}q^3\right)^3}$
Take the cube root.	$4p^{21}q^3$

TRY IT : : 8.23 Simplify: ⓐ $\sqrt{100a^2 b^2}$ ⓑ $\sqrt{144p^{12} q^{20}}$ ⓒ $\sqrt[3]{8x^{30} y^{12}}$

TRY IT : : 8.24 Simplify: ⓐ $\sqrt{225m^2 n^2}$ ⓑ $\sqrt{169x^{10} y^{14}}$ ⓒ $\sqrt[3]{27w^{36} z^{15}}$

MEDIA : :

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- **Simplifying Variables Exponents with Roots using Absolute Values (https://openstax.org/l/37SimVarAbVal)**

8.1 EXERCISES

Practice Makes Perfect

Simplify Expressions with Roots

In the following exercises, simplify.

1. ⓐ $\sqrt{64}$ ⓑ $-\sqrt{81}$

2. ⓐ $\sqrt{169}$ ⓑ $-\sqrt{100}$

3. ⓐ $\sqrt{196}$ ⓑ $-\sqrt{1}$

4. ⓐ $\sqrt{144}$ ⓑ $-\sqrt{121}$

5. ⓐ $\sqrt{\frac{4}{9}}$ ⓑ $-\sqrt{0.01}$

6. ⓐ $\sqrt{\frac{64}{121}}$ ⓑ $-\sqrt{0.16}$

7. ⓐ $\sqrt{-121}$ ⓑ $-\sqrt{289}$

8. ⓐ $-\sqrt{400}$ ⓑ $\sqrt{-36}$

9. ⓐ $-\sqrt{225}$ ⓑ $\sqrt{-9}$

10. ⓐ $\sqrt{-49}$ ⓑ $-\sqrt{256}$

11. ⓐ $\sqrt[3]{216}$ ⓑ $\sqrt[4]{256}$

12. ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$

13. ⓐ $\sqrt[3]{512}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{1}$

14. ⓐ $\sqrt[3]{125}$ ⓑ $\sqrt[4]{1296}$ ⓒ $\sqrt[5]{1024}$

15. ⓐ $\sqrt[3]{-8}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-32}$

16.
ⓐ $\sqrt[3]{-64}$
ⓑ $\sqrt[4]{-16}$
ⓒ $\sqrt[5]{-243}$

17.
ⓐ $\sqrt[3]{-125}$
ⓑ $\sqrt[4]{-1296}$
ⓒ $\sqrt[5]{-1024}$

18.
ⓐ $\sqrt[3]{-512}$
ⓑ $\sqrt[4]{-81}$
ⓒ $\sqrt[5]{-1}$

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

19. ⓐ $\sqrt{70}$ ⓑ $\sqrt[3]{71}$

20. ⓐ $\sqrt{55}$ ⓑ $\sqrt[3]{119}$

21. ⓐ $\sqrt{200}$ ⓑ $\sqrt[3]{137}$

22. ⓐ $\sqrt{172}$ ⓑ $\sqrt[3]{200}$

In the following exercises, approximate each root and round to two decimal places.

23. ⓐ $\sqrt{19}$ ⓑ $\sqrt[3]{89}$ ⓒ $\sqrt[4]{97}$

24. ⓐ $\sqrt{21}$ ⓑ $\sqrt[3]{93}$ ⓒ $\sqrt[4]{101}$

25. ⓐ $\sqrt{53}$ ⓑ $\sqrt[3]{147}$ ⓒ $\sqrt[4]{452}$

26. ⓐ $\sqrt{47}$ ⓑ $\sqrt[3]{163}$ ⓒ $\sqrt[4]{527}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

27. ⓐ $\sqrt[5]{u^5}$ ⓑ $\sqrt[8]{v^8}$

28. ⓐ $\sqrt[3]{a^3}$ ⓑ $\sqrt[9]{b^9}$

29. ⓐ $\sqrt[4]{y^4}$ ⓑ $\sqrt[7]{m^7}$

30. ⓐ $\sqrt[8]{k^8}$ ⓑ $\sqrt[6]{p^6}$

31. ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$

32. ⓐ $\sqrt{a^{14}}$ ⓑ $\sqrt{w^{24}}$

33. ⓐ $\sqrt{x^{24}}$ ⓑ $\sqrt{y^{22}}$

34. ⓐ $\sqrt{a^{12}}$ ⓑ $\sqrt{b^{26}}$

35. ⓐ $\sqrt[3]{x^9}$ ⓑ $\sqrt[4]{y^{12}}$

36. ⓐ $\sqrt[5]{a^{10}}$ ⓑ $\sqrt[3]{b^{27}}$

37. ⓐ $\sqrt[4]{m^8}$ ⓑ $\sqrt[5]{n^{20}}$

38. ⓐ $\sqrt[6]{r^{12}}$ ⓑ $\sqrt[3]{s^{30}}$

39. ⓐ $\sqrt{49x^2}$ ⓑ $-\sqrt{81x^{18}}$

40. ⓐ $\sqrt{100y^2}$ ⓑ $-\sqrt{100m^{32}}$

41. ⓐ $\sqrt{121m^{20}}$ ⓑ $-\sqrt{64a^2}$

42.
ⓐ $\sqrt{81x^{36}}$
ⓑ $-\sqrt{25x^2}$

43.
ⓐ $\sqrt[4]{16x^8}$
ⓑ $\sqrt[6]{64y^{12}}$

44.
ⓐ $\sqrt[3]{-8c^9}$
ⓑ $\sqrt[3]{125d^{15}}$

45.
ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

46.
ⓐ $\sqrt[7]{128r^{14}}$
ⓑ $\sqrt[4]{81s^{24}}$

47.
ⓐ $\sqrt{144x^2y^2}$
ⓑ $\sqrt{169w^8y^{10}}$
ⓒ $\sqrt[3]{8a^{51}b^6}$

48.
ⓐ $\sqrt{196a^2b^2}$
ⓑ $\sqrt{81p^{24}q^6}$
ⓒ $\sqrt[3]{27p^{45}q^9}$

49.
ⓐ $\sqrt{121a^2b^2}$
ⓑ $\sqrt{9c^8d^{12}}$
ⓒ $\sqrt[3]{64x^{15}y^{66}}$

50.
ⓐ $\sqrt{225x^2y^2z^2}$
ⓑ $\sqrt{36r^6s^{20}}$
ⓒ $\sqrt[3]{125y^{18}z^{27}}$

Writing Exercises

51. Why is there no real number equal to $\sqrt{-64}$?

52. What is the difference between 9^2 and $\sqrt{9}$?

53. Explain what is meant by the n^{th} root of a number.

54. Explain the difference of finding the n^{th} root of a number when the index is even compared to when the index is odd.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with roots.			
estimate and approximate roots.			
simplify variable expressions with roots.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

8.2 Simplify Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{x^9}{x^4}$.

   If you missed this problem, review Example 5.13.

2. Simplify: $\frac{y^3}{y^{11}}$.

   If you missed this problem, review Example 5.13.

3. Simplify: $\left(n^2\right)^6$.

   If you missed this problem, review Example 5.17.

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A radical expression, $\sqrt[n]{a}$, is considered simplified if it has no factors of m^n. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

Simplified Radical Expression

For real numbers *a* and *m*, and $n \geq 2$,

$$\sqrt[n]{a} \text{ is considered simplified i } \ a \text{ has no factors of } m^n$$

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

Similarly, $\sqrt[3]{4}$ is simplified because there are no perfect cube factors in 4. But $\sqrt[3]{24}$ is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that $(ab)^n = a^n b^n$. The corresponding of **Product Property of Roots** says that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Product Property of n^{th} Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $n \geq 2$ is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

We use the Product Property of Roots to remove all perfect square factors from a square root.

EXAMPLE 8.13 SIMPLIFY SQUARE ROOTS USING THE PRODUCT PROPERTY OF ROOTS

Simplify: $\sqrt{98}$.

Solution

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.	We see that 49 is the largest factor of 98 that has a power of 2. In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{98}$ $\sqrt{49 \cdot 2}$
Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Step 3. Simplify the root of the perfect power.		$7\sqrt{2}$

TRY IT : : 8.25 Simplify: $\sqrt{48}$.

TRY IT : : 8.26 Simplify: $\sqrt{45}$.

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

HOW TO : : SIMPLIFY A RADICAL EXPRESSION USING THE PRODUCT PROPERTY.

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.

Step 2. Use the product rule to rewrite the radical as the product of two radicals.

Step 3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

EXAMPLE 8.14

Simplify: ⓐ $\sqrt{500}$ ⓑ $\sqrt[3]{16}$ ⓒ $\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt{500}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{100 \cdot 5}$
Rewrite the radical as the product of two radicals	$\sqrt{100} \cdot \sqrt{5}$
Simplify.	$10\sqrt{5}$

ⓑ

	$\sqrt[3]{16}$
Rewrite the radicand as a product using the greatest perfect cube factor. $2^3 = 8$	$\sqrt[3]{8 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8} \cdot \sqrt[3]{2}$
Simplify.	$2\sqrt[3]{2}$

ⓒ

	$\sqrt[4]{243}$
Rewrite the radicand as a product using the greatest perfect fourth power factor. $3^4 = 81$	$\sqrt[4]{81 \cdot 3}$
Rewrite the radical as the product of two radicals	$\sqrt[4]{81} \cdot \sqrt[4]{3}$
Simplify.	$3\sqrt[4]{3}$

TRY IT : : 8.27 Simplify: ⓐ $\sqrt{288}$ ⓑ $\sqrt[3]{81}$ ⓒ $\sqrt[4]{64}$.

TRY IT : : 8.28 Simplify: ⓐ $\sqrt{432}$ ⓑ $\sqrt[3]{625}$ ⓒ $\sqrt[4]{729}$.

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

EXAMPLE 8.15

Simplify: ⓐ $\sqrt{x^3}$ ⓑ $\sqrt[3]{x^4}$ ⓒ $\sqrt[4]{x^7}$.

Solution

ⓐ

	$\sqrt{x^3}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{x^2 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt{x^2} \cdot \sqrt{x}$
Simplify.	$\lvert x \rvert \sqrt{x}$

ⓑ

	$\sqrt[3]{x^4}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{x^3 \cdot x}.$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$
Simplify.	$x\sqrt[3]{x}$

ⓒ

	$\sqrt[4]{x^7}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{x^4 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$
Simplify.	$\lvert x\rvert \sqrt[4]{x^3}$

TRY IT : : 8.29 Simplify: ⓐ $\sqrt{b^5}$ ⓑ $\sqrt[4]{y^6}$ ⓒ $\sqrt[3]{z^5}$

TRY IT : : 8.30 Simplify: ⓐ $\sqrt{p^9}$ ⓑ $\sqrt[5]{y^8}$ ⓒ $\sqrt[6]{q^{13}}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

EXAMPLE 8.16

Simplify: ⓐ $\sqrt{72n^7}$ ⓑ $\sqrt[3]{24x^7}$ ⓒ $\sqrt[4]{80y^{14}}$.

Solution

ⓐ

	$\sqrt{72n^7}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{36n^6 \cdot 2n}$
Rewrite the radical as the product of two radicals.	$\sqrt{36n^6} \cdot \sqrt{2n}$
Simplify.	$6\lvert n^3\rvert \sqrt{2n}$

ⓑ

	$\sqrt[3]{24x^7}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{8x^6 \cdot 3x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$
Rewrite the fir t radicand as $\left(2x^2\right)^3$.	$\sqrt[3]{\left(2x^2\right)^3} \cdot \sqrt[3]{3x}$
Simplify.	$2x^2 \sqrt[3]{3x}$

ⓒ

	$\sqrt[4]{80y^{14}}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{16y^{12} \cdot 5\,y^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{16y^{12}} \cdot \sqrt[4]{5y^2}$
Rewrite the fir t radicand as $\left(2y^3\right)^4$.	$\sqrt[4]{\left(2\,y^3\right)^4} \cdot \sqrt[4]{5y^2}$
Simplify.	$2\left\|y^3\right\| \sqrt[4]{5\,y^2}$

> **TRY IT : : 8.31** Simplify: ⓐ $\sqrt{32y^5}$ ⓑ $\sqrt[3]{54p^{10}}$ ⓒ $\sqrt[4]{64q^{10}}$.

> **TRY IT : : 8.32** Simplify: ⓐ $\sqrt{75a^9}$ ⓑ $\sqrt[3]{128m^{11}}$ ⓒ $\sqrt[4]{162n^7}$.

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

EXAMPLE 8.17

Simplify: ⓐ $\sqrt{63u^3v^5}$ ⓑ $\sqrt[3]{40x^4y^5}$ ⓒ $\sqrt[4]{48x^4y^7}$.

Solution

ⓐ

	$\sqrt{63u^3v^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9u^2v^4 \cdot 7uv}$
Rewrite the radical as the product of two radicals.	$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$
Rewrite the fir t radicand as $\left(3uv^2\right)^2$.	$\sqrt{\left(3uv^2\right)^2} \cdot \sqrt{7uv}$
Simplify.	$3\|u\|v^2\sqrt{7uv}$

ⓑ

	$\sqrt[3]{40x^4y^5}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{8x^3y^3 \cdot 5xy^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{5xy^2}$
Rewrite the fir t radicand as $(2xy)^3$.	$\sqrt[3]{(2xy)^3} \cdot \sqrt[3]{5xy^2}$
Simplify.	$2xy\sqrt[3]{5xy^2}$

ⓒ

	$\sqrt[4]{48x^4y^7}$
Rewrite the radicand as a product using the largest perfect fourth power factor.	$\sqrt[4]{16x^4y^4 \cdot 3y^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{16x^4y^4} \cdot \sqrt[4]{3y^3}$
Rewrite the fir t radicand as $(2xy)^4$.	$\sqrt[4]{(2xy)^4} \cdot \sqrt[4]{3y^3}$
Simplify.	$2\lvert xy\rvert \sqrt[4]{3y^3}$

TRY IT : : 8.33 Simplify: ⓐ $\sqrt{98a^7b^5}$ ⓑ $\sqrt[3]{56x^5y^4}$ ⓒ $\sqrt[4]{32x^5y^8}$.

TRY IT : : 8.34 Simplify: ⓐ $\sqrt{180m^9n^{11}}$ ⓑ $\sqrt[3]{72x^6y^5}$ ⓒ $\sqrt[4]{80x^7y^4}$.

EXAMPLE 8.18

Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-16}$.

Solution

ⓐ

	$\sqrt[3]{-27}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3}$
Take the cube root.	-3

ⓑ

	$\sqrt[4]{-16}$
There is no real number n where $n^4 = -16$.	Not a real number.

TRY IT : : 8.35 Simplify: ⓐ $\sqrt[3]{-64}$ ⓑ $\sqrt[4]{-81}$.

TRY IT : : 8.36 Simplify: ⓐ $\sqrt[3]{-625}$ ⓑ $\sqrt[4]{-324}$.

We have seen how to use the order of operations to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

EXAMPLE 8.19

Simplify: ⓐ $3 + \sqrt{32}$ ⓑ $\dfrac{4 - \sqrt{48}}{2}$.

Solution

ⓐ

	$3+\sqrt{32}$
Rewrite the radicand as a product using the largest perfect square factor.	$3+\sqrt{16\cdot 2}$
Rewrite the radical as the product of two radicals.	$3+\sqrt{16}\cdot\sqrt{2}$
Simplify.	$3+4\sqrt{2}$

The terms cannot be added as one has a radical and the other does not. Trying to add an integer and a radical is like trying to add an integer and a variable. They are not like terms!

ⓑ

	$\dfrac{4-\sqrt{48}}{2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\dfrac{4-\sqrt{16\cdot 3}}{2}$
Rewrite the radical as the product of two radicals.	$\dfrac{4-\sqrt{16}\cdot\sqrt{3}}{2}$
Simplify.	$\dfrac{4-4\sqrt{3}}{2}$
Factor the common factor from the numerator.	$\dfrac{4(1-\sqrt{3})}{2}$
Remove the common factor, 2, from the numerator and denominator.	$\dfrac{\cancel{2}\cdot 2(1-\sqrt{3})}{\cancel{2}}$
Simplify.	$2(1-\sqrt{3})$

> **TRY IT : :** 8.37 Simplify: ⓐ $5+\sqrt{75}$ ⓑ $\dfrac{10-\sqrt{75}}{5}$

> **TRY IT : :** 8.38 Simplify: ⓐ $2+\sqrt{98}$ ⓑ $\dfrac{6-\sqrt{45}}{3}$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

EXAMPLE 8.20

Simplify: ⓐ $\sqrt{\dfrac{45}{80}}$ ⓑ $\sqrt[3]{\dfrac{16}{54}}$ ⓒ $\sqrt[4]{\dfrac{5}{80}}$.

Solution

ⓐ

	$\sqrt{\dfrac{45}{80}}$
Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator.	$\sqrt{\dfrac{5\cdot 9}{5\cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt{\dfrac{9}{16}}$
Simplify. Note $\left(\dfrac{3}{4}\right)^2=\dfrac{9}{16}$.	$\dfrac{3}{4}$

ⓑ

	$\sqrt[3]{\frac{16}{54}}$
Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator.	$\sqrt[3]{\frac{2 \cdot 8}{2 \cdot 27}}$
Simplify the fraction by removing common factors.	$\sqrt[3]{\frac{8}{27}}$
Simplify. Note $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$.	$\frac{2}{3}$

ⓒ

	$\sqrt[4]{\frac{5}{80}}$
Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator.	$\sqrt[4]{\frac{5 \cdot 1}{5 \cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt[4]{\frac{1}{16}}$
Simplify. Note $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.	$\frac{1}{2}$

TRY IT : : 8.39 Simplify: ⓐ $\sqrt{\frac{75}{48}}$ ⓑ $\sqrt[3]{\frac{54}{250}}$ ⓒ $\sqrt[4]{\frac{32}{162}}$.

TRY IT : : 8.40 Simplify: ⓐ $\sqrt{\frac{98}{162}}$ ⓑ $\sqrt[3]{\frac{24}{375}}$ ⓒ $\sqrt[4]{\frac{4}{324}}$.

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

EXAMPLE 8.21

Simplify: ⓐ $\sqrt{\frac{m^6}{m^4}}$ ⓑ $\sqrt[3]{\frac{a^8}{a^5}}$ ⓒ $\sqrt[4]{\frac{a^{10}}{a^2}}$.

Solution

ⓐ

	$\sqrt{\frac{m^6}{m^4}}$
Simplify the fraction inside the radical fir t. Divide the like bases by subtracting the exponents.	$\sqrt{m^2}$
Simplify.	$\vert m \vert$

ⓑ

$$\sqrt[3]{\frac{a^8}{a^5}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical fir t. $\sqrt[3]{a^3}$

Simplify. a

ⓒ

$$\sqrt[4]{\frac{a^{10}}{a^2}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical fir t. $\sqrt[4]{a^8}$

Rewrite the radicand using perfect fourth power factors. $\sqrt[4]{\left(a^2\right)^4}$

Simplify. a^2

TRY IT : : 8.41 Simplify: ⓐ $\sqrt{\frac{a^8}{a^6}}$ ⓑ $\sqrt[4]{\frac{x^7}{x^3}}$ ⓒ $\sqrt[4]{\frac{y^{17}}{y^5}}$.

TRY IT : : 8.42 Simplify: ⓐ $\sqrt{\frac{x^{14}}{x^{10}}}$ ⓑ $\sqrt[3]{\frac{m^{13}}{m^7}}$ ⓒ $\sqrt[5]{\frac{n^{12}}{n^2}}$.

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$$

We can use a similar property to simplify a root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect power of the index, we simplify the numerator and denominator separately.

Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

EXAMPLE 8.22 HOW TO SIMPLIFY THE QUOTIENT OF RADICAL EXPRESSIONS

Simplify: $\sqrt{\frac{27m^3}{196}}$.

Solution

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$

Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2} \cdot \sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

TRY IT : : 8.43

Simplify: $\sqrt{\frac{24p^3}{49}}$.

TRY IT : : 8.44

Simplify: $\sqrt{\frac{48x^5}{100}}$.

HOW TO : : SIMPLIFY A SQUARE ROOT USING THE QUOTIENT PROPERTY.

Step 1. Simplify the fraction in the radicand, if possible.

Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.

Step 3. Simplify the radicals in the numerator and the denominator.

EXAMPLE 8.23

Simplify: ⓐ $\sqrt{\frac{45x^5}{y^4}}$ ⓑ $\sqrt[3]{\frac{24x^7}{y^3}}$ ⓒ $\sqrt[4]{\frac{48x^{10}}{y^8}}$.

Solution

ⓐ

	$\sqrt{\frac{45x^5}{y^4}}$
We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.	$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9x^4} \cdot \sqrt{5x}}{y^2}$
Simplify.	$\frac{3x^2\sqrt{5x}}{y^2}$

ⓑ

$$\sqrt[3]{\frac{24x^7}{y^3}}$$

The fraction in the radicand cannot be simplified. Use he Quotient Property to write as two radicals.

$$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$$

Rewrite each radicand as a product using perfect cube factors.

$$\frac{\sqrt[3]{8x^6 \cdot 3x}}{\sqrt[3]{y^3}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}}{\sqrt[3]{y^3}}$$

Simplify.

$$\frac{2x^2\sqrt[3]{3x}}{y}$$

ⓒ

$$\sqrt[4]{\frac{48x^{10}}{y^8}}$$

The fraction in the radicand cannot be simplified

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{(2x^2)^4} \cdot \sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$$

Simplify.

$$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$$

TRY IT : : 8.45

Simplify: ⓐ $\sqrt{\frac{80m^3}{n^6}}$ ⓑ $\sqrt[3]{\frac{108c^{10}}{d^6}}$ ⓒ $\sqrt[4]{\frac{80x^{10}}{y^4}}$.

TRY IT : : 8.46

Simplify: ⓐ $\sqrt{\frac{54u^7}{v^8}}$ ⓑ $\sqrt[3]{\frac{40r^3}{s^6}}$ ⓒ $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$.

Be sure to simplify the fraction in the radicand first, if possible.

EXAMPLE 8.24

Simplify: ⓐ $\sqrt{\frac{18p^5q^7}{32pq^2}}$ ⓑ $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$.

Solution

ⓐ

	$\sqrt{\dfrac{18p^5q^7}{32pq^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt{\dfrac{9p^4q^5}{16}}$
Rewrite using the Quotient Property.	$\dfrac{\sqrt{9p^4q^5}}{\sqrt{16}}$
Simplify the radicals in the numerator and the denominator.	$\dfrac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$
Simplify.	$\dfrac{3p^2q^2\sqrt{q}}{4}$

ⓑ

	$\sqrt[3]{\dfrac{16x^5y^7}{54x^2y^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt[3]{\dfrac{8x^3y^5}{27}}$
Rewrite using the Quotient Property.	$\dfrac{\sqrt[3]{8x^3y^5}}{\sqrt[3]{27}}$
Simplify the radicals in the numerator and the denominator.	$\dfrac{\sqrt[3]{8x^3y^3}\cdot\sqrt[3]{y^2}}{\sqrt[3]{27}}$
Simplify.	$\dfrac{2xy\sqrt[3]{y^2}}{3}$

ⓒ

	$\sqrt[4]{\dfrac{5a^8b^6}{80a^3b^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt[4]{\dfrac{a^5b^4}{16}}$
Rewrite using the Quotient Property.	$\dfrac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$
Simplify the radicals in the numerator and the denominator.	$\dfrac{\sqrt[4]{a^4b^4}\cdot\sqrt[4]{a}}{\sqrt[4]{16}}$
Simplify.	$\dfrac{\lvert ab\rvert\sqrt[4]{a}}{2}$

TRY IT : : 8.47

Simplify: ⓐ $\sqrt{\dfrac{50x^5y^3}{72x^4y}}$ ⓑ $\sqrt[3]{\dfrac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\dfrac{5a^8b^6}{80a^3b^2}}$.

TRY IT : : 8.48

Simplify: ⓐ $\sqrt{\dfrac{48m^7n^2}{100m^5n^8}}$ ⓑ $\sqrt[3]{\dfrac{54x^7y^5}{250x^2y^2}}$ ⓒ $\sqrt[4]{\dfrac{32a^9b^7}{162a^3b^3}}$.

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the Quotient Property again, to combine them into one radical. We will then look to see if we can simplify the expression.

EXAMPLE 8.25

Simplify: ⓐ $\dfrac{\sqrt{48a^7}}{\sqrt{3a}}$ ⓑ $\dfrac{\sqrt[3]{-108}}{\sqrt[3]{2}}$ ⓒ $\dfrac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$.

Solution

ⓐ

	$\dfrac{\sqrt{48a^7}}{\sqrt{3a}}$
The denominator cannot be simplified, s use the Quotient Property to write as one radical.	$\sqrt{\dfrac{48a^7}{3a}}$
Simplify the fraction under the radical.	$\sqrt{16a^6}$
Simplify.	$4\lvert a^3\rvert$

ⓑ

	$\dfrac{\sqrt[3]{-108}}{\sqrt[3]{2}}$
The denominator cannot be simplified, s use the Quotient Property to write as one radical.	$\sqrt[3]{\dfrac{-108}{2}}$
Simplify the fraction under the radical.	$\sqrt[3]{-54}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{(-3)^3} \cdot \sqrt[3]{2}$
Simplify.	$-3\sqrt[3]{2}$

ⓒ

	$\dfrac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$
The denominator cannot be simplified, s use the Quotient Property to write as one radical.	$\sqrt[4]{\dfrac{96x^7}{3x^2}}$
Simplify the fraction under the radical.	$\sqrt[4]{32x^5}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{16x^4} \cdot \sqrt[4]{2x}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$
Simplify.	$2\lvert x\rvert\sqrt[4]{2x}$

TRY IT : : 8.49

Simplify: ⓐ $\dfrac{\sqrt{98z^5}}{\sqrt{2z}}$ ⓑ $\dfrac{\sqrt[3]{-500}}{\sqrt[3]{2}}$ ⓒ $\dfrac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$.

TRY IT : : 8.50

Simplify: ⓐ $\dfrac{\sqrt{128m^9}}{\sqrt{2m}}$ ⓑ $\dfrac{\sqrt[3]{-192}}{\sqrt[3]{3}}$ ⓒ $\dfrac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$.

MEDIA : :

Access these online resources for additional instruction and practice with simplifying radical expressions.

- Simplifying Square Root and Cube Root with Variables (https://openstax.org/l/37SimRtwithVar1)
- Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents (https://openstax.org/l/37SimRtwithVar2)
- Simplifying Cube Roots (https://openstax.org/l/37SimRtwithVar3)

8.2 EXERCISES

Practice Makes Perfect

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

55. $\sqrt{27}$

56. $\sqrt{80}$

57. $\sqrt{125}$

58. $\sqrt{96}$

59. $\sqrt{147}$

60. $\sqrt{450}$

61. $\sqrt{800}$

62. $\sqrt{675}$

63. ⓐ $\sqrt[4]{32}$ ⓑ $\sqrt[5]{64}$

64. ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

65. ⓐ $\sqrt[5]{64}$ ⓑ $\sqrt[3]{256}$

66. ⓐ $\sqrt[4]{3125}$ ⓑ $\sqrt[3]{81}$

In the following exercises, simplify using absolute value signs as needed.

67.
ⓐ $\sqrt{y^{11}}$
ⓑ $\sqrt[3]{r^5}$
ⓒ $\sqrt[4]{s^{10}}$

68.
ⓐ $\sqrt{m^{13}}$
ⓑ $\sqrt[5]{u^7}$
ⓒ $\sqrt[6]{v^{11}}$

69.
ⓐ $\sqrt{n^{21}}$
ⓑ $\sqrt[3]{q^8}$
ⓒ $\sqrt[8]{n^{10}}$

70.
ⓐ $\sqrt{r^{25}}$
ⓑ $\sqrt[5]{p^8}$
ⓒ $\sqrt[4]{m^5}$

71.
ⓐ $\sqrt{125r^{13}}$
ⓑ $\sqrt[3]{108x^5}$
ⓒ $\sqrt[4]{48y^6}$

72.
ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
ⓒ $\sqrt[6]{128b^7}$

73.
ⓐ $\sqrt{242m^{23}}$
ⓑ $\sqrt[4]{405m10}$
ⓒ $\sqrt[5]{160n^8}$

74.
ⓐ $\sqrt{175n^{13}}$
ⓑ $\sqrt[5]{512p^5}$
ⓒ $\sqrt[4]{324q^7}$

75.
ⓐ $\sqrt{147m^7n^{11}}$
ⓑ $\sqrt[3]{48x^6y^7}$
ⓒ $\sqrt[4]{32x^5y^4}$

76.
ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
ⓒ $\sqrt[4]{80x^8y^9}$

77.
ⓐ $\sqrt{192q^3r^7}$
ⓑ $\sqrt[3]{54m^9n^{10}}$
ⓒ $\sqrt[4]{81a^9b^8}$

78.
ⓐ $\sqrt{150m^9n^3}$
ⓑ $\sqrt[3]{81p^7q^8}$
ⓒ $\sqrt[4]{162c^{11}d^{12}}$

79.
ⓐ $\sqrt[3]{-864}$
ⓑ $\sqrt[4]{-256}$

80.
ⓐ $\sqrt[5]{-486}$
ⓑ $\sqrt[6]{-64}$

81.
ⓐ $\sqrt[5]{-32}$
ⓑ $\sqrt[8]{-1}$

82.
ⓐ $\sqrt[3]{-8}$
ⓑ $\sqrt[4]{-16}$

83.
ⓐ $5+\sqrt{12}$
ⓑ $\dfrac{10-\sqrt{24}}{2}$

84.
ⓐ $8+\sqrt{96}$
ⓑ $\dfrac{8-\sqrt{80}}{4}$

85.
ⓐ $1+\sqrt{45}$
ⓑ $\dfrac{3+\sqrt{90}}{3}$

86.
ⓐ $3+\sqrt{125}$
ⓑ $\dfrac{15+\sqrt{75}}{5}$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

87. ⓐ $\sqrt{\dfrac{45}{80}}$ ⓑ $\sqrt[3]{\dfrac{8}{27}}$ ⓒ $\sqrt[4]{\dfrac{1}{81}}$

88. ⓐ $\sqrt{\dfrac{72}{98}}$ ⓑ $\sqrt[3]{\dfrac{24}{81}}$ ⓒ $\sqrt[4]{\dfrac{6}{96}}$

89. ⓐ $\sqrt{\dfrac{100}{36}}$ ⓑ $\sqrt[3]{\dfrac{81}{375}}$ ⓒ $\sqrt[4]{\dfrac{1}{256}}$

90. ⓐ $\sqrt{\dfrac{121}{16}}$ ⓑ $\sqrt[3]{\dfrac{16}{250}}$ ⓒ $\sqrt[4]{\dfrac{32}{162}}$

91. ⓐ $\sqrt{\dfrac{x^{10}}{x^6}}$ ⓑ $\sqrt[3]{\dfrac{p^{11}}{p^2}}$ ⓒ $\sqrt[4]{\dfrac{q^{17}}{q^{13}}}$

92. ⓐ $\sqrt{\dfrac{p^{20}}{p^{10}}}$ ⓑ $\sqrt[5]{\dfrac{d^{12}}{d^7}}$ ⓒ $\sqrt[8]{\dfrac{m^{12}}{m^4}}$

93. ⓐ $\sqrt{\dfrac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\dfrac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\dfrac{v^{30}}{v^{12}}}$

94. ⓐ $\sqrt{\dfrac{q^8}{q^{14}}}$ ⓑ $\sqrt[3]{\dfrac{r^{14}}{r^5}}$ ⓒ $\sqrt[4]{\dfrac{c^{21}}{c^9}}$

95. $\sqrt{\dfrac{96x^7}{121}}$

96. $\sqrt{\dfrac{108y^4}{49}}$

97. $\sqrt{\dfrac{300m^5}{64}}$

98. $\sqrt{\dfrac{125n^7}{169}}$

99. $\sqrt{\dfrac{98r^5}{100}}$

100. $\sqrt{\dfrac{180s^{10}}{144}}$

101. $\sqrt{\dfrac{28q^6}{225}}$

102. $\sqrt{\dfrac{150r^3}{256}}$

103.
ⓐ $\sqrt{\dfrac{75r^9}{s^8}}$
ⓑ $\sqrt[3]{\dfrac{54a^8}{b^3}}$
ⓒ $\sqrt[4]{\dfrac{64c^5}{d^4}}$

104.
ⓐ $\sqrt{\dfrac{72x^5}{y^6}}$
ⓑ $\sqrt[5]{\dfrac{96r^{11}}{s^5}}$
ⓒ $\sqrt[6]{\dfrac{128u^7}{v^{12}}}$

105.
ⓐ $\sqrt{\dfrac{28p^7}{q^2}}$
ⓑ $\sqrt[3]{\dfrac{81s^8}{t^3}}$
ⓒ $\sqrt[4]{\dfrac{64p^{15}}{q^{12}}}$

106.
ⓐ $\sqrt{\dfrac{45r^3}{s^{10}}}$
ⓑ $\sqrt[3]{\dfrac{625u^{10}}{v^3}}$
ⓒ $\sqrt[4]{\dfrac{729c^{21}}{d^8}}$

107.
ⓐ $\sqrt{\dfrac{32x^5y^3}{18x^3y}}$
ⓑ $\sqrt[3]{\dfrac{5x^6y^9}{40x^5y^3}}$
ⓒ $\sqrt[4]{\dfrac{5a^8b^6}{80a^3b^2}}$

108.
ⓐ $\sqrt{\dfrac{75r^6s^8}{48rs^4}}$
ⓑ $\sqrt[3]{\dfrac{24x^8y^4}{81x^2y}}$
ⓒ $\sqrt[4]{\dfrac{32m^9n^2}{162mn^2}}$

109.
ⓐ $\sqrt{\dfrac{27p^2q}{108p^4q^3}}$
ⓑ $\sqrt[3]{\dfrac{16c^5d^7}{250c^2d^2}}$
ⓒ $\sqrt[6]{\dfrac{2m^9n^7}{128m^3n}}$

110.
ⓐ $\sqrt{\dfrac{50r^5s^2}{128r^2s^6}}$
ⓑ $\sqrt[3]{\dfrac{24m^9n^7}{375m^4n}}$
ⓒ $\sqrt[4]{\dfrac{81m^2n^8}{256m^1n^2}}$

111.

ⓐ $\dfrac{\sqrt{45p^9}}{\sqrt{5q^2}}$

ⓑ $\dfrac{\sqrt[4]{64}}{\sqrt[4]{2}}$

ⓒ $\dfrac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

112.

ⓐ $\dfrac{\sqrt{80q^5}}{\sqrt{5q}}$

ⓑ $\dfrac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

ⓒ $\dfrac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

113.

ⓐ $\dfrac{\sqrt{50m^7}}{\sqrt{2m}}$

ⓑ $\sqrt[3]{\dfrac{1250}{2}}$

ⓒ $\sqrt[4]{\dfrac{486y^9}{2y^3}}$

114.

ⓐ $\dfrac{\sqrt{72n^{11}}}{\sqrt{2n}}$

ⓑ $\sqrt[3]{\dfrac{162}{6}}$

ⓒ $\sqrt[4]{\dfrac{160r^{10}}{5r^3}}$

Writing Exercises

115. Explain why $\sqrt{x^4} = x^2$. Then explain why $\sqrt{x^{16}} = x^8$.

116. Explain why $7 + \sqrt{9}$ is not equal to $\sqrt{7 + 9}$.

117. Explain how you know that $\sqrt[5]{x^{10}} = x^2$.

118. Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
use the Product Property to simplify radical expressions.			
use the Quotient Property to simplify radical expressions.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

8.3 Simplify Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15} + \frac{5}{12}$.
   If you missed this problem, review **Example 1.28**.
2. Simplify: $\left(4x^2 y^5\right)^3$.
   If you missed this problem, review **Example 5.18**.
3. Simplify: 5^{-3}.
   If you missed this problem, review **Example 5.14**.

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n = a^{m \cdot n}$ when *m* and *n* are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $(8^p)^3 = 8$. We will use the Power Property of Exponents to find the value of *p*.

	$(8^p)^3 = 8$
Multiply the exponents on the left.	$8^{3p} = 8$
Write the exponent 1 on the right.	$8^{3p} = 8^1$
Since the bases are the same, the exponents must be equal.	$3p = 1$
Solve for p.	$p = \frac{1}{3}$

So $\left(8^{\frac{1}{3}}\right)^3 = 8$. But we know also $\left(\sqrt[3]{8}\right)^3 = 8$. Then it must be that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This same logic can be used for any positive integer exponent *n* to show that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Rational Exponent $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

EXAMPLE 8.26

Write as a radical expression: ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$.

Solution

We want to write each expression in the form $\sqrt[n]{a}$.

ⓐ

	$x^{\frac{1}{2}}$
The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.	$\sqrt{x}$

ⓑ

	$y^{\frac{1}{3}}$
The denominator of the exponent is 3, so the index is 3.	$\sqrt[3]{y}$

ⓒ

	$z^{\frac{1}{4}}$
The denominator of the exponent is 4, so the index is 4.	$\sqrt[4]{z}$

TRY IT : : 8.51

Write as a radical expression: ⓐ $t^{\frac{1}{2}}$ ⓑ $m^{\frac{1}{3}}$ ⓒ $r^{\frac{1}{4}}$.

TRY IT : : 8.52

Write as a radial expression: ⓐ $b^{\frac{1}{6}}$ ⓑ $z^{\frac{1}{5}}$ ⓒ $p^{\frac{1}{4}}$.

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

EXAMPLE 8.27

Write with a rational exponent: ⓐ $\sqrt{5y}$ ⓑ $\sqrt[3]{4x}$ ⓒ $3\sqrt[4]{5z}$.

Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

	$\sqrt{5y}$
No index is shown, so it is 2. The denominator of the exponent will be 2. Put parentheses around the entire expression $5y$.	$(5y)^{\frac{1}{2}}$

ⓑ

	$\sqrt[3]{4x}$
The index is 3, so the denominator of the exponent is 3. Include parentheses $(4x)$.	$(4x)^{\frac{1}{3}}$

ⓒ

	$3\sqrt[4]{5z}$
The index is 4, so the denominator of the exponent is 4. Put parentheses only around the $5z$ since 3 is not under the radical sign.	$3(5z)^{\frac{1}{4}}$

TRY IT : : 8.53 Write with a rational exponent: ⓐ $\sqrt{10m}$ ⓑ $\sqrt[5]{3n}$ ⓒ $3\sqrt[4]{6y}$.

TRY IT : : 8.54 Write with a rational exponent: ⓐ $\sqrt[7]{3k}$ ⓑ $\sqrt[4]{5j}$ ⓒ $8\sqrt[3]{2a}$.

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

EXAMPLE 8.28

Simplify: ⓐ $25^{\frac{1}{2}}$ ⓑ $64^{\frac{1}{3}}$ ⓒ $256^{\frac{1}{4}}$.

Solution

ⓐ

	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5

ⓑ

	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

ⓒ

	$256^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{256}$
Recognize 256 is a perfect fourth power.	$\sqrt[4]{4^4}$
Simplify.	4

TRY IT : : 8.55 Simplify: ⓐ $36^{\frac{1}{2}}$ ⓑ $8^{\frac{1}{3}}$ ⓒ $16^{\frac{1}{4}}$.

TRY IT : : 8.56

Simplify: ⓐ $100^{\frac{1}{2}}$ ⓑ $27^{\frac{1}{3}}$ ⓒ $81^{\frac{1}{4}}$.

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

EXAMPLE 8.29

Simplify: ⓐ $(-16)^{\frac{1}{4}}$ ⓑ $-16^{\frac{1}{4}}$ ⓒ $(16)^{-\frac{1}{4}}$.

Solution

ⓐ

	$(-16)^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{-16}$
	$\sqrt[4]{(-2)^4}$
Simplify.	No real solution.

ⓑ

	$-16^{\frac{1}{4}}$
The exponent only applies to the 16. Rewrite as a fouth root.	$-\sqrt[4]{16}$
Rewrite 16 as 2^4.	$-\sqrt[4]{2^4}$
Simplify.	-2

ⓒ

	$(16)^{-\frac{1}{4}}$
Rewrite using the property $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{(16)^{\frac{1}{4}}}$
Rewrite as a fourth root.	$\frac{1}{\sqrt[4]{16}}$
Rewrite 16 as 2^4.	$\frac{1}{\sqrt[4]{2^4}}$
Simplify.	$\frac{1}{2}$

TRY IT : : 8.57

Simplify: ⓐ $(-64)^{-\frac{1}{2}}$ ⓑ $-64^{\frac{1}{2}}$ ⓒ $(64)^{-\frac{1}{2}}$.

TRY IT : : 8.58

Simplify: ⓐ $(-256)^{\frac{1}{4}}$ ⓑ $-256^{\frac{1}{4}}$ ⓒ $(256)^{-\frac{1}{4}}$.

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and

$\left(a^m\right)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the following definition.

Rational Exponent $a^{\frac{m}{n}}$

For any positive integers *m* and *n*,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

EXAMPLE 8.30

Write with a rational exponent: ⓐ $\sqrt{y^3}$ ⓑ $\left(\sqrt[3]{2x}\right)^4$ ⓒ $\sqrt{\left(\frac{3a}{4b}\right)^3}$.

Solution

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$.

ⓐ

	$\sqrt{y^3}$
The numerator of the exponent is the exponent, 3.	
The denominator of the exponent is the index of the radical, 2.	$y^{\frac{3}{2}}$

ⓑ

	$\left(\sqrt[3]{2x}\right)^4$
The numerator of the exponent is the exponent, 4.	
The denominator of the exponent is the index of the radical, 3.	$(2x)^{\frac{4}{3}}$

ⓒ

	$\sqrt{\left(\frac{3a}{4b}\right)^3}$
The numerator of the exponent is the exponent, 3.	
The denominator of the exponent is the index of the radical, 2.	$\left(\frac{3a}{4b}\right)^{\frac{3}{2}}$

TRY IT : : 8.59

Write with a rational exponent: ⓐ $\sqrt{x^5}$ ⓑ $\left(\sqrt[4]{3y}\right)^3$ ⓒ $\sqrt{\left(\frac{2m}{3n}\right)^5}$.

TRY IT : : 8.60

Write with a rational exponent: ⓐ $\sqrt[5]{a^2}$ ⓑ $\left(\sqrt[3]{5ab}\right)^5$ ⓒ $\sqrt{\left(\frac{7xy}{z}\right)^3}$.

Remember that $a^{-n} = \frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

EXAMPLE 8.31

Simplify: ⓐ $125^{\frac{2}{3}}$ ⓑ $16^{-\frac{3}{2}}$ ⓒ $32^{-\frac{2}{5}}$.

Solution

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

	$125^{\frac{2}{3}}$
The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.	$\left(\sqrt[3]{125}\right)^2$
Simplify.	$(5)^2$
	25

ⓑ We will rewrite each expression first using $a^{-n} = \frac{1}{a^n}$ and then change to radical form.

	$16^{-\frac{3}{2}}$
Rewrite using $a^{-n} = \frac{1}{a^n}$	$\frac{1}{16^{\frac{3}{2}}}$
Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.	$\frac{1}{(\sqrt{16})^3}$
Simplify.	$\frac{1}{4^3}$
	$\frac{1}{64}$

ⓒ

	$32^{-\frac{2}{5}}$
Rewrite using $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{32^{\frac{2}{5}}}$
Change to radical form.	$\frac{1}{\left(\sqrt[5]{32}\right)^2}$
Rewrite the radicand as a power.	$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$
Simplify.	$\frac{1}{2^2}$
	$\frac{1}{4}$

TRY IT : : 8.61

Simplify: ⓐ $27^{\frac{2}{3}}$ ⓑ $81^{-\frac{3}{2}}$ ⓒ $16^{-\frac{3}{4}}$.

TRY IT : : 8.62

Simplify: ⓐ $4^{\frac{3}{2}}$ ⓑ $27^{-\frac{2}{3}}$ ⓒ $625^{-\frac{3}{4}}$.

EXAMPLE 8.32

Simplify: ⓐ $-25^{\frac{3}{2}}$ ⓑ $-25^{-\frac{3}{2}}$ ⓒ $(-25)^{\frac{3}{2}}$.

Solution

ⓐ

	$-25^{\frac{3}{2}}$
Rewrite in radical form.	$-\left(\sqrt{25}\right)^3$
Simplify the radical.	$-(5)^3$
Simplify.	-125

ⓑ

	$-25^{-\frac{3}{2}}$
Rewrite using $a^{-n} = \frac{1}{a^n}$.	$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$
Rewrite in radical form.	$-\left(\frac{1}{(\sqrt{25})^3}\right)$
Simplify the radical.	$-\left(\frac{1}{(5)^3}\right)$
Simplify.	$-\frac{1}{125}$

ⓒ

	$(-25)^{\frac{3}{2}}$
Rewrite in radical form.	$(\sqrt{-25})^3$
There is no real number whose square root is -25.	Not a real number.

> **TRY IT : : 8.63** Simplify: ⓐ $-16^{\frac{3}{2}}$ ⓑ $-16^{-\frac{3}{2}}$ ⓒ $(-16)^{-\frac{3}{2}}$.

> **TRY IT : : 8.64** Simplify: ⓐ $-81^{\frac{3}{2}}$ ⓑ $-81^{-\frac{3}{2}}$ ⓒ $(-81)^{-\frac{3}{2}}$.

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponenets here to have them for reference as we simplify expressions.

Properties of Exponents

If *a* and *b* are real numbers and *m* and *n* are rational numbers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$
Quotient Property	$\frac{a^m}{a^n}$	$=$	$a^{m-n}, \ a \neq 0$
Zero Exponent Definitio	a^0	$=$	$1, \ a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$=$	$\frac{a^m}{b^m}, \ b \neq 0$
Negative Exponent Property	a^{-n}	$=$	$\frac{1}{a^n}, \ a \neq 0$

We will apply these properties in the next example.

EXAMPLE 8.33

Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$ ⓑ $\left(z^9\right)^{\frac{2}{3}}$ ⓒ $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$.

Solution

ⓐ The Product Property tells us that when we multiply the same base, we add the exponents.

	$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$
The bases are the same, so we add the exponents.	$x^{\frac{1}{2}+\frac{5}{6}}$
Add the fractions.	$x^{\frac{8}{6}}$
Simplify the exponent.	$x^{\frac{4}{3}}$

ⓑ The Power Property tells us that when we raise a power to a power, we multiply the exponents.

	$\left(z^9\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$z^{9 \cdot \frac{2}{3}}$
Simplify.	z^6

ⓒ The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

	$\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$
	$\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$
To divide with the same base, we subtract the exponents.	$\dfrac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$
Simplify.	$\dfrac{1}{x^{\frac{4}{3}}}$

TRY IT : : 8.65

Simplify: ⓐ $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$ ⓑ $\left(x^6\right)^{\frac{4}{3}}$ ⓒ $\dfrac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$.

TRY IT : : 8.66

Simplify: ⓐ $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$ ⓑ $\left(m^9\right)^{\frac{2}{9}}$ ⓒ $\dfrac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$.

Sometimes we need to use more than one property. In the next example, we will use both the Product to a Power Property and then the Power Property.

EXAMPLE 8.34

Simplify: ⓐ $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ ⓑ $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$.

Solution

ⓐ

	$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
Rewrite 27 as a power of 3.	$\left(3^3\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$\left(3^2\right)\left(u^{\frac{1}{3}}\right)$
Simplify.	$9u^{\frac{1}{3}}$

ⓑ

	$\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$
First we use the Product to a Power Property.	$\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$
To raise a power to a power, we multiply the exponents.	$mn^{\frac{3}{4}}$

TRY IT : : 8.67

Simplify: ⓐ $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ ⓑ $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$.

TRY IT : : 8.68

Simplify: ⓐ $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$ ⓑ $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$.

We will use both the Product Property and the Quotient Property in the next example.

EXAMPLE 8.35

Simplify: ⓐ $\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ ⓑ $\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$.

Solution

ⓐ

$$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$$

Use the Quotient Property, subtract the exponents.

$$x^{\frac{8}{4}}$$

Simplify.

$$x^2$$

ⓑ Follow the order of operations to simplify inside the parenthese first.

$$\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$$

Use the Quotient Property, subtract the exponents.

$$\left(\frac{16x^{\frac{6}{3}}}{y^{\frac{6}{6}}}\right)^{\frac{1}{2}}$$

Simplify.

$$\left(\frac{16x^2}{y}\right)^{\frac{1}{2}}$$

Use the Product to a Power Property, multiply the exponents.

$$\frac{4x}{y^{\frac{1}{2}}}$$

TRY IT : : 8.69

Simplify: ⓐ $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$ ⓑ $\left(\frac{25m^{\frac{1}{6}}n^{\frac{11}{6}}}{m^{\frac{2}{3}}n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$.

TRY IT : : 8.70

Simplify: ⓐ $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$ ⓑ $\left(\frac{27x^{\frac{4}{5}}y^{\frac{1}{6}}}{x^{\frac{1}{5}}y^{-\frac{5}{6}}}\right)^{\frac{1}{3}}$.

MEDIA : :

Access these online resources for additional instruction and practice with simplifying rational exponents.

- **Review-Rational Exponents (https://openstax.org/l/37RatExpont1)**
- **Using Laws of Exponents on Radicals: Properties of Rational Exponents (https://openstax.org/l/37RatExpont2)**

8.3 EXERCISES

Practice Makes Perfect

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

119. ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$

120. ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

121. ⓐ $u^{\frac{1}{5}}$ ⓑ $v^{\frac{1}{9}}$ ⓒ $w^{\frac{1}{20}}$

122. ⓐ $g^{\frac{1}{7}}$ ⓑ $h^{\frac{1}{5}}$ ⓒ $j^{\frac{1}{25}}$

In the following exercises, write with a rational exponent.

123. ⓐ $\sqrt[7]{x}$ ⓑ $\sqrt[9]{y}$ ⓒ $\sqrt[5]{f}$

124. ⓐ $\sqrt[8]{r}$ ⓑ $\sqrt[10]{s}$ ⓒ $\sqrt[4]{t}$

125. ⓐ $\sqrt[3]{7c}$ ⓑ $\sqrt[7]{12d}$ ⓒ $2\sqrt[4]{6b}$

126. ⓐ $\sqrt[4]{5x}$ ⓑ $\sqrt[8]{9y}$ ⓒ $7\sqrt[5]{3z}$

127. ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $4\sqrt[6]{36r}$

128. ⓐ $\sqrt[3]{25a}$ ⓑ $\sqrt{3b}$ ⓒ $\sqrt[8]{40c}$

In the following exercises, simplify.

129.
ⓐ $81^{\frac{1}{2}}$
ⓑ $125^{\frac{1}{3}}$
ⓒ $64^{\frac{1}{2}}$

130.
ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
ⓒ $32^{\frac{1}{5}}$

131.
ⓐ $16^{\frac{1}{4}}$
ⓑ $16^{\frac{1}{2}}$
ⓒ $625^{\frac{1}{4}}$

132.
ⓐ $64^{\frac{1}{3}}$
ⓑ $32^{\frac{1}{5}}$
ⓒ $81^{\frac{1}{4}}$

133.
ⓐ $(-216)^{\frac{1}{3}}$
ⓑ $-216^{\frac{1}{3}}$
ⓒ $(216)^{-\frac{1}{3}}$

134.
ⓐ $(-1000)^{\frac{1}{3}}$
ⓑ $-1000^{\frac{1}{3}}$
ⓒ $(1000)^{-\frac{1}{3}}$

135.
ⓐ $(-81)^{\frac{1}{4}}$
ⓑ $-81^{\frac{1}{4}}$
ⓒ $(81)^{-\frac{1}{4}}$

136.
ⓐ $(-49)^{\frac{1}{2}}$
ⓑ $-49^{\frac{1}{2}}$
ⓒ $(49)^{-\frac{1}{2}}$

137.
ⓐ $(-36)^{\frac{1}{2}}$
ⓑ $-36^{\frac{1}{2}}$
ⓒ $(36)^{-\frac{1}{2}}$

138.
ⓐ $(-16)^{\frac{1}{4}}$
ⓑ $-16^{\frac{1}{4}}$
ⓒ $16^{-\frac{1}{4}}$

139.
ⓐ $(-100)^{\frac{1}{2}}$
ⓑ $-100^{\frac{1}{2}}$
ⓒ $(100)^{-\frac{1}{2}}$

140.
ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

141.
ⓐ $\sqrt{m^5}$
ⓑ $\left(\sqrt[3]{3y}\right)^7$
ⓒ $\sqrt[5]{\left(\frac{4x}{5y}\right)^3}$

142.
ⓐ $\sqrt[4]{r^7}$
ⓑ $\left(\sqrt[5]{2pq}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

143.
ⓐ $\sqrt[5]{u^2}$
ⓑ $\left(\sqrt[3]{6x}\right)^5$
ⓒ $\sqrt[4]{\left(\frac{18a}{5b}\right)^7}$

144.
ⓐ $\sqrt[3]{a}$
ⓑ $\left(\sqrt[4]{21v}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{2xy}{5z}\right)^2}$

In the following exercises, simplify.

145.
ⓐ $64^{\frac{5}{2}}$
ⓑ $81^{\frac{-3}{2}}$
ⓒ $(-27)^{\frac{2}{3}}$

146.
ⓐ $25^{\frac{3}{2}}$
ⓑ $9^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{2}{3}}$

147.
ⓐ $32^{\frac{2}{5}}$
ⓑ $27^{-\frac{2}{3}}$
ⓒ $(-25)^{\frac{1}{2}}$

148.
ⓐ $100^{\frac{3}{2}}$
ⓑ $49^{-\frac{5}{2}}$
ⓒ $(-100)^{\frac{3}{2}}$

149.
ⓐ $-9^{\frac{3}{2}}$
ⓑ $-9^{-\frac{3}{2}}$
ⓒ $(-9)^{\frac{3}{2}}$

150.
ⓐ $-64^{\frac{3}{2}}$
ⓑ $-64^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{3}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

151.
ⓐ $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$
ⓑ $\left(p^{12}\right)^{\frac{3}{4}}$
ⓒ $\frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

152.
ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$
ⓒ $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

153.
ⓐ $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$
ⓑ $\left(x^{12}\right)^{\frac{2}{3}}$
ⓒ $\frac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$

154.

ⓐ $q^{\frac{2}{3}} \cdot q^{\frac{5}{6}}$

ⓑ $\left(h^6\right)^{\frac{4}{3}}$

ⓒ $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

155.

ⓐ $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$

ⓑ $\left(a^{\frac{1}{3}} b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

156.

ⓐ $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$

ⓑ $\left(m^{\frac{4}{3}} n^{\frac{1}{2}}\right)^{\frac{3}{4}}$

157.

ⓐ $\left(16\,u^{\frac{1}{3}}\right)^{\frac{3}{4}}$

ⓑ $\left(4\,p^{\frac{1}{3}} q^{\frac{1}{2}}\right)^{\frac{3}{2}}$

158.

ⓐ $\left(625\,n^{\frac{8}{3}}\right)^{\frac{3}{4}}$

ⓑ $\left(9\,x^{\frac{2}{5}} y^{\frac{3}{5}}\right)^{\frac{5}{2}}$

159.

ⓐ $\dfrac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$

ⓑ $\left(\dfrac{36\,s^{\frac{1}{5}} t^{-\frac{3}{2}}}{s^{-\frac{9}{5}} t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$

160.

ⓐ $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

ⓑ $\left(\dfrac{27\,b^{\frac{2}{3}} c^{-\frac{5}{2}}}{b^{-\frac{7}{3}} c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

161.

ⓐ $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$

ⓑ $\left(\dfrac{8\,x^{\frac{5}{3}} y^{-\frac{1}{2}}}{27\,x^{-\frac{4}{3}} y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$

162.

ⓐ $\dfrac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

ⓑ $\left(\dfrac{16\,m^{\frac{1}{5}} n^{\frac{3}{2}}}{81\,m^{\frac{9}{5}} n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

Writing Exercises

163. Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

164. Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with $a^{\frac{1}{n}}$.			
simplify expressions with $a^{\frac{m}{n}}$.			
use the Laws of Exponents to simply expressions with rational exponents.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

8.4 Add, Subtract, and Multiply Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $3x^2 + 9x - 5 - \left(x^2 - 2x + 3\right)$.
   If you missed this problem, review Example 5.5.
2. Simplify: $(2 + a)(4 - a)$.
   If you missed this problem, review Example 5.28.
3. Simplify: $(9 - 5y)^2$.
   If you missed this problem, review Example 5.31.

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Like Radicals

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x + 8x$ is $11x$. Similarly we add $3\sqrt{x} + 8\sqrt{x}$ and the result is $11\sqrt{x}$.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

EXAMPLE 8.36

Simplify: ⓐ $2\sqrt{2} - 7\sqrt{2}$ ⓑ $5\sqrt[3]{y} + 4\sqrt[3]{y}$ ⓒ $7\sqrt[4]{x} - 2\sqrt[4]{y}$.

Solution

ⓐ

	$2\sqrt{2} - 7\sqrt{2}$
Since the radicals are like, we subtract the coefficient	$-5\sqrt{2}$

ⓑ

	$5\sqrt[3]{y} + 4\sqrt[3]{y}$
Since the radicals are like, we add the coefficient	$9\sqrt[3]{y}$

ⓒ

$$7\sqrt[4]{x} - 2\sqrt[4]{y}$$

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

TRY IT : : 8.71 Simplify: ⓐ $8\sqrt{2} - 9\sqrt{2}$ ⓑ $4\sqrt[3]{x} + 7\sqrt[3]{x}$ ⓒ $3\sqrt[4]{x} - 5\sqrt[4]{y}$.

TRY IT : : 8.72 Simplify: ⓐ $5\sqrt{3} - 9\sqrt{3}$ ⓑ $5\sqrt[3]{y} + 3\sqrt[3]{y}$ ⓒ $5\sqrt[4]{m} - 2\sqrt[3]{m}$.

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

EXAMPLE 8.37

Simplify: ⓐ $2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$ ⓑ $\sqrt[4]{3xy} + 5\sqrt[4]{3xy} - 4\sqrt[4]{3xy}$.

Solution

ⓐ

	$2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$
Since the radicals are like, we combine them.	$0\sqrt{5n}$
Simplify.	0

ⓑ

	$\sqrt[4]{3xy} + 5\sqrt[4]{3xy} - 4\sqrt[4]{3xy}$
Since the radicals are like, we combine them.	$2\sqrt[4]{3xy}$

TRY IT : : 8.73 Simplify: ⓐ $\sqrt{7x} - 7\sqrt{7x} + 4\sqrt{7x}$ ⓑ $4\sqrt[4]{5xy} + 2\sqrt[4]{5xy} - 7\sqrt[4]{5xy}$.

TRY IT : : 8.74 Simplify: ⓐ $4\sqrt{3y} - 7\sqrt{3y} + 2\sqrt{3y}$ ⓑ $6\sqrt[3]{7mn} + \sqrt[3]{7mn} - 4\sqrt[3]{7mn}$.

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

EXAMPLE 8.38

Simplify: ⓐ $\sqrt{20} + 3\sqrt{5}$ ⓑ $\sqrt[3]{24} - \sqrt[3]{375}$ ⓒ $\frac{1}{2}\sqrt[4]{48} - \frac{2}{3}\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt{20} + 3\sqrt{5}$
Simplify the radicals, when possible.	$\sqrt{4} \cdot \sqrt{5} + 3\sqrt{5}$
	$2\sqrt{5} + 3\sqrt{5}$
Combine the like radicals.	$5\sqrt{5}$

ⓑ

	$\sqrt[3]{24} - \sqrt[3]{375}$
Simplify the radicals.	$\sqrt[3]{8} \cdot \sqrt[3]{3} - \sqrt[3]{125} \cdot \sqrt[3]{3}$
	$2\sqrt[3]{3} - 5\sqrt[3]{3}$
Combine the like radicals.	$-3\sqrt[3]{3}$

ⓒ

	$\frac{1}{2}\sqrt[4]{48} - \frac{2}{3}\sqrt[4]{243}$
Simplify the radicals.	$\frac{1}{2}\sqrt[4]{16}\cdot\sqrt[4]{3} - \frac{2}{3}\sqrt[4]{81}\cdot\sqrt[4]{3}$
	$\frac{1}{2}\cdot 2\cdot\sqrt[4]{3} - \frac{2}{3}\cdot 3\cdot\sqrt[4]{3}$
	$\sqrt[4]{3} - 2\sqrt[4]{3}$
Combine the like radicals.	$-\sqrt[4]{3}$

TRY IT : : 8.75 Simplify: ⓐ $\sqrt{18} + 6\sqrt{2}$ ⓑ $6\sqrt[3]{16} - 2\sqrt[3]{250}$ ⓒ $\frac{2}{3}\sqrt[3]{81} - \frac{1}{2}\sqrt[3]{24}$.

TRY IT : : 8.76 Simplify: ⓐ $\sqrt{27} + 4\sqrt{3}$ ⓑ $4\sqrt[3]{5} - 7\sqrt[3]{40}$ ⓒ $\frac{1}{2}\sqrt[3]{128} - \frac{5}{3}\sqrt[3]{54}$.

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

EXAMPLE 8.39

Simplify: ⓐ $9\sqrt{50m^2} - 6\sqrt{48m^2}$ ⓑ $\sqrt[3]{54n^5} - \sqrt[3]{16n^5}$.

Solution

ⓐ

	$9\sqrt{50m^2} - 6\sqrt{48m^2}$
Simplify the radicals.	$9\sqrt{25m^2}\cdot\sqrt{2} - 6\sqrt{16m^2}\cdot\sqrt{3}$
	$9\cdot 5m\cdot\sqrt{2} - 6\cdot 4m\cdot\sqrt{3}$
	$45m\sqrt{2} - 24m\sqrt{3}$
The radicals are not like and so cannot be combined.	

ⓑ

	$\sqrt[3]{54n^5} - \sqrt[3]{16n^5}$
Simplify the radicals.	$\sqrt[3]{27n^3}\cdot\sqrt[3]{2n^2} - \sqrt[3]{8n^3}\cdot\sqrt[3]{2n^2}$
	$3n\sqrt[3]{2n^2} - 2n\sqrt[3]{2n^2}$
Combine the like radicals.	$n\sqrt[3]{2n^2}$

TRY IT : : 8.77 Simplify: ⓐ $\sqrt{32m^7} - \sqrt{50m^7}$ ⓑ $\sqrt[3]{135x^7} - \sqrt[3]{40x^7}$.

TRY IT : : 8.78 Simplify: ⓐ $\sqrt{27p^3} - \sqrt{48p^3}$ ⓑ $\sqrt[3]{256y^5} - \sqrt[3]{32n^5}$.

Multiply Radical Expressions

We have used the Product Property of Roots to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots ‘in reverse’ to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Product Property of Roots

For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is 12*xy*. Keep this in mind as you do these examples.

EXAMPLE 8.40

Simplify: ⓐ $(6\sqrt{2})(3\sqrt{10})$ ⓑ $\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$.

Solution

ⓐ

	$(6\sqrt{2})(3\sqrt{10})$
Multiply using the Product Property.	$18\sqrt{20}$
Simplify the radical.	$18\sqrt{4} \cdot \sqrt{5}$
Simplify.	$18 \cdot 2 \cdot \sqrt{5}$
	$36\sqrt{5}$

ⓑ

	$\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$
Multiply using the Product Property.	$20\sqrt[3]{24}$
Simplify the radical.	$20\sqrt[3]{8} \cdot \sqrt[3]{3}$
Simplify.	$20 \cdot 2 \cdot \sqrt[3]{3}$
	$40\sqrt[3]{3}$

TRY IT : : 8.79 Simplify: ⓐ $(3\sqrt{2})(2\sqrt{30})$ ⓑ $\left(2\sqrt[3]{18}\right)\left(-3\sqrt[3]{6}\right)$.

TRY IT : : 8.80 Simplify: ⓐ $(3\sqrt{3})(3\sqrt{6})$ ⓑ $\left(-4\sqrt[3]{9}\right)\left(3\sqrt[3]{6}\right)$.

We follow the same procedures when there are variables in the radicands.

EXAMPLE 8.41

Simplify: ⓐ $\left(10\sqrt{6p^3}\right)(4\sqrt{3p})$ ⓑ $\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$.

Solution

ⓐ

	$\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$
Multiply.	$40\sqrt{18p^4}$
Simplify the radical.	$40\sqrt{9p^4}\cdot\sqrt{2}$
Simplify.	$40\cdot 3p^2\cdot\sqrt{3}$
	$120p^2\sqrt{3}$

ⓑ When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

	$\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$
Multiply.	$6\sqrt[4]{4\cdot 5\cdot 4\cdot 7y^5}$
Simplify the radical.	$6\sqrt[4]{16y^4}\cdot\sqrt[4]{35y}$
Simplify.	$6\cdot 2y\sqrt[4]{35y}$
Multiply.	$12y\sqrt[4]{35y}$

TRY IT : : 8.81 Simplify: ⓐ $\left(6\sqrt{6x^2}\right)\left(8\sqrt{30x^4}\right)$ ⓑ $\left(-4\sqrt[4]{12y^3}\right)\left(-\sqrt[4]{8y^3}\right)$.

TRY IT : : 8.82 Simplify: ⓐ $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$ ⓑ $\left(-4\sqrt[4]{9a^3}\right)\left(3\sqrt[4]{27a^2}\right)$.

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the Distributive Property to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

EXAMPLE 8.42

Simplify: ⓐ $\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$ ⓑ $\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$.

Solution

ⓐ

	$\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$
Multiply.	$\sqrt{12}+\sqrt{108}$
Simplify.	$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$
Simplify.	$2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$8\sqrt{3}$

ⓑ

	$\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$
Distribute.	$5\sqrt[3]{9}-\sqrt[3]{162}$
Simplify.	$5\sqrt[3]{9}-\sqrt[3]{27}\cdot\sqrt[3]{6}$
Simplify.	$5\sqrt[3]{9}-3\sqrt[3]{6}$

TRY IT : : 8.83 Simplify: ⓐ $\sqrt{6}(1+3\sqrt{6})$ ⓑ $\sqrt[3]{4}\left(-2-\sqrt[3]{6}\right)$.

TRY IT : : 8.84 Simplify: ⓐ $\sqrt{8}(2-5\sqrt{8})$ ⓑ $\sqrt[3]{3}\left(-\sqrt[3]{9}-\sqrt[3]{6}\right)$.

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

EXAMPLE 8.43

Simplify: ⓐ $(3-2\sqrt{7})(4-2\sqrt{7})$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$.

Solution

ⓐ

	$(3-2\sqrt{7})(4-2\sqrt{7})$
Multiply	$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$
Simplify.	$12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$40-14\sqrt{7}$

ⓑ

	$\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$
Multiply.	$\sqrt[3]{x^2}+4\sqrt[3]{x}-2\sqrt[3]{x}-8$
Combine like terms.	$\sqrt[3]{x^2}+2\sqrt[3]{x}-8$

TRY IT : : 8.85 Simplify: ⓐ $(6-3\sqrt{7})(3+4\sqrt{7})$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}-3\right)$.

TRY IT : : 8.86 Simplify: ⓐ $(2-3\sqrt{11})(4-\sqrt{11})$ ⓑ $\left(\sqrt[3]{x}+1\right)\left(\sqrt[3]{x}+3\right)$.

EXAMPLE 8.44

Simplify: $\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$.

Solution

	$\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$
Multiply.	$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$
Simplify.	$6+12\sqrt{10}-\sqrt{10}-20$
Combine like terms.	$-14+11\sqrt{10}$

TRY IT : : 8.87 Simplify: $\left(5\sqrt{3}-\sqrt{7}\right)\left(\sqrt{3}+2\sqrt{7}\right)$

TRY IT : : 8.88 Simplify: $\left(\sqrt{6}-3\sqrt{8}\right)\left(2\sqrt{6}+\sqrt{8}\right)$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Special Products

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a+b)(a-b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

We will use the special product formulas in the next few examples. We will start with the Product of Binomial Squares Pattern.

EXAMPLE 8.45

Simplify: ⓐ $\left(2+\sqrt{3}\right)^2$ ⓑ $\left(4-2\sqrt{5}\right)^2$.

Solution

Be sure to include the $2ab$ term when squaring a binomial.

ⓐ

	$\overset{(a\ +\ b)^2}{\left(2+\sqrt{3}\right)^2}$
Multiply, using the Product of Binomial Squares Pattern.	$\overset{a^2+2\ \ a\ \ \ b\ +\ b^2}{2^2+2\cdot 2\cdot\sqrt{3}+\left(\sqrt{3}\right)^2}$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

ⓑ

	$\overset{(a\ -\ b)^2}{\left(4-2\sqrt{5}\right)^2}$
Multiply, using the Product of Binomial Squares Pattern.	$\overset{a^2-2\ \ a\ \ \ b\ +\ b^2}{4^2-2\cdot 4\cdot 2\sqrt{5}+\left(2\sqrt{5}\right)^2}$
Simplify.	$16-16\sqrt{5}+4\cdot 5$
	$16-16\sqrt{5}+20$
Combine like terms.	$36-16\sqrt{5}$

TRY IT : : 8.89 Simplify: ⓐ $\left(10+\sqrt{2}\right)^2$ ⓑ $\left(1+3\sqrt{6}\right)^2$.

TRY IT : : 8.90 Simplify: ⓐ $\left(6-\sqrt{5}\right)^2$ ⓑ $\left(9-2\sqrt{10}\right)^2$.

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

EXAMPLE 8.46

Simplify: $\left(5-2\sqrt{3}\right)\left(5+2\sqrt{3}\right)$.

Solution

	$\overset{(a-\ \ b)\ (a+\ \ b)}{\left(5-2\sqrt{3}\right)\left(5+2\sqrt{3}\right)}$
Multiply, using the Product of Conjugates Pattern.	$\overset{a^2-b^2}{5^2-\left(2\sqrt{3}\right)^2}$
Simplify.	$25-4\cdot 3$
	13

TRY IT : : 8.91 Simplify: $\left(3-2\sqrt{5}\right)\left(3+2\sqrt{5}\right)$

TRY IT : : 8.92 Simplify: $\left(4+5\sqrt{7}\right)\left(4-5\sqrt{7}\right)$.

MEDIA : :

Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- **Multiplying Adding Subtracting Radicals (https://openstax.org/l/37Radicals1)**
- **Multiplying Special Products: Square Binomials Containing Square Roots (https://openstax.org/l/37Radicals2)**
- **Multiplying Conjugates (https://openstax.org/l/37Radicals3)**

8.4 EXERCISES

Practice Makes Perfect

Add and Subtract Radical Expressions

In the following exercises, simplify.

165.
ⓐ $8\sqrt{2} - 5\sqrt{2}$
ⓑ $5\sqrt[3]{m} + 2\sqrt[3]{m}$
ⓒ $8\sqrt[4]{m} - 2\sqrt[4]{n}$

166.
ⓐ $7\sqrt{2} - 3\sqrt{2}$
ⓑ $7\sqrt[3]{p} + 2\sqrt[3]{p}$
ⓒ $5\sqrt[3]{x} - 3\sqrt[3]{x}$

167.
ⓐ $3\sqrt{5} + 6\sqrt{5}$
ⓑ $9\sqrt[3]{a} + 3\sqrt[3]{a}$
ⓒ $5\sqrt[4]{2z} + \sqrt[4]{2z}$

168.
ⓐ $4\sqrt{5} + 8\sqrt{5}$
ⓑ $\sqrt[3]{m} - 4\sqrt[3]{m}$
ⓒ $\sqrt{n} + 3\sqrt{n}$

169.
ⓐ $3\sqrt{2a} - 4\sqrt{2a} + 5\sqrt{2a}$
ⓑ $5\sqrt[4]{3ab} - 3\sqrt[4]{3ab} - 2\sqrt[4]{3ab}$

170.
ⓐ $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$
ⓑ $8\sqrt[4]{11cd} + 5\sqrt[4]{11cd} - 9\sqrt[4]{11cd}$

171.
ⓐ $8\sqrt{3c} + 2\sqrt{3c} - 9\sqrt{3c}$
ⓑ $2\sqrt[3]{4pq} - 5\sqrt[3]{4pq} + 4\sqrt[3]{4pq}$

172.
ⓐ $3\sqrt{5d} + 8\sqrt{5d} - 11\sqrt{5d}$
ⓑ $11\sqrt[3]{2rs} - 9\sqrt[3]{2rs} + 3\sqrt[3]{2rs}$

173.
ⓐ $\sqrt{27} - \sqrt{75}$
ⓑ $\sqrt[3]{40} - \sqrt[3]{320}$
ⓒ $\frac{1}{2}\sqrt[4]{32} + \frac{2}{3}\sqrt[4]{162}$

174.
ⓐ $\sqrt{72} - \sqrt{98}$
ⓑ $\sqrt[3]{24} + \sqrt[3]{81}$
ⓒ $\frac{1}{2}\sqrt[4]{80} - \frac{2}{3}\sqrt[4]{405}$

175.
ⓐ $\sqrt{48} + \sqrt{27}$
ⓑ $\sqrt[3]{54} + \sqrt[3]{128}$
ⓒ $6\sqrt[4]{5} - \frac{3}{2}\sqrt[4]{320}$

176.
ⓐ $\sqrt{45} + \sqrt{80}$
ⓑ $\sqrt[3]{81} - \sqrt[3]{192}$
ⓒ $\frac{5}{2}\sqrt[4]{80} + \frac{7}{3}\sqrt[4]{405}$

177.
ⓐ $\sqrt{72a^5} - \sqrt{50a^5}$
ⓑ $9\sqrt[4]{80p^4} - 6\sqrt[4]{405p^4}$

178.
ⓐ $\sqrt{48b^5} - \sqrt{75b^5}$
ⓑ $8\sqrt[3]{64q^6} - 3\sqrt[3]{125q^6}$

179.
ⓐ $\sqrt{80c^7} - \sqrt{20c^7}$
ⓑ $2\sqrt[4]{162r^{10}} + 4\sqrt[4]{32r^{10}}$

180.
ⓐ $\sqrt{96d^9} - \sqrt{24d^9}$
ⓑ $5\sqrt[4]{243s^6} + 2\sqrt[4]{3s^6}$

181.
$3\sqrt{128y^2} + 4y\sqrt{162} - 8\sqrt{98y^2}$

182. $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

Multiply Radical Expressions

In the following exercises, simplify.

183.
ⓐ $(-2\sqrt{3})(3\sqrt{18})$
ⓑ $\left(8\sqrt[3]{4}\right)\left(-4\sqrt[3]{18}\right)$

184.
ⓐ $(-4\sqrt{5})(5\sqrt{10})$
ⓑ $\left(-2\sqrt[3]{9}\right)\left(7\sqrt[3]{9}\right)$

185.
ⓐ $(5\sqrt{6})(-\sqrt{12})$
ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

186.
ⓐ $(-2\sqrt{7})(-2\sqrt{14})$
ⓑ $\left(-3\sqrt[4]{8}\right)\left(-5\sqrt[4]{6}\right)$

187.
ⓐ $\left(4\sqrt{12z^3}\right)\left(3\sqrt{9z}\right)$
ⓑ $\left(5\sqrt[3]{3x^3}\right)\left(3\sqrt[3]{18x^3}\right)$

188.
ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$
ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

189.
ⓐ $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$
ⓑ $\left(2\sqrt[4]{8y^2}\right)\left(-2\sqrt[4]{12y^3}\right)$

190.
ⓐ $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$
ⓑ $\left(-\sqrt[4]{6b^3}\right)\left(3\sqrt[4]{8b^3}\right)$

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

191.
ⓐ $\sqrt{7}\left(5+2\sqrt{7}\right)$
ⓑ $\sqrt[3]{6}\left(4+\sqrt[3]{18}\right)$

192.
ⓐ $\sqrt{11}\left(8+4\sqrt{11}\right)$
ⓑ $\sqrt[3]{3}\left(\sqrt[3]{9}+\sqrt[3]{18}\right)$

193.
ⓐ $\sqrt{11}\left(-3+4\sqrt{11}\right)$
ⓑ $\sqrt[4]{3}\left(\sqrt[4]{54}+\sqrt[4]{18}\right)$

194.
ⓐ $\sqrt{2}\left(-5+9\sqrt{2}\right)$
ⓑ $\sqrt[4]{2}\left(\sqrt[4]{12}+\sqrt[4]{24}\right)$

195. $\left(7+\sqrt{3}\right)\left(9-\sqrt{3}\right)$

196. $\left(8-\sqrt{2}\right)\left(3+\sqrt{2}\right)$

197.
ⓐ $\left(9-3\sqrt{2}\right)\left(6+4\sqrt{2}\right)$
ⓑ $\left(\sqrt[3]{x}-3\right)\left(\sqrt[3]{x}+1\right)$

198.
ⓐ $(3-2\sqrt{7})(5-4\sqrt{7})$
ⓑ $\left(\sqrt[3]{x}-5\right)\left(\sqrt[3]{x}-3\right)$

199.
ⓐ $\left(1+3\sqrt{10}\right)\left(5-2\sqrt{10}\right)$
ⓑ $\left(2\sqrt[3]{x}+6\right)\left(\sqrt[3]{x}+1\right)$

200.
ⓐ $\left(7-2\sqrt{5}\right)\left(4+9\sqrt{5}\right)$
ⓑ $\left(3\sqrt[3]{x}+2\right)\left(\sqrt[3]{x}-2\right)$

201. $\left(\sqrt{3}+\sqrt{10}\right)\left(\sqrt{3}+2\sqrt{10}\right)$

202. $\left(\sqrt{11}+\sqrt{5}\right)\left(\sqrt{11}+6\sqrt{5}\right)$

203. $\left(2\sqrt{7}-5\sqrt{11}\right)\left(4\sqrt{7}+9\sqrt{11}\right)$

204. $\left(4\sqrt{6}+7\sqrt{13}\right)\left(8\sqrt{6}-3\sqrt{13}\right)$

205. ⓐ $\left(3+\sqrt{5}\right)^2$ ⓑ $\left(2-5\sqrt{3}\right)^2$

206. ⓐ $\left(4+\sqrt{11}\right)^2$ ⓑ $\left(3-2\sqrt{5}\right)^2$

207. ⓐ $\left(9-\sqrt{6}\right)^2$ ⓑ $\left(10+3\sqrt{7}\right)^2$

208. ⓐ $\left(5-\sqrt{10}\right)^2$ ⓑ $\left(8+3\sqrt{2}\right)^2$

209. $\left(4+\sqrt{2}\right)\left(4-\sqrt{2}\right)$

210. $\left(7+\sqrt{10}\right)\left(7-\sqrt{10}\right)$

211. $\left(4+9\sqrt{3}\right)\left(4-9\sqrt{3}\right)$

212. $\left(1+8\sqrt{2}\right)\left(1-8\sqrt{2}\right)$

213. $\left(12-5\sqrt{5}\right)\left(12+5\sqrt{5}\right)$

214. $\left(9-4\sqrt{3}\right)\left(9+4\sqrt{3}\right)$

215. $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

216. $\left(\sqrt[3]{4x}+3\right)\left(\sqrt[3]{4x}-3\right)$

Mixed Practice

217. $\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$

218. $\sqrt{175k^4}-\sqrt{63k^4}$

219. $\frac{5}{6}\sqrt{162}+\frac{3}{16}\sqrt{128}$

220. $\sqrt[3]{24} + \sqrt[3]{/81}$

221. $\frac{1}{2}\sqrt[4]{80} - \frac{2}{3}\sqrt[4]{405}$

222. $8\sqrt[4]{13} - 4\sqrt[4]{13} - 3\sqrt[4]{13}$

223. $5\sqrt{12c^4} - 3\sqrt{27c^6}$

224. $\sqrt{80a^5} - \sqrt{45a^5}$

225. $\frac{3}{5}\sqrt{75} - \frac{1}{4}\sqrt{48}$

226. $21\sqrt[3]{9} - 2\sqrt[3]{9}$

227. $8\sqrt[3]{64q^6} - 3\sqrt[3]{125q^6}$

228. $11\sqrt{11} - 10\sqrt{11}$

229. $\sqrt{3} \cdot \sqrt{21}$

230. $(4\sqrt{6})(-\sqrt{18})$

231. $\left(7\sqrt[3]{4}\right)\left(-3\sqrt[3]{18}\right)$

232. $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$

233. $(\sqrt{29})^2$

234. $(-4\sqrt{17})(-3\sqrt{17})$

235. $(-4 + \sqrt{17})(-3 + \sqrt{17})$

236. $\left(3\sqrt[4]{8a^2}\right)\left(\sqrt[4]{12a^3}\right)$

237. $(6 - 3\sqrt{2})^2$

238. $\sqrt{3}(4 - 3\sqrt{3})$

239. $\sqrt[3]{3}\left(2\sqrt[3]{9} + \sqrt[3]{18}\right)$

240. $(\sqrt{6} + \sqrt{3})(\sqrt{6} + 6\sqrt{3})$

Writing Exercises

241. Explain the when a radical expression is in simplest form.

242. Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

243.
ⓐ Explain why $(-\sqrt{n})^2$ is always non-negative, for $n \geq 0$.
ⓑ Explain why $-(\sqrt{n})^2$ is always non-positive, for $n \geq 0$.

244. Use the binomial square pattern to simplify $(3 + \sqrt{2})^2$. Explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
add and subtract radical expressions.			
multiply radical expressions.			
use polynomial multiplication to multiply radical expressions.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

8.5 Divide Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{30}{48}$.
   If you missed this problem, review **Example 1.24**.
2. Simplify: $x^2 \cdot x^4$.
   If you missed this problem, review **Example 5.12**.
3. Multiply: $(7+3x)(7-3x)$.
   If you missed this problem, review **Example 5.32**.

Divide Radical Expressions

We have used the Quotient Property of Radical Expressions to simplify roots of fractions. We will need to use this property ‘in reverse’ to simplify a fraction with radicals.

We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

EXAMPLE 8.47

Simplify: ⓐ $\frac{\sqrt{72x^3}}{\sqrt{162x}}$ ⓑ $\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$.

Solution

ⓐ

	$\dfrac{\sqrt{72x^3}}{\sqrt{162x}}$
Rewrite using the quotient property, $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.	$\sqrt{\dfrac{72x^3}{162x}}$
Remove common factors.	$\sqrt{\dfrac{\cancel{18} \cdot 4 \cdot x^2 \cdot \cancel{x}}{\cancel{18} \cdot 9 \cdot \cancel{x}}}$
Simplify.	$\sqrt{\dfrac{4x^2}{9}}$
Simplify the radical.	$\dfrac{2x}{3}$

ⓑ

	$\dfrac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$
Rewrite using the quotient property, $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.	$\sqrt[3]{\dfrac{32x^2}{4x^5}}$
Simplify the fraction under the radical.	$\sqrt[3]{\dfrac{8}{x^3}}$
Simplify the radical.	$\dfrac{2}{x}$

TRY IT : : 8.93

Simplify: ⓐ $\dfrac{\sqrt{50s^3}}{\sqrt{128s}}$ ⓑ $\dfrac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$.

TRY IT : : 8.94

Simplify: ⓐ $\dfrac{\sqrt{75q^5}}{\sqrt{108q}}$ ⓑ $\dfrac{\sqrt[3]{72b^2}}{\sqrt[3]{9b^5}}$.

EXAMPLE 8.48

Simplify: ⓐ $\dfrac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$ ⓑ $\dfrac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$.

Solution

ⓐ

	$\dfrac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$
Rewrite using the quotient property.	$\sqrt{\dfrac{147ab^8}{3a^3b^4}}$
Remove common factors in the fraction.	$\sqrt{\dfrac{49b^4}{a^2}}$
Simplify the radical.	$\dfrac{7b^2}{a}$

ⓑ

	$\dfrac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$
Rewrite using the quotient property.	$\sqrt[3]{\dfrac{-250mn^{-2}}{2m^{-2}n^4}}$
Simplify the fraction under the radical.	$\sqrt[3]{\dfrac{-125m^3}{n^6}}$
Simplify the radical.	$-\dfrac{5m}{n^2}$

TRY IT : : 8.95

Simplify: ⓐ $\dfrac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$ ⓑ $\dfrac{\sqrt[3]{-128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$.

TRY IT : : 8.96

Simplify: ⓐ $\dfrac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$ ⓑ $\dfrac{\sqrt[3]{-81pq^{-1}}}{\sqrt[3]{3p^{-2}q^5}}$.

EXAMPLE 8.49

Simplify: $\dfrac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$.

Solution

	$\dfrac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$
Rewrite using the quotient property.	$\sqrt{\dfrac{54x^5y^3}{3x^2y}}$
Remove common factors in the fraction.	$\sqrt{18x^3y^2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9x^2y^2 \cdot 2x}$
Rewrite the radical as the product of two radicals.	$\sqrt{9x^2y^2} \cdot \sqrt{2x}$
Simplify.	$3xy\sqrt{2x}$

TRY IT : : 8.97

Simplify: $\dfrac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$.

TRY IT : : 8.98

Simplify: $\dfrac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$.

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator.

This process is still used today, and is useful in other areas of mathematics, too.

Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a radical expression is not considered simplified if the radicand contains a fraction.

Simplified Radical Expressions

A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

EXAMPLE 8.50

Simplify: ⓐ $\frac{4}{\sqrt{3}}$ ⓑ $\sqrt{\frac{3}{20}}$ ⓒ $\frac{3}{\sqrt{6x}}$.

Solution

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt{\frac{3}{20}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{3}}{\sqrt{20}}$
Simplify the denominator.	$\frac{\sqrt{3}}{2\sqrt{5}}$
Multiply the numerator and denominator by $\sqrt{5}$.	$\frac{\sqrt{3} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}}$

Simplify.	$\dfrac{\sqrt{15}}{2 \cdot 5}$
Simplify.	$\dfrac{\sqrt{15}}{10}$

ⓒ

	$\dfrac{3}{\sqrt{6x}}$
Multiply the numerator and denominator by $\sqrt{6x}$.	$\dfrac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$
Simplify.	$\dfrac{3\sqrt{6x}}{6x}$
Simplify.	$\dfrac{\sqrt{6x}}{2x}$

TRY IT : : 8.99 Simplify: ⓐ $\dfrac{5}{\sqrt{3}}$ ⓑ $\sqrt{\dfrac{3}{32}}$ ⓒ $\dfrac{2}{\sqrt{2x}}$.

TRY IT : : 8.100 Simplify: ⓐ $\dfrac{6}{\sqrt{5}}$ ⓑ $\sqrt{\dfrac{7}{18}}$ ⓒ $\dfrac{5}{\sqrt{5x}}$.

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,

	$\dfrac{1}{\sqrt[3]{2}}$	$\dfrac{1}{\sqrt[4]{5}}$	1 power of 5, need 3 more to get a perfect fourth
Multiply numerator and denominator by a radical to get a perfect power.	$\dfrac{1 \cdot \sqrt[3]{2^2}}{\sqrt[3]{2} \cdot \sqrt[3]{2^2}}$	$\dfrac{1 \cdot \sqrt[4]{5^3}}{\sqrt[4]{5} \cdot \sqrt[4]{5^3}}$	
1 power of 2, need 2 more to get a perfect cube	$\dfrac{\sqrt[3]{4}}{\sqrt[3]{2^3}}$	$\dfrac{\sqrt[4]{5^3}}{\sqrt[4]{5^4}}$	
Simplify the denominator.	$\dfrac{\sqrt[3]{4}}{2}$	$\dfrac{\sqrt[4]{5^3}}{5}$	

We will use this technique in the next examples.

EXAMPLE 8.51

Simplify ⓐ $\dfrac{1}{\sqrt[3]{6}}$ ⓑ $\sqrt[3]{\dfrac{7}{24}}$ ⓒ $\dfrac{3}{\sqrt[3]{4x}}$.

Solution

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\dfrac{1}{\sqrt[3]{6}}$
The radical in the denominator has one factor of 6. Multiply both the numerator and denominator by $\sqrt[3]{6^2}$, which gives us 2 more factors of 6.	$\dfrac{1 \cdot \sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \sqrt[3]{6^2}}$
Multiply. Notice the radicand in the denominator has 3 powers of 6.	$\dfrac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$
Simplify the cube root in the denominator.	$\dfrac{\sqrt[3]{36}}{6}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt[3]{\dfrac{7}{24}}$
The fraction is not a perfect cube, so rewrite using the Quotient Property.	$\dfrac{\sqrt[3]{7}}{\sqrt[3]{24}}$
Simplify the denominator.	$\dfrac{\sqrt[3]{7}}{2\sqrt[3]{3}}$
Multiply the numerator and denominator by $\sqrt[3]{3^2}$. This will give us 3 factors of 3.	$\dfrac{\sqrt[3]{7} \cdot \sqrt[3]{3^2}}{2\sqrt[3]{3} \cdot \sqrt[3]{3^2}}$
Simplify.	$\dfrac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$
Remember, $\sqrt[3]{3^3} = 3$.	$\dfrac{\sqrt[3]{63}}{2 \cdot 3}$
Simplify.	$\dfrac{\sqrt[3]{63}}{6}$

ⓒ

	$\dfrac{3}{\sqrt[3]{4x}}$
Rewrite the radicand to show the factors.	$\dfrac{3}{\sqrt[3]{2^2 \cdot x}}$
Multiply the numerator and denominator by $\sqrt[3]{2 \cdot x^2}$. This will get us 3 factors of 2 and 3 factors of x.	$\dfrac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$
Simplify.	$\dfrac{3\sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$
Simplify the radical in the denominator.	$\dfrac{3\sqrt[3]{2x^2}}{2x}$

> **TRY IT : : 8.101**
>
> Simplify: ⓐ $\dfrac{1}{\sqrt[3]{7}}$ ⓑ $\sqrt[3]{\dfrac{5}{12}}$ ⓒ $\dfrac{5}{\sqrt[3]{9y}}$.

> **TRY IT : : 8.102**
>
> Simplify: ⓐ $\dfrac{1}{\sqrt[3]{2}}$ ⓑ $\sqrt[3]{\dfrac{3}{20}}$ ⓒ $\dfrac{2}{\sqrt[3]{25n}}$.

EXAMPLE 8.52

Simplify: ⓐ $\dfrac{1}{\sqrt[4]{2}}$ ⓑ $\sqrt[4]{\dfrac{5}{64}}$ ⓒ $\dfrac{2}{\sqrt[4]{8x}}$.

Solution

To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\dfrac{1}{\sqrt[4]{2}}$
The radical in the denominator has one factor of 2. Multiply both the numerator and denominator by $\sqrt[4]{2^3}$, which gives us 3 more factors of 2.	$\dfrac{1 \cdot \sqrt[4]{2^3}}{\sqrt[4]{2} \cdot \sqrt[4]{2^3}}$
Multiply. Notice the radicand in the denominator has 4 powers of 2.	$\dfrac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$
Simplify the fourth root in the denominator.	$\dfrac{\sqrt[4]{8}}{2}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt[4]{\dfrac{5}{64}}$

The fraction is not a perfect fourth power, so rewrite using the Quotient Property.	$\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$
Rewrite the radicand in the denominator to show the factors.	$\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$
Simplify the denominator.	$\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$
Multiply the numerator and denominator by $\sqrt[4]{2^2}$. This will give us 4 factors of 2.	$\frac{\sqrt[4]{5}\cdot\sqrt[4]{2^2}}{2\sqrt[4]{2^2}\cdot\sqrt[4]{2^2}}$
Simplify.	$\frac{\sqrt[4]{5}\cdot\sqrt[4]{4}}{2\sqrt[4]{2^4}}$
Remember, $\sqrt[4]{2^4}=2$.	$\frac{\sqrt[4]{20}}{2\cdot 2}$
Simplify.	$\frac{\sqrt[4]{20}}{4}$

ⓒ

	$\frac{2}{\sqrt[4]{8x}}$
Rewrite the radicand to show the factors.	$\frac{2}{\sqrt[4]{2^3\cdot x}}$
Multiply the numerator and denominator by $\sqrt[4]{2\cdot x^3}$. This will get us 4 factors of 2 and 4 factors of *x*.	$\frac{2\cdot\sqrt[4]{2\cdot x^3}}{\sqrt[4]{2^3x}\cdot\sqrt[4]{2\cdot x^3}}$
Simplify.	$\frac{2\sqrt[4]{2x^3}}{\sqrt[4]{2^4x^4}}$
Simplify the radical in the denominator.	$\frac{2\sqrt[4]{2x^3}}{2^4x^4}$
Simplify the fraction.	$\frac{\sqrt[4]{2x^3}}{x}$

TRY IT : : 8.103 Simplify: ⓐ $\frac{1}{\sqrt[4]{3}}$ ⓑ $\sqrt[4]{\frac{3}{64}}$ ⓒ $\frac{3}{\sqrt[4]{125x}}$.

TRY IT : : 8.104 Simplify: ⓐ $\frac{1}{\sqrt[4]{5}}$ ⓑ $\sqrt[4]{\frac{7}{128}}$ ⓒ $\frac{4}{\sqrt[4]{4x}}$

Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.

$$\begin{array}{cc} (a-b)(a+b) & \left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right) \\ a^2-b^2 & 2^2-\left(\sqrt{5}\right)^2 \\ & 4-5 \\ & -1 \end{array}$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

EXAMPLE 8.53

Simplify: $\dfrac{5}{2-\sqrt{3}}$.

Solution

	$\dfrac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\dfrac{5\left(2+\sqrt{3}\right)}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)}$
Multiply the conjugates in the denominator.	$\dfrac{5\left(2+\sqrt{3}\right)}{2^2-\left(\sqrt{3}\right)^2}$
Simplify the denominator.	$\dfrac{5\left(2+\sqrt{3}\right)}{4-3}$
Simplify the denominator.	$\dfrac{5\left(2+\sqrt{3}\right)}{1}$
Simplify.	$5\left(2+\sqrt{3}\right)$

TRY IT : : 8.105 Simplify: $\dfrac{3}{1-\sqrt{5}}$.

TRY IT : : 8.106 Simplify: $\dfrac{2}{4-\sqrt{6}}$.

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

EXAMPLE 8.54

Simplify: $\dfrac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution

	$\dfrac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\dfrac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}-\sqrt{6}\right)\left(\sqrt{u}+\sqrt{6}\right)}$
Multiply the conjugates in the denominator.	$\dfrac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}\right)^2-\left(\sqrt{6}\right)^2}$

Simplify the denominator.	$\dfrac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

TRY IT : : 8.107 Simplify: $\dfrac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

TRY IT : : 8.108 Simplify: $\dfrac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

EXAMPLE 8.55

Simplify: $\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution

	$\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\dfrac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$\dfrac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\dfrac{(\sqrt{x}+\sqrt{7})^2}{x-7}$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

TRY IT : : 8.109 Simplify: $\dfrac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

TRY IT : : 8.110 Simplify: $\dfrac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$

MEDIA : :

Access these online resources for additional instruction and practice with dividing radical expressions.

- **Rationalize the Denominator (https://openstax.org/l/37RatDenom1)**
- **Dividing Radical Expressions and Rationalizing the Denominator (https://openstax.org/l/37RatDenom2)**
- **Simplifying a Radical Expression with a Conjugate (https://openstax.org/l/37RatDenom3)**
- **Rationalize the Denominator of a Radical Expression (https://openstax.org/l/37RatDenom4)**

8.5 EXERCISES

Practice Makes Perfect

Divide Square Roots

In the following exercises, simplify.

245. ⓐ $\frac{\sqrt{128}}{\sqrt{72}}$ ⓑ $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$

246. ⓐ $\frac{\sqrt{48}}{\sqrt{75}}$ ⓑ $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$

247. ⓐ $\frac{\sqrt{200m^5}}{\sqrt{98m}}$ ⓑ $\frac{\sqrt[3]{54y^2}}{\sqrt[3]{2y^5}}$

248. ⓐ $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$ ⓑ $\frac{\sqrt[3]{54y}}{\sqrt[3]{16y^4}}$

249. ⓐ $\frac{\sqrt{75r^3}}{\sqrt{108r^7}}$ ⓑ $\frac{\sqrt[3]{24x^7}}{\sqrt[3]{81x^4}}$

250. ⓐ $\frac{\sqrt{196q}}{\sqrt{484q^5}}$ ⓑ $\frac{\sqrt[3]{16m^4}}{\sqrt[3]{54m}}$

251. ⓐ $\frac{\sqrt{108p^5q^2}}{\sqrt{3p^3q^6}}$ ⓑ $\frac{\sqrt[3]{-16a^4b^{-2}}}{\sqrt[3]{2a^{-2}b}}$

252. ⓐ $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$ ⓑ $\frac{\sqrt[3]{-375y^4z^{-2}}}{\sqrt[3]{3y^{-2}z^4}}$

253. ⓐ $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$ ⓑ $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

254. ⓐ $\frac{\sqrt{810c^{-3}d^7}}{\sqrt{1000cd^{-1}}}$ ⓑ $\frac{\sqrt[3]{24a^7b^{-1}}}{\sqrt[3]{-81a^{-2}b^2}}$

255. $\frac{\sqrt{56x^5y^4}}{\sqrt{2xy^3}}$

256. $\frac{\sqrt{72a^3b^6}}{\sqrt{3ab^3}}$

257. $\frac{\sqrt[3]{48a^3b^6}}{\sqrt[3]{3a^{-1}b^3}}$

258. $\frac{\sqrt[3]{162x^{-3}y^6}}{\sqrt[3]{2x^3y^{-2}}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

259. ⓐ $\frac{10}{\sqrt{6}}$ ⓑ $\sqrt{\frac{4}{27}}$ ⓒ $\frac{10}{\sqrt{5x}}$

260. ⓐ $\frac{8}{\sqrt{3}}$ ⓑ $\sqrt{\frac{7}{40}}$ ⓒ $\frac{8}{\sqrt{2y}}$

261. ⓐ $\frac{6}{\sqrt{7}}$ ⓑ $\sqrt{\frac{8}{45}}$ ⓒ $\frac{12}{\sqrt{3p}}$

262. ⓐ $\frac{4}{\sqrt{5}}$ ⓑ $\sqrt{\frac{27}{80}}$ ⓒ $\frac{18}{\sqrt{6q}}$

263. ⓐ $\frac{1}{\sqrt[3]{5}}$ ⓑ $\sqrt[3]{\frac{5}{24}}$ ⓒ $\frac{4}{\sqrt[3]{36a}}$

264. ⓐ $\frac{1}{\sqrt[3]{3}}$ ⓑ $\sqrt[3]{\frac{5}{32}}$ ⓒ $\frac{7}{\sqrt[3]{49b}}$

265. ⓐ $\frac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\frac{7}{54}}$ ⓒ $\frac{3}{\sqrt[3]{3x^2}}$

266. ⓐ $\frac{1}{\sqrt[3]{13}}$ ⓑ $\sqrt[3]{\frac{3}{128}}$ ⓒ $\frac{3}{\sqrt[3]{6y^2}}$

267. ⓐ $\frac{1}{\sqrt[4]{7}}$ ⓑ $\sqrt[4]{\frac{5}{32}}$ ⓒ $\frac{4}{\sqrt[4]{4x^2}}$

268. ⓐ $\frac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\frac{9}{32}}$ ⓒ $\frac{6}{\sqrt[4]{9x^3}}$

269. ⓐ $\frac{1}{\sqrt[4]{9}}$ ⓑ $\sqrt[4]{\frac{25}{128}}$ ⓒ $\frac{6}{\sqrt[4]{27a}}$

270. ⓐ $\frac{1}{\sqrt[4]{8}}$ ⓑ $\sqrt[4]{\frac{27}{128}}$ ⓒ $\frac{16}{\sqrt[4]{64b^2}}$

Rationalize a Two Term Denominator

In the following exercises, simplify.

271. $\frac{8}{1-\sqrt{5}}$

272. $\frac{7}{2-\sqrt{6}}$

273. $\frac{6}{3-\sqrt{7}}$

274. $\frac{5}{4-\sqrt{11}}$

275. $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

276. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

277. $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

278. $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

279. $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

280. $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

281. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

282. $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Writing Exercises

283.

ⓐ Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

ⓑ Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

ⓒ Why are the two methods of simplifying square roots different?

284. Explain what is meant by the word rationalize in the phrase, "rationalize a denominator."

285. Explain why multiplying $\sqrt{2x}-3$ by its conjugate results in an expression with no radicals.

286. Explain why multiplying $\frac{7}{\sqrt[3]{x}}$ by $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$ does not rationalize the denominator.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
divide radical expressions.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

8.6 Solve Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
   If you missed this problem, review **Example 5.31**.
2. Solve: $2x-5=0$.
   If you missed this problem, review **Example 2.2**.
3. Solve $n^2-6n+8=0$.
   If you missed this problem, review **Example 6.45**.

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index *n* to the *n*th power. This will eliminate the radical.

$$\text{For } a \geq 0,\ (\sqrt[n]{a})^n = a.$$

EXAMPLE 8.56 HOW TO SOLVE A RADICAL EQUATION

Solve: $\sqrt{5n-4}-9=0$.

Solution

Step 1. Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Step 2. Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$\left(\sqrt{5n-4}\right)^2=(9)^2$
Step 3. Solve the new equation.	Remember, $\left(\sqrt{a}\right)^2=a$.	$5n-4=81$ $5n=85$ $n=17$

Step 4. Check the answer in the original equation.		Check the answer. $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9\stackrel{?}{=}0$ $0=0\checkmark$ The solution is $n=17$.

TRY IT : : 8.111 Solve: $\sqrt{3m+2}-5=0$.

TRY IT : : 8.112 Solve: $\sqrt{10z+1}-2=0$.

HOW TO : : SOLVE A RADICAL EQUATION WITH ONE RADICAL.

Step 1. Isolate the radical on one side of the equation.
Step 2. Raise both sides of the equation to the power of the index.
Step 3. Solve the new equation.
Step 4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

EXAMPLE 8.57

Solve: $\sqrt{9k-2}+1=0$.

Solution

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

TRY IT : : 8.113 Solve: $\sqrt{2r-3}+5=0$.

TRY IT : : 8.114 Solve: $\sqrt{7s-3}+2=0$.

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Binomial Squares

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

Don't forget the middle term!

EXAMPLE 8.58

Solve: $\sqrt{p-1} + 1 = p$.

Solution

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right. Then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the Zero Product Property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	

$p=1$: $\sqrt{p-1}+1=p$, $\sqrt{1-1}+1\stackrel{?}{=}1$, $\sqrt{0}+1\stackrel{?}{=}1$, $1=1\checkmark$

$p=2$: $\sqrt{p-1}+1=p$, $\sqrt{2-1}+1\stackrel{?}{=}2$, $\sqrt{1}+1\stackrel{?}{=}2$, $2=2\checkmark$

The solutions are $p=1, \quad p=2$.

TRY IT : : 8.115 Solve: $\sqrt{x-2}+2=x$.

TRY IT : : 8.116 Solve: $\sqrt{y-5}+5=y$.

When the index of the radical is 3, we cube both sides to remove the radical.

$$\left(\sqrt[3]{a}\right)^3 = a$$

EXAMPLE 8.59

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution

	$\sqrt[3]{5x+1}+8=4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1}=-4$
Cube both sides of the equation.	$\left(\sqrt[3]{5x+1}\right)^3=(-4)^3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	
$x=-13$ $\quad \sqrt[3]{5x+1}+8=4$ $\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$ $\sqrt[3]{-64}+8\stackrel{?}{=}4$ $-4+8\stackrel{?}{=}4$ $4=4\checkmark$	
	The solution is $x=-13$.

> **TRY IT : : 8.117** Solve: $\sqrt[3]{4x-3}+8=5$

> **TRY IT : : 8.118** Solve: $\sqrt[3]{6x-10}+1=-3$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n=a^{m\cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2=x,\quad \left(x^{\frac{1}{3}}\right)^3=x$$

Remember, $x^{\frac{1}{2}}=\sqrt{x}$ and $x^{\frac{1}{3}}=\sqrt[3]{x}$.

EXAMPLE 8.60

Solve: $(3x-2)^{\frac{1}{4}}+3=5$.

Solution

$$(3x-2)^{\frac{1}{4}}+3=5$$

To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}} = 2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4 = (2)^4$
Simplify.	$3x - 2 = 16$
Solve the equation.	$3x = 18$
	$x = 6$
Check the answer.	
$x = 6 \quad (3x-2)^{\frac{1}{4}} + 3 = 5$ $(3 \cdot 6 - 2)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $(16)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $2 + 3 \stackrel{?}{=} 5$ $5 = 5 ✓$	
	The solution is $x = 6$.

> **TRY IT : :** 8.119
>
> Solve: $(9x+9)^{\frac{1}{4}} - 2 = 1$.

> **TRY IT : :** 8.120
>
> Solve: $(4x-8)^{\frac{1}{4}} + 5 = 7$.

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an extraneous solution!

EXAMPLE 8.61

Solve: $\sqrt{r+4} - r + 2 = 0$.

Solution

	$\sqrt{r+4} - r + 2 = 0$
Isolate the radical.	$\sqrt{r+4} = r - 2$
Square both sides of the equation.	$\left(\sqrt{r+4}\right)^2 = (r-2)^2$
Simplify and then solve the equation	$r + 4 = r^2 - 4r + 4$
It is a quadratic equation, so get zero on one side.	$0 = r^2 - 5r$
Factor the right side.	$0 = r(r-5)$
Use the Zero Product Property.	$0 = r \qquad 0 = r - 5$

Solve the equation.	$r=0 \quad r=5$
Check your answer.	
$r=0,\ \sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2\stackrel{?}{=}0$ $\sqrt{4}+2\stackrel{?}{=}0$ $4\neq 0$ $r=5,\ \sqrt{r+4}-r+2=0$ $\sqrt{5+4}-5+2\stackrel{?}{=}0$ $\sqrt{9}-3\stackrel{?}{=}0$ $0=0\checkmark$	The solution is $r=5$.
	$r=0$ is an extraneous solution.

> **TRY IT : : 8.121** Solve: $\sqrt{m+9}-m+3=0$.

> **TRY IT : : 8.122** Solve: $\sqrt{n+1}-n+1=0$.

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

EXAMPLE 8.62

Solve: $3\sqrt{3x-5}-8=4$.

Solution

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$
Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$\left(\sqrt{3x-5}\right)^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	
$x=7 \quad 3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4\checkmark$	
	The solution is $x=7$.

> **TRY IT : : 8.123** Solve: $2\sqrt{4a+4}-16=16$.

TRY IT : : 8.124 Solve: $3\sqrt{2b+3}-25=50.$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

EXAMPLE 8.63

Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}.$

Solution

The radical terms are isolated.	$\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$
Since the index is 3, cube both sides of the equation.	$\left(\sqrt[3]{4x-3}\right)^3=\left(\sqrt[3]{3x+2}\right)^3$
Simplify, then solve the new equation.	$4x-3=3x+2$
	$x-3=2$
	$x=5$
	The solution is $x=5.$
Check the answer. We leave it to you to show that 5 checks!	

TRY IT : : 8.125 Solve: $\sqrt[3]{5x-4}=\sqrt[3]{2x+5}.$

TRY IT : : 8.126 Solve: $\sqrt[3]{7x+1}=\sqrt[3]{2x-5}.$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

EXAMPLE 8.64 HOW TO SOLVE A RADICAL EQUATION

Solve: $\sqrt{m}+1=\sqrt{m+9}.$

Solution

Step 1. Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2. Raise both sides of the equation to the power of the index.	We square both sides. Simplify—be very careful as you multiply!	$\left(\sqrt{m}+1\right)^2=\left(\sqrt{m+9}\right)^2$
Step 3. Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$m+2\sqrt{m}+1=m+9$ $2\sqrt{m}=8$ $\sqrt{m}=4$ $\left(\sqrt{m}\right)^2=(4)^2$ $m=16$

Step 4. Check the answer in the original equation.		$\sqrt{m}+1=\sqrt{m+9}$ $\sqrt{16}+1\stackrel{?}{=}\sqrt{16+9}$ $4+1\stackrel{?}{=}5$ $5=5\checkmark$ The solution is $m=16$.

TRY IT : : 8.127 Solve: $3-\sqrt{x}=\sqrt{x-3}$.

TRY IT : : 8.128 Solve: $\sqrt{x}+2=\sqrt{x+16}$.

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

HOW TO : : SOLVE A RADICAL EQUATION.

Step 1. Isolate one of the radical terms on one side of the equation.

Step 2. Raise both sides of the equation to the power of the index.

Step 3. Are there any more radicals?
If yes, repeat Step 1 and Step 2 again.
If no, solve the new equation.

Step 4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern is $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

EXAMPLE 8.65

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$(\sqrt{q-2}+3)^2=(\sqrt{4q+1})^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$(6\sqrt{q-2})^2=\left(\overset{a-b}{3q-6}\right)^2$ $6^2(\sqrt{q-2})^2=\overset{a^2}{(3q)^2}-\overset{2ab}{2\cdot 3q\cdot 6}+\overset{b^2}{6^2}$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$

Distribute.	$36q - 72 = 9q^2 - 36q + 36$
It is a quadratic equation, so get zero on one side.	$0 = 9q^2 - 72q + 108$
Factor the right side.	$0 = 9(q^2 - 8q + 12)$ $0 = 9(q - 6)(q - 2)$
Use the Zero Product Property.	$q - 6 = 0 \quad q - 2 = 0$ $q = 6 \quad q = 2$
The checks are left to you.	The solutions are $q = 6$ and $q = 2$.

TRY IT : : 8.129 Solve: $\sqrt{x-1} + 2 = \sqrt{2x+6}$

TRY IT : : 8.130 Solve: $\sqrt{x} + 2 = \sqrt{3x+4}$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

HOW TO : : USE A PROBLEM SOLVING STRATEGY FOR APPLICATIONS WITH FORMULAS.

Step 1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve the equation** using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula

$$t = \frac{\sqrt{h}}{4}.$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

	$t=\frac{\sqrt{h}}{4}$
	$t=\frac{\sqrt{64}}{4}$
Take the square root of 64.	$t=\frac{8}{4}$
Simplify the fraction.	$t=2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

EXAMPLE 8.66

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the time it takes for the sunglasses to reach the river
Step 3. Name what we are looking.	Let $t=$ time.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Step 5. Solve the equation.	$t=\frac{20}{4}$
	$t=5$
Step 6. Check the answer in the problem and make sure it makes sense.	$5\overset{?}{=}\frac{\sqrt{400}}{4}$ $5\overset{?}{=}\frac{20}{4}$ $5=5\checkmark$
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7. Answer the question.	It will take 5 seconds for the sunglasses to reach the river.

TRY IT : : 8.131

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

TRY IT : : 8.132

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Skid Marks and Speed of a Car

If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

EXAMPLE 8.67

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution

Step 1. Read the problem	
Step 2. Identify what we are looking for.	the speed of a car
Step 3. Name what weare looking for,	Let $s =$ the speed.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5. Solve the equation.	$s = \sqrt{4{,}560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \overset{?}{\approx} \sqrt{24(190)}$ $67.5 \overset{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277... \checkmark$
	The speed of the car before the brakes were applied was 67.5 miles per hour.

TRY IT : : 8.133

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

TRY IT : : 8.134

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

MEDIA : :

Access these online resources for additional instruction and practice with solving radical equations.

- **Solving an Equation Involving a Single Radical (https://openstax.org/l/37RadEquat1)**
- **Solving Equations with Radicals and Rational Exponents (https://openstax.org/l/37RadEquat2)**
- **Solving Radical Equations (https://openstax.org/l/37RadEquat3)**
- **Solve Radical Equations (https://openstax.org/l/37RadEquat4)**
- **Radical Equation Application (https://openstax.org/l/37RadEquat5)**

8.6 EXERCISES

Practice Makes Perfect

Solve Radical Equations

In the following exercises, solve.

287. $\sqrt{5x-6}=8$

288. $\sqrt{4x-3}=7$

289. $\sqrt{5x+1}=-3$

290. $\sqrt{3y-4}=-2$

291. $\sqrt[3]{2x}=-2$

292. $\sqrt[3]{4x-1}=3$

293. $\sqrt{2m-3}-5=0$

294. $\sqrt{2n-1}-3=0$

295. $\sqrt{6v-2}-10=0$

296. $\sqrt{12u+1}-11=0$

297. $\sqrt{4m+2}+2=6$

298. $\sqrt{6n+1}+4=8$

299. $\sqrt{2u-3}+2=0$

300. $\sqrt{5v-2}+5=0$

301. $\sqrt{u-3}-3=u$

302. $\sqrt{v-10}+10=v$

303. $\sqrt{r-1}=r-1$

304. $\sqrt{s-8}=s-8$

305. $\sqrt[3]{6x+4}=4$

306. $\sqrt[3]{11x+4}=5$

307. $\sqrt[3]{4x+5}-2=-5$

308. $\sqrt[3]{9x-1}-1=-5$

309. $(6x+1)^{\frac{1}{2}}-3=4$

310. $(3x-2)^{\frac{1}{2}}+1=6$

311. $(8x+5)^{\frac{1}{3}}+2=-1$

312. $(12x-5)^{\frac{1}{3}}+8=3$

313. $(12x-3)^{\frac{1}{4}}-5=-2$

314. $(5x-4)^{\frac{1}{4}}+7=9$

315. $\sqrt{x+1}-x+1=0$

316. $\sqrt{y+4}-y+2=0$

317. $\sqrt{z+100}-z=-10$

318. $\sqrt{w+25}-w=-5$

319. $3\sqrt{2x-3}-20=7$

320. $2\sqrt{5x+1}-8=0$

321. $2\sqrt{8r+1}-8=2$

322. $3\sqrt{7y+1}-10=8$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

323. $\sqrt{3u+7}=\sqrt{5u+1}$

324. $\sqrt{4v+1}=\sqrt{3v+3}$

325. $\sqrt{8+2r}=\sqrt{3r+10}$

326. $\sqrt{10+2c}=\sqrt{4c+16}$

327. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$

328. $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$

329. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

330. $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$

331. $\sqrt{a}+2=\sqrt{a+4}$

332. $\sqrt{r}+6=\sqrt{r+8}$

333. $\sqrt{u}+1=\sqrt{u+4}$

334. $\sqrt{x}+1=\sqrt{x+2}$

335. $\sqrt{a+5}-\sqrt{a}=1$

336. $-2=\sqrt{d-20}-\sqrt{d}$

337. $\sqrt{2x+1}=1+\sqrt{x}$

338. $\sqrt{3x+1}=1+\sqrt{2x-1}$

339. $\sqrt{2x-1}-\sqrt{x-1}=1$

340. $\sqrt{x+1}-\sqrt{x-2}=1$

341. $\sqrt{x+7} - \sqrt{x-5} = 2$

342. $\sqrt{x+5} - \sqrt{x-3} = 2$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

343. Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

344. Landscaping Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

345. Gravity A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.

346. Gravity A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

347. Accident investigation The skid marks for a car involved in an accident measured 216 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

348. Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Writing Exercises

349. Explain why an equation of the form $\sqrt{x} + 1 = 0$ has no solution.

350.

ⓐ Solve the equation $\sqrt{r+4} - r + 2 = 0.$

ⓑ Explain why one of the "solutions" that was found was not actually a solution to the equation.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
solve radical equations.			
solve radical equations with two radicals.			
use radicals in applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

8.7 Use Radicals in Functions

Learning Objectives

By the end of this section, you will be able to:

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $1 - 2x \geq 0.$
   If you missed this problem, review **Example 2.50**.
2. For $f(x) = 3x - 4,$ evaluate $f(2),\ f(-1),\ f(0).$
   If you missed this problem, review **Example 3.48**.
3. Graph $f(x) = \sqrt{x}.$ State the domain and range of the function in interval notation.
   If you missed this problem, review **Example 3.56**.

Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

The square root function is $f(x) = \sqrt{x}.$

The cube root function is $f(x) = \sqrt[3]{x}.$

Radical Function

A **radical function** is a function that is defined by a radical expression.

To evaluate a radical function, we find the value of *f*(*x*) for a given value of *x* just as we did in our previous work with functions.

EXAMPLE 8.68

For the function $f(x) = \sqrt{2x - 1},$ find ⓐ $f(5)$ ⓑ $f(-2).$

Solution

ⓐ

	$f(x) = \sqrt{2x - 1}$
To evaluate $f(5)$, substitute 5 for x.	$f(5) = \sqrt{2 \cdot 5 - 1}$
Simplify.	$f(5) = \sqrt{9}$
Take the square root.	$f(5) = 3$

ⓑ

	$f(x) = \sqrt{2x - 1}$
To evaluate $f(-2)$, substitute -2 for x.	$f(-2) = \sqrt{2(-2) - 1}$
Simplify.	$f(-2) = \sqrt{-5}$

Since the square root of a negative number is not a real number, the function does not have a value at $x = -2.$

TRY IT : : 8.135 For the function $f(x) = \sqrt{3x - 2},$ find ⓐ $f(6)$ ⓑ $f(0).$

TRY IT : : 8.136 For the function $g(x)=\sqrt{5x+5}$, find ⓐ $g(4)$ ⓑ $g(-3)$.

We follow the same procedure to evaluate cube roots.

EXAMPLE 8.69

For the function $g(x)=\sqrt[3]{x-6}$, find ⓐ $g(14)$ ⓑ $g(-2)$.

Solution

ⓐ

	$g(x)=\sqrt[3]{x-6}$
To evaluate $g(14)$, substitute 14 for x.	$g(14)=\sqrt[3]{14-6}$
Simplify.	$g(14)=\sqrt[3]{8}$
Take the cube root.	$g(14)=2$

ⓑ

	$g(x)=\sqrt[3]{x-6}$
To evaluate $g(-2)$, substitute -2 for x.	$g(-2)=\sqrt[3]{-2-6}$
Simplify.	$g(-2)=\sqrt[3]{-8}$
Take the cube root.	$g(-2)=-2$

TRY IT : : 8.137 For the function $g(x)=\sqrt[3]{3x-4}$, find ⓐ $g(4)$ ⓑ $g(1)$.

TRY IT : : 8.138 For the function $h(x)=\sqrt[3]{5x-2}$, find ⓐ $h(2)$ ⓑ $h(-5)$.

The next example has fourth roots.

EXAMPLE 8.70

For the function $f(x)=\sqrt[4]{5x-4}$, find ⓐ $f(4)$ ⓑ $f(-12)$

Solution

ⓐ

	$f(x)=\sqrt[4]{5x-4}$
To evaluate $f(4)$, substitute 4 for x.	$f(4)=\sqrt[4]{5\cdot 4-4}$
Simplify.	$f(4)=\sqrt[4]{16}$
Take the fourth root.	$f(4)=2$

ⓑ

	$f(x)=\sqrt[4]{5x-4}$
To evaluate $f(-12)$, substitute -12 for x.	$f(-12)=\sqrt[4]{5(-12)-4}$
Simplify.	$f(-12)=\sqrt[4]{-64}$

Since the fourth root of a negative number is not a real number, the function does not have a value at $x=-12$.

TRY IT : : 8.139 For the function $f(x)=\sqrt[4]{3x+4}$, find ⓐ $f(4)$ ⓑ $f(-1)$.

TRY IT : : 8.140 For the function $g(x)=\sqrt[4]{5x+1}$, find ⓐ $g(16)$ ⓑ $g(3)$.

Find the Domain of a Radical Function

To find the domain and range of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Properties of $\sqrt[n]{a}$

When *n* is an **even** number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When *n* is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of *a*.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

Domain of a Radical Function

When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
When the **index** of the radical is **odd**, the radicand can be any real number.

EXAMPLE 8.71

Find the domain of the function, $f(x)=\sqrt{3x-4}$. Write the domain in interval notation.

Solution

Since the function, $f(x)=\sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

$$\begin{array}{lrcl} & 3x-4 & \geq & 0 \\ \text{Solve.} & 3x & \geq & 4 \\ & x & \geq & \frac{4}{3} \end{array}$$

The domain of $f(x)=\sqrt{3x-4}$ is all values $x \geq \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

TRY IT : : 8.141 Find the domain of the function, $f(x)=\sqrt{6x-5}$. Write the domain in interval notation.

TRY IT : : 8.142 Find the domain of the function, $f(x)=\sqrt{4-5x}$. Write the domain in interval notation.

EXAMPLE 8.72

Find the domain of the function, $g(x)=\sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution

Since the function, $g(x)=\sqrt{\frac{6}{x-1}}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

The radicand cannot be zero since the numerator is not zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive since the numerator is positive. We know a positive divided by a positive is positive.

We set $x - 1 > 0$ and solve.

$$\begin{array}{lrcl} & x-1 & > & 0 \\ \text{Solve.} & x & > & 1 \end{array}$$

Also, since the radicand is a fraction, we must realize that the denominator cannot be zero.

We solve $x - 1 = 0$ to find the value that must be eliminated from the domain.

$$\begin{array}{lrcl} & x-1 & = & 0 \\ \text{Solve.} & x & = & 1 \text{ so } x \neq 1 \text{ in the domain.} \end{array}$$

Putting this together we get the domain is $x > 1$ and we write it as $(1, \infty)$.

TRY IT : : 8.143 Find the domain of the function, $f(x) = \sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

TRY IT : : 8.144 Find the domain of the function, $h(x) = \sqrt{\frac{9}{x-5}}$. Write the domain in interval notation.

The next example involves a cube root and so will require different thinking.

EXAMPLE 8.73

Find the domain of the function, $f(x) = \sqrt[3]{2x^2 + 3}$. Write the domain in interval notation.

Solution

Since the function, $f(x) = \sqrt[3]{2x^2 + 3}$ has a radical with an index of 3, which is odd, we know the radicand can be any real number. This tells us the domain is any real number. In interval notation, we write $(-\infty, \infty)$.

The domain of $f(x) = \sqrt[3]{2x^2 + 3}$ is all real numbers and we write it in interval notation as $(-\infty, \infty)$.

TRY IT : : 8.145 Find the domain of the function, $f(x) = \sqrt[3]{3x^2 - 1}$. Write the domain in interval notation.

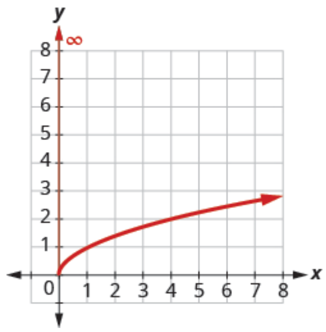
TRY IT : : 8.146 Find the domain of the function, $g(x) = \sqrt[3]{5x - 4}$. Write the domain in interval notation.

Graph Radical Functions

Before we graph any radical function, we first find the domain of the function. For the function, $f(x) = \sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x \geq 0$ and we write this in interval notation as $[0, \infty)$.

Previously we used point plotting to graph the function, $f(x) = \sqrt{x}$. We chose *x*-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.



x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Once we see the graph, we can find the range of the function. The *y*-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

EXAMPLE 8.74

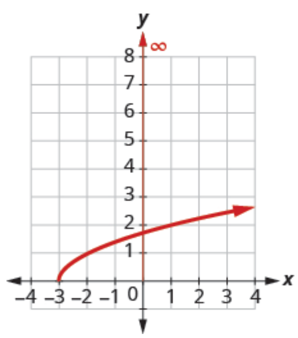
For the function $f(x)=\sqrt{x+3}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution

ⓐ Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x+3 \geq 0,$ then $x \geq -3.$ This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty).$

ⓑ To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.



x	$f(x)=\sqrt{x+3}$	$(x, f(x))$
–3	0	(–3, 0)
–2	1	(–2, 1)
1	2	(1, 2)
6	3	(6, 3)

ⓒ Looking at the graph, we see the *y*-values of the function are greater than or equal to zero. The range then is $[0, \infty).$

TRY IT : : 8.147

For the function $f(x)=\sqrt{x+2},$ ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

TRY IT : : 8.148

For the function $f(x)=\sqrt{x-2},$ ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

In our previous work graphing functions, we graphed $f(x)=x^3$ but we did not graph the function $f(x)=\sqrt[3]{x}.$ We will do this now in the next example.

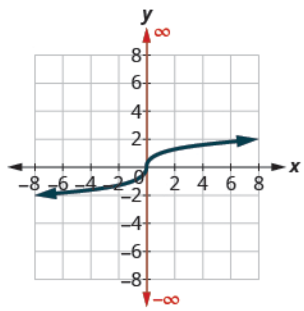
EXAMPLE 8.75

For the function $f(x)=\sqrt[3]{x}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution

ⓐ Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as $(-\infty,\ \infty)$

ⓑ To graph the function, we choose points in the interval $(-\infty,\ \infty)$ that will also give us a radicand which will be easy to take the cube root.



x	$f(x)=\sqrt[3]{x}$	$(x, f(x))$
−8	−2	(−8, −2)
−1	−1	(−1, −1)
0	0	(0, 0)
1	1	(1, 1)
8	2	(8, 2)

ⓒ Looking at the graph, we see the *y*-values of the function are all real numbers. The range then is $(-\infty,\ \infty)$.

TRY IT : : 8.149

For the function $f(x)=-\sqrt[3]{x}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

TRY IT : : 8.150

For the function $f(x)=\sqrt[3]{x-2}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

MEDIA : :

Access these online resources for additional instruction and practice with radical functions.

- **Domain of a Radical Function (https://openstax.org/l/37RadFuncDom1)**
- **Domain of a Radical Function 2 (https://openstax.org/l/37RadFuncDom2)**
- **Finding Domain of a Radical Function (https://openstax.org/l/37RadFuncDom3)**

8.7 EXERCISES

Practice Makes Perfect

Evaluate a Radical Function

In the following exercises, evaluate each function.

351. $f(x) = \sqrt{4x - 4}$, find
ⓐ $f(5)$
ⓑ $f(0)$.

352. $f(x) = \sqrt{6x - 5}$, find
ⓐ $f(5)$
ⓑ $f(-1)$.

353. $g(x) = \sqrt{6x + 1}$, find
ⓐ $g(4)$
ⓑ $g(8)$.

354. $g(x) = \sqrt{3x + 1}$, find
ⓐ $g(8)$
ⓑ $g(5)$.

355. $F(x) = \sqrt{3 - 2x}$, find
ⓐ $F(1)$
ⓑ $F(-11)$.

356. $F(x) = \sqrt{8 - 4x}$, find
ⓐ $F(1)$
ⓑ $F(-2)$.

357. $G(x) = \sqrt{5x - 1}$, find
ⓐ $G(5)$
ⓑ $G(2)$.

358. $G(x) = \sqrt{4x + 1}$, find
ⓐ $G(11)$
ⓑ $G(2)$.

359. $g(x) = \sqrt[3]{2x - 4}$, find
ⓐ $g(6)$
ⓑ $g(-2)$.

360. $g(x) = \sqrt[3]{7x - 1}$, find
ⓐ $g(4)$
ⓑ $g(-1)$.

361. $h(x) = \sqrt[3]{x^2 - 4}$, find
ⓐ $h(-2)$
ⓑ $h(6)$.

362. $h(x) = \sqrt[3]{x^2 + 4}$, find
ⓐ $h(-2)$
ⓑ $h(6)$.

363. For the function $f(x) = \sqrt[4]{2x^3}$, find
ⓐ $f(0)$
ⓑ $f(2)$.

364. For the function $f(x) = \sqrt[4]{3x^3}$, find
ⓐ $f(0)$
ⓑ $f(3)$.

365. For the function $g(x) = \sqrt[4]{4 - 4x}$, find
ⓐ $g(1)$
ⓑ $g(-3)$.

366. For the function $g(x) = \sqrt[4]{8 - 4x}$, find
ⓐ $g(-6)$
ⓑ $g(2)$.

Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

367. $f(x) = \sqrt{3x - 1}$

368. $f(x) = \sqrt{4x - 2}$

369. $g(x) = \sqrt{2 - 3x}$

370. $g(x) = \sqrt{8 - x}$

371. $h(x) = \sqrt{\dfrac{5}{x - 2}}$

372. $h(x) = \sqrt{\dfrac{6}{x + 3}}$

373. $f(x) = \sqrt{\dfrac{x + 3}{x - 2}}$

374. $f(x) = \sqrt{\dfrac{x - 1}{x + 4}}$

375. $g(x) = \sqrt[3]{8x - 1}$

376. $g(x) = \sqrt[3]{6x + 5}$

377. $f(x) = \sqrt[3]{4x^2 - 16}$

378. $f(x) = \sqrt[3]{6x^2 - 25}$

379. $F(x) = \sqrt[4]{8x + 3}$

380. $F(x) = \sqrt[4]{10 - 7x}$

381. $G(x) = \sqrt[5]{2x - 1}$

382. $G(x) = \sqrt[5]{6x - 3}$

Graph Radical Functions

In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

383. $f(x) = \sqrt{x + 1}$

384. $f(x) = \sqrt{x - 1}$

385. $g(x) = \sqrt{x + 4}$

386. $g(x) = \sqrt{x - 4}$

387. $f(x) = \sqrt{x} + 2$

388. $f(x) = \sqrt{x} - 2$

389. $g(x) = 2\sqrt{x}$

390. $g(x) = 3\sqrt{x}$

391. $f(x) = \sqrt{3 - x}$

392. $f(x) = \sqrt{4 - x}$

393. $g(x) = -\sqrt{x}$

394. $g(x) = -\sqrt{x} + 1$

395. $f(x) = \sqrt[3]{x + 1}$

396. $f(x) = \sqrt[3]{x - 1}$

397. $g(x) = \sqrt[3]{x + 2}$

398. $g(x) = \sqrt[3]{x - 2}$

399. $f(x) = \sqrt[3]{x} + 3$

400. $f(x) = \sqrt[3]{x} - 3$

401. $g(x) = \sqrt[3]{x}$

402. $g(x) = -\sqrt[3]{x}$

403. $f(x) = 2\sqrt[3]{x}$

404. $f(x) = -2\sqrt[3]{x}$

Writing Exercises

405. Explain how to find the domain of a fourth root function.

406. Explain how to find the domain of a fifth root function.

407. Explain why $y = \sqrt[3]{x}$ is a function.

408. Explain why the process of finding the domain of a radical function with an even index is different from the process when the index is odd.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate a radical function.			
find the domain of a radical function.			
graph a radical function.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

8.8 Use the Complex Number System

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Be Prepared!

Before you get started, take this readiness quiz.

1. Given the numbers $-4, \ -\sqrt{7}, 0.\overline{5}\,, \frac{7}{3}, 3, \sqrt{81},$ list the ⓐ rational numbers, ⓑ irrational numbers, ⓒ real numbers.
If you missed this problem, review **Example 1.42**.
2. Multiply: $(x-3)(2x+5)$.
If you missed this problem, review **Example 5.28**.
3. Rationalize the denominator: $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$.
If you missed this problem, review **Example 5.32**.

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1}$, we are looking for a real number x so that $x^2 = -1$. Since all real numbers squared are positive numbers, there is no real number that equals –1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is –1.

Imaginary Unit

The **imaginary unit** i is the number whose square is –1.

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

We will use the imaginary unit to simplify the square roots of negative numbers.

Square Root of a Negative Number

If b is a positive real number, then

$$\sqrt{-b} = \sqrt{b}\,i$$

We will use this definition in the next example. Be careful that it is clear that the i is not under the radical. Sometimes you will see this written as $\sqrt{-b} = i\sqrt{b}$ to emphasize the i is not under the radical. But the $\sqrt{-b} = \sqrt{b}\,i$ is considered standard form.

EXAMPLE 8.76

Write each expression in terms of i and simplify if possible:

ⓐ $\sqrt{-25}$ ⓑ $\sqrt{-7}$ ⓒ $\sqrt{-12}$.

Solution

ⓐ

	$\sqrt{-25}$
Use the definition of he square root of negative numbers.	$\sqrt{25}\,i$
Simplify.	$5i$

ⓑ

	$\sqrt{-7}$
Use the definition of he square root of negative numbers.	$\sqrt{7}i$
Simplify.	Be careful that it is clear that i is not under the radical sign.

ⓒ

	$\sqrt{-12}$
Use the definition of he square root of negative numbers.	$\sqrt{12}\,i$
Simplify $\sqrt{12}$.	$2\sqrt{3}\,i$

TRY IT : : 8.151 Write each expression in terms of *i* and simplify if possible:

ⓐ $\sqrt{-81}$ ⓑ $\sqrt{-5}$ ⓒ $\sqrt{-18}$.

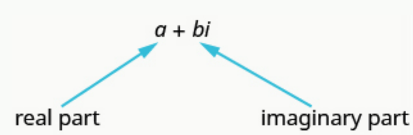
TRY IT : : 8.152 Write each expression in terms of *i* and simplify if possible:

ⓐ $\sqrt{-36}$ ⓑ $\sqrt{-3}$ ⓒ $\sqrt{-27}$.

Now that we are familiar with the imaginary number *i*, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form *a* + *bi*, where *a, b* are real numbers. We call *a* the real part and *b* the imaginary part.

Complex Number

A **complex number** is of the form *a* + *bi*, where *a* and *b* are real numbers.



A complex number is in standard form when written as $a + bi$, where *a* and *b* are real numbers.

If $b = 0$, then $a + bi$ becomes $a + 0 \cdot i = a$, and is a real number.

If $b \neq 0$, then $a + bi$ is an imaginary number.

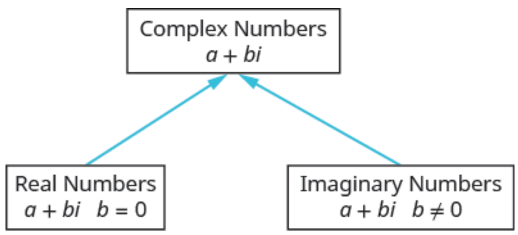
If $a = 0$, then $a + bi$ becomes $0 + bi = bi$, and is called a pure imaginary number.

We summarize this here.

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bi$ bi	Pure imaginary number

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $\sqrt{-b}=\sqrt{b}i$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.



Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

EXAMPLE 8.77

Add: $\sqrt{-12}+\sqrt{-27}$.

Solution

	$\sqrt{-12}+\sqrt{-27}$
Use the definition of he square root of negative numbers.	$\sqrt{12}\,i+\sqrt{27}\,i$
Simplify the square roots.	$2\sqrt{3}\,i+3\sqrt{3}\,i$
Add.	$5\sqrt{3}\,i$

TRY IT : : 8.153 Add: $\sqrt{-8}+\sqrt{-32}$.

TRY IT : : 8.154 Add: $\sqrt{-27}+\sqrt{-48}$.

Remember to add both the real parts and the imaginary parts in this next example.

EXAMPLE 8.78

Simplify: ⓐ $(4-3i)+(5+6i)$ ⓑ $(2-5i)-(5-2i)$.

Solution

ⓐ

	$(4-3i)+(5+6i)$
Use the Associative Property to put the real parts and the imaginary parts together.	$(4+5)+(-3i+6i)$
Simplify.	$9+3i$

ⓑ

	$(2-5i)-(5-2i)$
Distribute.	$2-5i-5+2i$
Use the Associative Property to put the real parts and the imaginary parts together.	$2-5-5i+2i$
Simplify.	$-3-3i$

TRY IT : : 8.155 Simplify: ⓐ $(2+7i)+(4-2i)$ ⓑ $(8-4i)-(2-i)$.

TRY IT : : 8.156 Simplify: ⓐ $(3-2i)+(-5-4i)$ ⓑ $(4+3i)-(2-6i)$.

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

EXAMPLE 8.79

Multiply: $2i(7-5i)$.

Solution

	$2i(7-5i)$
Distribute.	$14i-10i^2$
Simplify i^2.	$14i-10(-1)$
Multiply.	$14i+10$
Write in standard form.	$10+14i$

TRY IT : : 8.157 Multiply: $4i(5-3i)$.

TRY IT : : 8.158 Multiply: $-3i(2+4i)$.

In the next example, we multiply the binomials using the Distributive Property or FOIL.

EXAMPLE 8.80

Multiply: $(3+2i)(4-3i)$.

Solution

	$(3+2i)(4-3i)$
Use FOIL.	$12-9i+8i-6i^2$
Simplify i^2 and combine like terms.	$12-i-6(-1)$
Multiply.	$12-i+6$
Combine the real parts.	$18-i$

TRY IT : : 8.159 Multiply: $(5-3i)(-1-2i)$.

TRY IT : : 8.160 Multiply: $(-4-3i)(2+i)$.

In the next example, we could use FOIL or the Product of Binomial Squares Pattern.

EXAMPLE 8.81

Multiply: $(3+2i)^2$

Solution

	$\left(\begin{matrix} a+b \\ 3+2i \end{matrix}\right)^2$
Use the Product of Binomial Squares Pattern, $(a+b)^2 = a^2 + 2ab + b^2$.	$\begin{matrix} a^2 + 2 \quad a \quad b + b^2 \\ 3^2 + 2 \cdot 3 \cdot 2i + (2i)^2 \end{matrix}$
Simplify.	$9 + 12i + 4i^2$
Simplify i^2.	$9 + 12i + 4(-1)$
Simplify.	$5 + 12i$

TRY IT : : 8.161 Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

TRY IT : : 8.162 Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b} = \sqrt{b}i$. This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

EXAMPLE 8.82

Multiply: $\sqrt{-36} \cdot \sqrt{-4}$.

Solution

To multiply square roots of negative numbers, we first write them as complex numbers.

	$\sqrt{-36} \cdot \sqrt{-4}$
Write as complex numbers using $\sqrt{-b} = \sqrt{b}i$.	$\sqrt{36}\, i \cdot \sqrt{4}\, i$
Simplify.	$6i \cdot 2i$
Multiply.	$12i^2$
Simplify i^2 and multiply.	-12

TRY IT : : 8.163 Multiply: $\sqrt{-49} \cdot \sqrt{-4}$.

TRY IT : : 8.164 Multiply: $\sqrt{-36} \cdot \sqrt{-81}$.

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

EXAMPLE 8.83

Multiply: $\left(3-\sqrt{-12}\right)\left(5+\sqrt{-27}\right)$.

Solution

To multiply square roots of negative numbers, we first write them as complex numbers.

	$\left(3-\sqrt{-12}\right)\left(5+\sqrt{-27}\right)$
Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.	$\left(3-2\sqrt{3}\,i\right)\left(5+3\sqrt{3}\,i\right)$
Use FOIL.	$15+9\sqrt{3}\,i-10\sqrt{3}\,i-6\cdot 3i^2$
Combine like terms and simplify i^2.	$15-\sqrt{3}\,i-6\cdot(-3)$
Multiply and combine like terms.	$33-\sqrt{3}\,i$

TRY IT : : 8.165 Multiply: $\left(4-\sqrt{-12}\right)\left(3-\sqrt{-48}\right)$.

TRY IT : : 8.166 Multiply: $\left(-2+\sqrt{-8}\right)\left(3-\sqrt{-18}\right)$.

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),\ (a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Complex Conjugate Pair

A **complex conjugate pair** is of the form $a+bi,\quad a-bi$.

We will multiply a complex conjugate pair in the next example.

EXAMPLE 8.84

Multiply: $(3-2i)(3+2i)$.

Solution

	$(3-2i)(3+2i)$
Use FOIL.	$9+6i-6i-4i^2$
Combine like terms and simplify i^2.	$9-4(-1)$
Multiply and combine like terms.	13

TRY IT : : 8.167 Multiply: $(4-3i)\cdot(4+3i)$.

TRY IT : : 8.168 Multiply: $(-2+5i)\cdot(-2-5i)$.

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$. The result is called a difference of squares. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the Product of Conjugates Pattern.

$$\overset{a\ -\ b}{(3 - 2i)}\overset{a\ +\ b}{(3 + 2i)}$$

$$\overset{a^2\ -\ b^2}{(3)^2 - (2i)^2}$$

$$9 - 4i^2$$

$$9 - 4(-1)$$

$$13$$

Notice this is the same result we found in **Example 8.84**.

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a - bi)(a + bi)$$
$$a^2 - (bi)^2$$
$$a^2 - b^2 i^2$$
$$a^2 - b^2(-1)$$
$$a^2 + b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a - bi)(a + bi) = a^2 + b^2$

Product of Complex Conjugates

If *a* and *b* are real numbers, then

$$(a - bi)(a + bi) = a^2 + b^2$$

EXAMPLE 8.85

Multiply using the Product of Complex Conjugates Pattern: $(8 - 2i)(8 + 2i)$.

Solution

	$\overset{(a - b)(a + b)}{(8 - 2i)(8 + 2i)}$
Use the Product of Complex Conjugates Pattern, $(a - bi)(a + bi) = a^2 + b^2$.	$\overset{a^2 + b^2}{8^2 + 2^2}$
Simplify the squares.	$64 + 4$
Add.	68

TRY IT : : 8.169 Multiply using the Product of Complex Conjugates Pattern: $(3 - 10i)(3 + 10i)$.

TRY IT : : 8.170 Multiply using the Product of Complex Conjugates Pattern: $(-5 + 4i)(-5 - 4i)$.

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

EXAMPLE 8.86 HOW TO DIVIDE COMPLEX NUMBERS

Divide: $\dfrac{4 + 3i}{3 - 4i}$.

Solution

Step 1. Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$
Step 3. Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator.	$\frac{12+16i+9i+12i^2}{9+16}$
	Combine like terms.	$\frac{12+25i-12}{25}$
	Simplify.	$\frac{25i}{25}$
	Write the result in standard form.	i

TRY IT : : 8.171 Divide: $\frac{2+5i}{5-2i}$.

TRY IT : : 8.172 Divide: $\frac{1+6i}{6-i}$.

We summarize the steps here.

HOW TO : : HOW TO DIVIDE COMPLEX NUMBERS.

Step 1. Write both the numerator and denominator in standard form.

Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.

Step 3. Simplify and write the result in standard form.

EXAMPLE 8.87

Divide, writing the answer in standard form: $\frac{-3}{5+2i}$.

Solution

	$\frac{-3}{5+2i}$
Multiply the numerator and denominator by the complex conjugate of the denominator.	$\frac{-3(5-2i)}{(5+2i)(5-2i)}$
Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator.	$\frac{-15+6i}{5^2+2^2}$
Simplify.	$\frac{-15+6i}{29}$
Write in standard form.	$-\frac{15}{29}+\frac{6}{29}i$

TRY IT : : 8.173 Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

TRY IT : : 8.174 Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Be careful as you find the conjugate of the denominator.

EXAMPLE 8.88

Divide: $\frac{5+3i}{4i}$.

Solution

	$\frac{5+3i}{4i}$
Write the denominator in standard form.	$\frac{5+3i}{0+4i}$
Multiply the numerator and denominator by the complex conjugate of the denominator.	$\frac{(5+3i)(0-4i)}{(0+4i)(0-4i)}$
Simplify.	$\frac{(5+3i)(-4i)}{(4i)(-4i)}$
Multiply.	$\frac{-20i-12i^2}{-16i^2}$
Simplify the i^2.	$\frac{-20i+12}{16}$
Rewrite in standard form.	$\frac{12}{16}-\frac{20}{16}i$
Simplify the fractions.	$\frac{3}{4}-\frac{5}{4}i$

TRY IT : : 8.175 Divide: $\frac{3+3i}{2i}$.

TRY IT : : 8.176 Divide: $\frac{2+4i}{5i}$.

Simplify Powers of *i*

The powers of i make an interesting pattern that will help us simplify higher powers of *i*. Let's evaluate the powers of i to see the pattern.

i^1	i^2	i^3	i^4
i	-1	$i^2 \cdot i$	$i^2 \cdot i^2$
		$-1 \cdot i$	$(-1)(-1)$
		$-i$	1
i^5	i^6	i^7	i^8
$i^4 \cdot i$	$i^4 \cdot i^2$	$i^4 \cdot i^3$	$i^4 \cdot i^4$
$1 \cdot i$	$1 \cdot i^2$	$1 \cdot i^3$	$1 \cdot 1$
i	i^2	i^3	1
	-1	$-i$	

We summarize this now.

$$\begin{array}{rclcrcl} i^1 &=& i & \qquad & i^5 &=& i \\ i^2 &=& -1 & & i^6 &=& -1 \\ i^3 &=& -i & & i^7 &=& -i \\ i^4 &=& 1 & & i^8 &=& 1 \end{array}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of i. Since $i^4 = 1$, we rewrite each power, i^n, as a product using i^4 to a power and another power of i.

We rewrite it in the form $i^n = \left(i^4\right)^q \cdot i^r$, where the exponent, q, is the quotient of n divided by 4 and the exponent, r, is the remainder from this division. For example, to simplify i^{57}, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, $57 = 4 \cdot 14 + 1$. So we write $i^{57} = \left(1^4\right)^{14} \cdot i^1$ and then simplify from there.

$$\begin{array}{r} 14 \\ 4\overline{)57} \\ \underline{4} \\ 17 \\ \underline{16} \\ 1 \end{array} \qquad \begin{array}{c} i^{57} \\ \left(1^4\right)^{14} \cdot i^1 \\ 1 \cdot i \\ i \end{array}$$

EXAMPLE 8.89

Simplify: i^{86}.

Solution

	i^{86}
Divide 86 by 4 and rewrite i^{86} in the $i^n = \left(i^4\right)^q \cdot i^r$ form.	$\left(1^4\right)^{21} \cdot i^2$
	$\begin{array}{r} 21 \\ 4\overline{)86} \\ \underline{8} \\ 6 \\ \underline{4} \\ 2 \end{array}$
Simplify.	$(1)^{21} \cdot (-1)$
Simplify.	-1

TRY IT : : 8.177 Simplify: i^{75}.

TRY IT : : 8.178 Simplify: i^{92}.

MEDIA : :

Access these online resources for additional instruction and practice with the complex number system.

- Expressing Square Roots of Negative Numbers with i (https://openstax.org/l/37CompNumb1)
- Subtract and Multiply Complex Numbers (https://openstax.org/l/37CompNumb2)
- Dividing Complex Numbers (https://openstax.org/l/37CompNumb3)
- Rewriting Powers of i (https://openstax.org/l/37CompNumb4)

8.8 EXERCISES

Practice Makes Perfect

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

409.
ⓐ $\sqrt{-16}$
ⓑ $\sqrt{-11}$
ⓒ $\sqrt{-8}$

410.
ⓐ $\sqrt{-121}$
ⓑ $\sqrt{-1}$
ⓒ $\sqrt{-20}$

411.
ⓐ $\sqrt{-100}$
ⓑ $\sqrt{-13}$
ⓒ $\sqrt{-45}$

412.
ⓐ $\sqrt{-49}$
ⓑ $\sqrt{-15}$
ⓒ $\sqrt{-75}$

Add or Subtract Complex Numbers

In the following exercises, add or subtract.

413. $\sqrt{-75}+\sqrt{-48}$

414. $\sqrt{-12}+\sqrt{-75}$

415. $\sqrt{-50}+\sqrt{-18}$

416. $\sqrt{-72}+\sqrt{-8}$

417. $(1+3i)+(7+4i)$

418. $(6+2i)+(3-4i)$

419. $(8-i)+(6+3i)$

420. $(7-4i)+(-2-6i)$

421. $(1-4i)-(3-6i)$

422. $(8-4i)-(3+7i)$

423. $(6+i)-(-2-4i)$

424. $(-2+5i)-(-5+6i)$

425. $\left(5-\sqrt{-36}\right)+\left(2-\sqrt{-49}\right)$

426. $\left(-3+\sqrt{-64}\right)+\left(5-\sqrt{-16}\right)$

427. $\left(-7-\sqrt{-50}\right)-\left(-32-\sqrt{-18}\right)$

428. $\left(-5+\sqrt{-27}\right)-\left(-4-\sqrt{-48}\right)$

Multiply Complex Numbers

In the following exercises, multiply.

429. $4i(5-3i)$

430. $2i(-3+4i)$

431. $-6i(-3-2i)$

432. $-i(6+5i)$

433. $(4+3i)(-5+6i)$

434. $(-2-5i)(-4+3i)$

435. $(-3+3i)(-2-7i)$

436. $(-6-2i)(-3-5i)$

In the following exercises, multiply using the Product of Binomial Squares Pattern.

437. $(3+4i)^2$

438. $(-1+5i)^2$

439. $(-2-3i)^2$

440. $(-6-5i)^2$

In the following exercises, multiply.

441. $\sqrt{-25}\cdot\sqrt{-36}$

442. $\sqrt{-4}\cdot\sqrt{-16}$

443. $\sqrt{-9}\cdot\sqrt{-100}$

444. $\sqrt{-64} \cdot \sqrt{-9}$

445. $(-2 - \sqrt{-27})(4 - \sqrt{-48})$

446. $(5 - \sqrt{-12})(-3 + \sqrt{-75})$

447. $(2 + \sqrt{-8})(-4 + \sqrt{-18})$

448. $(5 + \sqrt{-18})(-2 - \sqrt{-50})$

449. $(2 - i)(2 + i)$

450. $(4 - 5i)(4 + 5i)$

451. $(7 - 2i)(7 + 2i)$

452. $(-3 - 8i)(-3 + 8i)$

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

453. $(7 - i)(7 + i)$

454. $(6 - 5i)(6 + 5i)$

455. $(9 - 2i)(9 + 2i)$

456. $(-3 - 4i)(-3 + 4i)$

Divide Complex Numbers

In the following exercises, divide.

457. $\dfrac{3 + 4i}{4 - 3i}$

458. $\dfrac{5 - 2i}{2 + 5i}$

459. $\dfrac{2 + i}{3 - 4i}$

460. $\dfrac{3 - 2i}{6 + i}$

461. $\dfrac{3}{2 - 3i}$

462. $\dfrac{2}{4 - 5i}$

463. $\dfrac{-4}{3 - 2i}$

464. $\dfrac{-1}{3 + 2i}$

465. $\dfrac{1 + 4i}{3i}$

466. $\dfrac{4 + 3i}{7i}$

467. $\dfrac{-2 - 3i}{4i}$

468. $\dfrac{-3 - 5i}{2i}$

Simplify Powers of *i*

In the following exercises, simplify.

469. i^{41}

470. i^{39}

471. i^{66}

472. i^{48}

473. i^{128}

474. i^{162}

475. i^{137}

476. i^{255}

Writing Exercises

477. Explain the relationship between real numbers and complex numbers.

478. Aniket multiplied as follows and he got the wrong answer. What is wrong with his reasoning?

$$\begin{array}{c} \sqrt{-7} \cdot \sqrt{-7} \\ \sqrt{49} \\ 7 \end{array}$$

479. Why is $\sqrt{-64} = 8i$ but $\sqrt[3]{-64} = -4$.

480. Explain how dividing complex numbers is similar to rationalizing a denominator.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
evaluate the square root of a negative number.			
add or subtract complex numbers.			
multiply complex numbers.			
divide complex numbers.			
simplify powers of *i*.			

ⓑ *On a scale of* $1-10,$ *how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

CHAPTER 8 REVIEW

KEY TERMS

complex conjugate pair A complex conjugate pair is of the form $a + bi$, $a - bi$.

complex number A complex number is of the form $a + bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit The imaginary unit i is the number whose square is –1. $i^2 = -1$ or $i = \sqrt{-1}$.

like radicals Like radicals are radical expressions with the same index and the same radicand.

radical equation An equation in which a variable is in the radicand of a radical expression is called a radical equation.

radical function A radical function is a function that is defined by a radical expression.

rationalizing the denominator Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

square of a number If $n^2 = m$, then m is the square of n.

square root of a number If $n^2 = m$, then n is a square root of m.

standard form A complex number is in standard form when written as $a + bi$, where a, b are real numbers.

KEY CONCEPTS

8.1 Simplify Expressions with Roots

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of m'
  - If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

  - The square root of m, $\sqrt{m}$, is a positive number whose square is m.
- **n^{th} Root of a Number**
  - If $b^n = a$, then b is an n^{th} root of a.
  - The principal n^{th} root of a is written $\sqrt[n]{a}$.
  - n is called the *index* of the radical.
- **Properties of $\sqrt[n]{a}$**
  - When n is an even number and
    - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
    - $a < 0$, then $\sqrt[n]{a}$ is not a real number
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Simplifying Odd and Even Roots**
  - For any integer $n \geq 2$,
    - when n is odd $\sqrt[n]{a^n} = a$
    - when n is even $\sqrt[n]{a^n} = |a|$
  - We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

8.2 Simplify Radical Expressions

- **Simplified Radical Expression**

- For real numbers *a*, *m* and $n \geq 2$

  $\sqrt[n]{a}$ is considered simplified if *a* has no factors of m^n

- **Product Property of n^th Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

    $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

- **How to simplify a radical expression using the Product Property**

  Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.

  Step 2. Use the product rule to rewrite the radical as the product of two radicals.

  Step 3. Simplify the root of the perfect power.

- **Quotient Property of Radical Expressions**
  - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,

    $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

- **How to simplify a radical expression using the Quotient Property.**

  Step 1. Simplify the fraction in the radicand, if possible.

  Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.

  Step 3. Simplify the radicals in the numerator and the denominator.

8.3 Simplify Rational Exponents

- **Rational Exponent** $a^{\frac{1}{n}}$
  - If $\sqrt[n]{a}$ is a real number and $n \geq 2,$ then $a^{\frac{1}{n}} = \sqrt[n]{a}.$
- **Rational Exponent** $a^{\frac{m}{n}}$
  - For any positive integers *m* and *n*,

    $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

- **Properties of Exponents**
  - If *a, b* are real numbers and *m, n* are rational numbers, then
    - **Product Property** $a^m \cdot a^n = a^{m+n}$
    - **Power Property** $(a^m)^n = a^{m \cdot n}$
    - **Product to a Power** $(ab)^m = a^m b^m$
    - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n},\ a \neq 0$
    - **Zero Exponent Definition** $a^0 = 1, \quad a \neq 0$
    - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$
    - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n},\ a \neq 0$

8.4 Add, Subtract, and Multiply Radical Expressions

- **Product Property of Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

- **Special Products**

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a+b)(a-b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

8.5 Divide Radical Expressions

- **Quotient Property of Radical Expressions**
    - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,
      $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- **Simplified Radical Expressions**
    - A radical expression is considered simplified if there are:
        - no factors in the radicand that have perfect powers of the index
        - no fractions in the radicand
        - no radicals in the denominator of a fraction

8.6 Solve Radical Equations

- **Binomial Squares**
  $(a+b)^2 = a^2 + 2ab + b^2$
  $(a-b)^2 = a^2 - 2ab + b^2$
- **Solve a Radical Equation**
  Step 1. Isolate one of the radical terms on one side of the equation.
  Step 2. Raise both sides of the equation to the power of the index.
  Step 3. Are there any more radicals?
  If yes, repeat Step 1 and Step 2 again.
  If no, solve the new equation.
  Step 4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**
  Step 1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  Step 2. Identify what we are looking for.
  Step 3. Name what we are looking for by choosing a variable to represent it.
  Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  Step 5. Solve the equation using good algebra techniques.
  Step 6. Check the answer in the problem and make sure it makes sense.
  Step 7. Answer the question with a complete sentence.
- **Falling Objects**
    - On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
    - If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

8.7 Use Radicals in Functions

- **Properties of $\sqrt[n]{a}$**
  - When *n* is an **even** number and:
    $a \geq 0,$ then $\sqrt[n]{a}$ is a real number.
    $a < 0,$ then $\sqrt[n]{a}$ is not a real number.
  - When *n* is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of *a*.
- **Domain of a Radical Function**
  - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
  - When the **index** of the radical is **odd**, the radicand can be any real number.

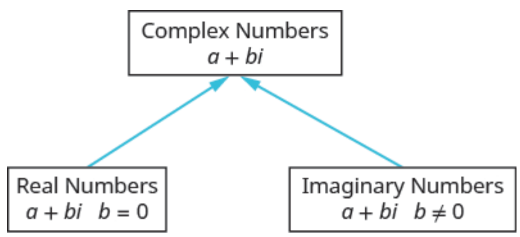
8.8 Use the Complex Number System

- **Square Root of a Negative Number**
  - If *b* is a positive real number, then $\sqrt{-b} = \sqrt{b}i$

	$a + bi$	
$b = 0$	$a + 0 \cdot i$ a	Real number
$b \neq 0$	$a + bi$	Imaginary number
$a = 0$	$0 + bi$ bi	Pure imaginary number

Table 8.32

  - A complex number is in **standard form** when written as *a* + *bi*, where *a, b* are real numbers.



- **Product of Complex Conjugates**
  - If *a, b* are real numbers, then
    $(a - bi)(a + bi) = a^2 + b^2$
- **How to Divide Complex Numbers**

Step 1. Write both the numerator and denominator in standard form.
Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.
Step 3. Simplify and write the result in standard form.

REVIEW EXERCISES

8.1 8.1 Simplify Expressions with Roots

Simplify Expressions with Roots

In the following exercises, simplify.

481. ⓐ $\sqrt{225}$ ⓑ $-\sqrt{16}$

482. ⓐ $-\sqrt{169}$ ⓑ $\sqrt{-8}$

483. ⓐ $\sqrt[3]{8}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{243}$

484. ⓐ $\sqrt[3]{-512}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1}$

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

485. ⓐ $\sqrt{68}$ ⓑ $\sqrt[3]{84}$

In the following exercises, approximate each root and round to two decimal places.

486. ⓐ $\sqrt{37}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{125}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

487.
ⓐ $\sqrt[3]{a^3}$
ⓑ $\sqrt[7]{b^7}$

488.
ⓐ $\sqrt{a^{14}}$
ⓑ $\sqrt{w^{24}}$

489.
ⓐ $\sqrt[4]{m^8}$
ⓑ $\sqrt[5]{n^{20}}$

490.
ⓐ $\sqrt{121m^{20}}$
ⓑ $-\sqrt{64a^2}$

491.
ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

492.
ⓐ $\sqrt{144x^2y^2}$
ⓑ $\sqrt{169w^8y^{10}}$
ⓒ $\sqrt[3]{8a^{51}b^6}$

8.2 8.2 Simplify Radical Expressions

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

493. $\sqrt{125}$

494. $\sqrt{675}$

495. ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

In the following exercises, simplify using absolute value signs as needed.

496.
ⓐ $\sqrt{a^{23}}$
ⓑ $\sqrt[3]{b^8}$
ⓒ $\sqrt[8]{c^{13}}$

497.
ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
ⓒ $\sqrt[6]{128b^7}$

498.
ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
ⓒ $\sqrt[4]{80x^8y^9}$

499.
ⓐ $\sqrt[5]{-32}$
ⓑ $\sqrt[8]{-1}$

500.
ⓐ $8+\sqrt{96}$
ⓑ $\dfrac{2+\sqrt{40}}{2}$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

501. ⓐ $\sqrt{\frac{72}{98}}$ ⓑ $\sqrt[3]{\frac{24}{81}}$ ⓒ $\sqrt[4]{\frac{6}{96}}$

502. ⓐ $\sqrt{\frac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

503. $\sqrt{\frac{300m^5}{64}}$

504.
ⓐ $\sqrt{\frac{28p^7}{q^2}}$
ⓑ $\sqrt[3]{\frac{81s^8}{t^3}}$
ⓒ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

505.
ⓐ $\sqrt{\frac{27p^2q}{108p^4q^3}}$
ⓑ $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$
ⓒ $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

506.
ⓐ $\frac{\sqrt{80q^5}}{\sqrt{5q}}$
ⓑ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$
ⓒ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

8.3 8.3 Simplify Rational Exponents

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

507. ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

In the following exercises, write with a rational exponent.

508. ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $4\sqrt[6]{36r}$

In the following exercises, simplify.

509.
ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
ⓒ $32^{\frac{1}{5}}$

510.
ⓐ $(-1{,}000)^{\frac{1}{3}}$
ⓑ $-1{,}000^{\frac{1}{3}}$
ⓒ $(1{,}000)^{-\frac{1}{3}}$

511.
ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

512.
ⓐ $\sqrt[4]{r^7}$
ⓑ $\left(\sqrt[5]{2pq}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

In the following exercises, simplify.

513.

ⓐ $25^{\frac{3}{2}}$

ⓑ $9^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{2}{3}}$

514.

ⓐ $-64^{\frac{3}{2}}$

ⓑ $-64^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{3}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

515.

ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$

ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$

ⓒ $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

516.

ⓐ $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

ⓑ $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

8.4 8.4 Add, Subtract and Multiply Radical Expressions

Add and Subtract Radical Expressions

In the following exercises, simplify.

517.

ⓐ $7\sqrt{2} - 3\sqrt{2}$

ⓑ $7\sqrt[3]{p} + 2\sqrt[3]{p}$

ⓒ $5\sqrt[3]{x} - 3\sqrt[3]{x}$

518.

ⓐ $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$

ⓑ $8\sqrt[4]{11cd} + 5\sqrt[4]{11cd} - 9\sqrt[4]{11cd}$

519.

ⓐ $\sqrt{48} + \sqrt{27}$

ⓑ $\sqrt[3]{54} + \sqrt[3]{128}$

ⓒ $6\sqrt[4]{5} - \frac{3}{2}\sqrt[4]{320}$

520.

ⓐ $\sqrt{80c^7} - \sqrt{20c^7}$

ⓑ $2\sqrt[4]{162r^{10}} + 4\sqrt[4]{32r^{10}}$

521. $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

Multiply Radical Expressions

In the following exercises, simplify.

522.

ⓐ $(5\sqrt{6})(-\sqrt{12})$

ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

523.

ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$

ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

524.

ⓐ $\sqrt{11}\left(8 + 4\sqrt{11}\right)$

ⓑ $\sqrt[3]{3}\left(\sqrt[3]{9} + \sqrt[3]{18}\right)$

525.

ⓐ $(3 - 2\sqrt{7})(5 - 4\sqrt{7})$

ⓑ $\left(\sqrt[3]{x} - 5\right)\left(\sqrt[3]{x} - 3\right)$

526. $\left(2\sqrt{7} - 5\sqrt{11}\right)\left(4\sqrt{7} + 9\sqrt{11}\right)$

527.
ⓐ $\left(4+\sqrt{11}\right)^2$
ⓑ $\left(3-2\sqrt{5}\right)^2$

528. $\left(7+\sqrt{10}\right)\left(7-\sqrt{10}\right)$

529. $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

8.5 8.5 Divide Radical Expressions

Divide Square Roots

In the following exercises, simplify.

530.
ⓐ $\dfrac{\sqrt{48}}{\sqrt{75}}$
ⓑ $\dfrac{\sqrt[3]{81}}{\sqrt[3]{24}}$

531.
ⓐ $\dfrac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$
ⓑ $\dfrac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

532. ⓐ $\dfrac{8}{\sqrt{3}}$ ⓑ $\sqrt{\dfrac{7}{40}}$ ⓒ $\dfrac{8}{\sqrt{2y}}$

533. ⓐ $\dfrac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\dfrac{7}{54}}$ ⓒ $\dfrac{3}{\sqrt[3]{3x^2}}$

534. ⓐ $\dfrac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\dfrac{9}{32}}$ ⓒ $\dfrac{6}{\sqrt[4]{9x^3}}$

Rationalize a Two Term Denominator

In the following exercises, simplify.

535. $\dfrac{7}{2-\sqrt{6}}$

536. $\dfrac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

537. $\dfrac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

8.6 8.6 Solve Radical Equations

Solve Radical Equations

In the following exercises, solve.

538. $\sqrt{4x-3}=7$

539. $\sqrt{5x+1}=-3$

540. $\sqrt[3]{4x-1}=3$

541. $\sqrt{u-3}+3=u$

542. $\sqrt[3]{4x+5}-2=-5$

543. $(8x+5)^{\frac{1}{3}}+2=-1$

544. $\sqrt{y+4}-y+2=0$

545. $2\sqrt{8r+1}-8=2$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

546. $\sqrt{10+2c}=\sqrt{4c+16}$

547.
$\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

548. $\sqrt{r}+6=\sqrt{r+8}$

549. $\sqrt{x+1}-\sqrt{x-2}=1$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

550. Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

551. Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

8.7 8.7 Radical Functions

Evaluate a Radical Function

In the following exercises, evaluate each function.

552. $g(x) = \sqrt{6x+1}$, find

ⓐ $g(4)$

ⓑ $g(8)$

553. $G(x) = \sqrt{5x-1}$, find

ⓐ $G(5)$

ⓑ $G(2)$

554. $h(x) = \sqrt[3]{x^2-4}$, find

ⓐ $h(-2)$

ⓑ $h(6)$

555. For the function $g(x) = \sqrt[4]{4-4x}$, find

ⓐ $g(1)$

ⓑ $g(-3)$

Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

556. $g(x) = \sqrt{2-3x}$

557. $F(x) = \sqrt{\dfrac{x+3}{x-2}}$

558. $f(x) = \sqrt[3]{4x^2-16}$

559. $F(x) = \sqrt[4]{10-7x}$

Graph Radical Functions

In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

560. $g(x) = \sqrt{x+4}$

561. $g(x) = 2\sqrt{x}$

562. $f(x) = \sqrt[3]{x-1}$

563. $f(x) = \sqrt[3]{x}+3$

8.8 8.8 The Complex Number System

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

564.

ⓐ $\sqrt{-100}$

ⓑ $\sqrt{-13}$

ⓒ $\sqrt{-45}$

Add or Subtract Complex Numbers

In the following exercises, add or subtract.

565. $\sqrt{-50}+\sqrt{-18}$

566. $(8-i)+(6+3i)$

567. $(6+i)-(-2-4i)$

568. $\left(-7-\sqrt{-50}\right)-\left(-32-\sqrt{-18}\right)$

Multiply Complex Numbers

In the following exercises, multiply.

569. $(-2-5i)(-4+3i)$

570. $-6i(-3-2i)$

571. $\sqrt{-4}\cdot\sqrt{-16}$

572. $\left(5-\sqrt{-12}\right)\left(-3+\sqrt{-75}\right)$

In the following exercises, multiply using the Product of Binomial Squares Pattern.

573. $(-2-3i)^2$

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

574. $(9-2i)(9+2i)$

Divide Complex Numbers

In the following exercises, divide.

575. $\dfrac{2+i}{3-4i}$

576. $\dfrac{-4}{3-2i}$

Simplify Powers of *i*

In the following exercises, simplify.

577. i^{48}

578. i^{255}

PRACTICE TEST

In the following exercises, simplify using absolute values as necessary.

579. $\sqrt[3]{125x^9}$

580. $\sqrt{169x^8y^6}$

581. $\sqrt[3]{72x^8y^4}$

582. $\sqrt{\dfrac{45x^3y^4}{180x^5y^2}}$

In the following exercises, simplify. Assume all variables are positive.

583. ⓐ $216^{-\frac{1}{4}}$ ⓑ $-49^{\frac{3}{2}}$

584. $\sqrt{-45}$

585. $\dfrac{x^{-\frac{1}{4}} \cdot x^{\frac{5}{4}}}{x^{-\frac{3}{4}}}$

586. $\left(\dfrac{8x^{\frac{2}{3}}y^{-\frac{5}{2}}}{x^{-\frac{7}{3}}y^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

587. $\sqrt{48x^5} - \sqrt{75x^5}$

588. $\sqrt{27x^2} - 4x\sqrt{12} + \sqrt{108x^2}$

589. $2\sqrt{12x^5} \cdot 3\sqrt{6x^3}$

590. $\sqrt[3]{4}\left(\sqrt[3]{16} - \sqrt[3]{6}\right)$

591. $\left(4 - 3\sqrt{3}\right)\left(5 + 2\sqrt{3}\right)$

592. $\dfrac{\sqrt[3]{128}}{\sqrt[3]{54}}$

593. $\dfrac{\sqrt{245xy^{-4}}}{\sqrt{45x^{-4}y^3}}$

594. $\dfrac{1}{\sqrt[3]{5}}$

595. $\dfrac{3}{2 + \sqrt{3}}$

596. $\sqrt{-4} \cdot \sqrt{-9}$

597. $-4i(-2 - 3i)$

598. $\dfrac{4 + i}{3 - 2i}$

599. i^{172}

In the following exercises, solve.

600. $\sqrt{2x + 5} + 8 = 6$

601. $\sqrt{x + 5} + 1 = x$

602. $\sqrt[3]{2x^2 - 6x - 23} = \sqrt[3]{x^2 - 3x + 5}$

In the following exercise, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

603. $g(x) = \sqrt{x + 2}$