9 QUADRATIC EQUATIONS AND FUNCTIONS

Figure 9.1 Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology. (credit: “intographics”/Pixabay)

Chapter Outline

Introduction

Blink your eyes. You’ve taken a photo. That’s what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

9.1 Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x - h)^2 = k$ using the Square Root Property

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{128}$.
   If you missed this problem, review Example 8.13.
2. Simplify: $\sqrt{\frac{32}{5}}$.

If you missed this problem, review **Example 8.50**.

3. Factor: $9x^2 - 12x + 4$.
   If you missed this problem, review **Example 6.23**.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the diffe ence of squares.	$(x-3)(x+3) = 0$
Use the Zero Product Property.	$x - 3 = 0 \quad x - 3 = 0$
Solve each equation.	$x = 3 \quad x = -3$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4,\ x = -4$ and $x = 5,\ x = -5$.

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so −13 is also a square root of 169. Therefore, both 13 and −13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}.$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

	$x^2 = 9$
Use the Square Root Property.	$x = \pm\sqrt{9}$
	$x = \pm 3$
	So $x = 3$ or $x = -3$.

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

	$x^2 = 7$
Use the Square Root Property.	$x = \sqrt{7}, \quad x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

EXAMPLE 9.1 HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $AX^2 = K$ USING THE SQUARE ROOT PROPERTY

Solve: $x^2 - 50 = 0$.

Solution

Step 1. Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$
Step 2. Use Square Root Property.	Remember to write the $\pm$ symbol.	$x = \pm\sqrt{50}$
Step 3. Simplify the radical.	Rewrite to show two solutions.	$x = \pm\sqrt{25} \cdot \sqrt{2}$ $x = \pm 5\sqrt{2}$ $x = 5\sqrt{2},\ x = -5\sqrt{2}$
Step 4. Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0\ \checkmark$ $x^2 - 50 = 0$ $(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0\ \checkmark$

TRY IT : : 9.1 Solve: $x^2 - 48 = 0$.

TRY IT : : 9.2 Solve: $y^2 - 27 = 0$.

The steps to take to use the Square Root Property to solve a quadratic equation are listed here.

HOW TO : : SOLVE A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY.

Step 1. Isolate the quadratic term and make its coefficient one.
Step 2. Use Square Root Property.
Step 3. Simplify the radical.
Step 4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

EXAMPLE 9.2

Solve: $3z^2 = 108$.

Solution

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6, \quad z = -6$
Check the solutions:	

$$\begin{array}{rr} 3z^2 = 108 & 3z^2 = 108 \\ 3(6)^2 \stackrel{?}{=} 108 & 3(-6)^2 \stackrel{?}{=} 108 \\ 3(36) \stackrel{?}{=} 108 & 3(36) \stackrel{?}{=} 108 \\ 108 = 108\,\checkmark & 108 = 108\,\checkmark \end{array}$$

TRY IT : : 9.3 Solve: $2x^2 = 98$.

TRY IT : : 9.4 Solve: $5m^2 = 80$.

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

EXAMPLE 9.3

Solve: $x^2 + 72 = 0$.

Solution

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm\sqrt{-72}$
Simplify using complex numbers.	$x = \pm\sqrt{72}\,i$
Simplify the radical.	$x = \pm 6\sqrt{2}\,i$
Rewrite to show two solutions.	$x = 6\sqrt{2}\,i, \quad x = -6\sqrt{2}\,i$

Check the solutions:

$$\begin{array}{rr} x^2+72=0 & x^2+72=0 \\ \left(6\sqrt{2}\,i\right)^2+72\stackrel{?}{=}0 & \left(6\sqrt{2}\,i\right)^2+72\stackrel{?}{=}0 \\ 6^2\left(\sqrt{2}\right)^2 i^2+72\stackrel{?}{=}0 & (-6)^2\left(\sqrt{2}\right)^2 i^2+72\stackrel{?}{=}0 \\ 36\cdot 2\cdot(-1)+72\stackrel{?}{=}0 & 36\cdot 2\cdot(-1)+72\stackrel{?}{=}0 \\ 0=0\checkmark & 0=0\checkmark \end{array}$$

TRY IT : : 9.5 Solve: $c^2+12=0$.

TRY IT : : 9.6 Solve: $q^2+24=0$.

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

EXAMPLE 9.4

Solve: $\frac{2}{3}u^2+5=17$.

Solution

	$\frac{2}{3}u^2+5=17$
Isolate the quadratic term.	$\frac{2}{3}u^2=12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2}\cdot\frac{2}{3}u^2=\frac{3}{2}\cdot 12$
Simplify.	$u^2=18$
Use the Square Root Property.	$u=\pm\sqrt{18}$
Simplify the radical.	$u=\pm\sqrt{9\cdot 2}$
Simplify.	$u=\pm 3\sqrt{2}$
Rewrite to show two solutions.	$u=3\sqrt{2},\quad u=-3\sqrt{2}$

Check:

$$\begin{array}{rr} \frac{2}{3}u^2+5=17 & \frac{2}{3}u^2+5=17 \\ \frac{2}{3}\left(3\sqrt{2}\right)^2+5\stackrel{?}{=}17 & \frac{2}{3}\left(-3\sqrt{2}\right)^2+5\stackrel{?}{=}17 \\ \frac{2}{3}\cdot 18+5\stackrel{?}{=}17 & \frac{2}{3}\cdot 18+5\stackrel{?}{=}17 \\ 12+5\stackrel{?}{=}17 & 12+5\stackrel{?}{=}17 \\ 17=17\checkmark & 17=17\checkmark \end{array}$$

TRY IT : : 9.7 Solve: $\frac{1}{2}x^2+4=24$.

TRY IT : : 9.8 Solve: $\frac{3}{4}y^2 - 3 = 18$.

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

EXAMPLE 9.5

Solve: $2x^2 - 8 = 41$.

Solution

	$2x^2 - 8 = 41$
Isolate the quadratic term.	$2x^2 = 49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2} = \frac{49}{2}$
Simplify.	$x^2 = \frac{49}{2}$
Use the Square Root Property.	$x = \pm\sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x = \pm\frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$x = \pm\frac{\sqrt{49} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$
Simplify.	$x = \pm\frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$x = \frac{7\sqrt{2}}{2}, \quad x = -\frac{7\sqrt{2}}{2}$
Check: We leave the check for you.	

TRY IT : : 9.9 Solve: $5r^2 - 2 = 34$.

TRY IT : : 9.10 Solve: $3t^2 + 6 = 70$.

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation of the form $a(x - h)^2 = k$ as well. Notice that the quadratic term, x, in the original form $ax^2 = k$ is replaced with $(x - h)$.

$$ax^2 = k \qquad a(x - h)^2 = k$$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x - h)^2$.

EXAMPLE 9.6

Solve: $4(y - 7)^2 = 48$.

Solution

	$4(y-7)^2 = 48$
Divide both sides by the coefficient 4.	$(y-7)^2 = 12$
Use the Square Root Property on the binomial	$y - 7 = \pm\sqrt{12}$
Simplify the radical.	$y - 7 = \pm 2\sqrt{3}$
Solve for y.	$y = 7 \pm 2\sqrt{3}$
Rewrite to show two solutions.	$y = 7 + 2\sqrt{3}, \quad y = 7 - 2\sqrt{3}$

Check:

$$\begin{array}{rr} 4(y-7)^2 = 48 & 4(y-7)^2 = 48 \\ 4\left(7+2\sqrt{3}-7\right)^2 \stackrel{?}{=} 48 & 4\left(7-2\sqrt{3}-7\right)^2 \stackrel{?}{=} 48 \\ 4\left(2\sqrt{3}\right)^2 \stackrel{?}{=} 48 & 4\left(-2\sqrt{3}\right)^2 \stackrel{?}{=} 48 \\ 4(12) \stackrel{?}{=} 48 & 4(12) \stackrel{?}{=} 48 \\ 48 = 48\,\checkmark & 48 = 48\,\checkmark \end{array}$$

TRY IT : : 9.11 Solve: $3(a-3)^2 = 54.$

TRY IT : : 9.12 Solve: $2(b+2)^2 = 80.$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

EXAMPLE 9.7

Solve: $\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}.$

Solution

$$\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$$

Use the Square Root Property. $x - \frac{1}{3} = \pm\sqrt{\frac{5}{9}}$

Rewrite the radical as a fraction of square roots. $x - \frac{1}{3} = \pm\frac{\sqrt{5}}{\sqrt{9}}$

Simplify the radical. $x - \frac{1}{3} = \pm\frac{\sqrt{5}}{3}$

Solve for x. $x = \frac{1}{3} \pm \frac{\sqrt{5}}{3}$

Rewrite to show two solutions. $x = \frac{1}{3} + \frac{\sqrt{5}}{3}, \quad x = \frac{1}{3} - \frac{\sqrt{5}}{3}$

Check:
We leave the check for you.

TRY IT : : 9.13 Solve: $\left(x - \frac{1}{2}\right)^2 = \frac{5}{4}$.

TRY IT : : 9.14 Solve: $\left(y + \frac{3}{4}\right)^2 = \frac{7}{16}$.

We will start the solution to the next example by isolating the binomial term.

EXAMPLE 9.8

Solve: $2(x - 2)^2 + 3 = 57$.

Solution

$$2(x - 2)^2 + 3 = 57$$

Subtract 3 from both sides to isolate the binomial term. $2(x - 2)^2 = 54$

Divide both sides by 2. $(x - 2)^2 = 27$

Use the Square Root Property. $x - 2 = \pm\sqrt{27}$

Simplify the radical. $x - 2 = \pm 3\sqrt{3}$

Solve for x. $x = 2 \pm 3\sqrt{3}$

Rewrite to show two solutions. $x = 2 + 3\sqrt{3}, \quad x = 2 - 3\sqrt{3}$

Check:
We leave the check for you.

TRY IT : : 9.15 Solve: $5(a - 5)^2 + 4 = 104$.

TRY IT : : 9.16 Solve: $3(b + 3)^2 - 8 = 88$.

Sometimes the solutions are complex numbers.

EXAMPLE 9.9

Solve: $(2x-3)^2 = -12$.

Solution

	$(2x-3)^2 = -12$
Use the Square Root Property.	$2x-3 = \pm\sqrt{-12}$
Simplify the radical.	$2x-3 = \pm 2\sqrt{3}\,i$
Add 3 to both sides.	$2x = 3 \pm 2\sqrt{3}\,i$
Divide both sides by 2.	$x = \dfrac{3 \pm 2\sqrt{3}\,i}{2}$
Rewrite in standard form.	$x = \dfrac{3}{2} \pm \dfrac{2\sqrt{3}\,i}{2}$
Simplify.	$x = \dfrac{3}{2} \pm \sqrt{3}\,i$
Rewrite to show two solutions.	$x = \dfrac{3}{2} + \sqrt{3}\,i, \quad x = \dfrac{3}{2} - \sqrt{3}\,i$

Check:
We leave the check for you.

TRY IT : : 9.17 Solve: $(3r+4)^2 = -8$.

TRY IT : : 9.18 Solve: $(2t-8)^2 = -10$.

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

EXAMPLE 9.10

Solve: $4n^2 + 4n + 1 = 16$.

Solution

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2 + 4n + 1 = 16$
Factor the perfect square trinomial.	$(2n+1)^2 = 16$
Use the Square Root Property.	$2n+1 = \pm\sqrt{16}$
Simplify the radical.	$2n+1 = \pm 4$
Solve for n.	$2n = -1 \pm 4$
Divide each side by 2.	$\dfrac{2n}{2} = \dfrac{-1 \pm 4}{2}$ $n = \dfrac{-1 \pm 4}{2}$
Rewrite to show two solutions.	$n = \dfrac{-1+4}{2}, \quad n = \dfrac{-1-4}{2}$
Simplify each equation.	$n = \dfrac{3}{2}, \qquad n = -\dfrac{5}{2}$

Check:

$$4n^2 + 4n + 1 = 16 \qquad 4n^2 + 4n + 1 = 16$$
$$4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad 4\left(-\frac{5}{2}\right)^2 + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$
$$4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad 4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$
$$9 + 6 + 1 \stackrel{?}{=} 16 \qquad 25 - 10 + 1 \stackrel{?}{=} 16$$
$$16 = 16\checkmark \qquad 16 = 16\checkmark$$

TRY IT : : 9.19 Solve: $9m^2 - 12m + 4 = 25$.

TRY IT : : 9.20 Solve: $16n^2 + 40n + 25 = 4$.

MEDIA : :

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- **Solving Quadratic Equations: The Square Root Property (https://openstax.org/l/37SqRtProp1)**
- **Using the Square Root Property to Solve Quadratic Equations (https://openstax.org/l/37SqRtProp2)**

9.1 EXERCISES

Practice Makes Perfect

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

1. $a^2 = 49$

2. $b^2 = 144$

3. $r^2 - 24 = 0$

4. $t^2 - 75 = 0$

5. $u^2 - 300 = 0$

6. $v^2 - 80 = 0$

7. $4m^2 = 36$

8. $3n^2 = 48$

9. $\frac{4}{3}x^2 = 48$

10. $\frac{5}{3}y^2 = 60$

11. $x^2 + 25 = 0$

12. $y^2 + 64 = 0$

13. $x^2 + 63 = 0$

14. $y^2 + 45 = 0$

15. $\frac{4}{3}x^2 + 2 = 110$

16. $\frac{2}{3}y^2 - 8 = -2$

17. $\frac{2}{5}a^2 + 3 = 11$

18. $\frac{3}{2}b^2 - 7 = 41$

19. $7p^2 + 10 = 26$

20. $2q^2 + 5 = 30$

21. $5y^2 - 7 = 25$

22. $3x^2 - 8 = 46$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

23. $(u - 6)^2 = 64$

24. $(v + 10)^2 = 121$

25. $(m - 6)^2 = 20$

26. $(n + 5)^2 = 32$

27. $\left(r - \frac{1}{2}\right)^2 = \frac{3}{4}$

28. $\left(x + \frac{1}{5}\right)^2 = \frac{7}{25}$

29. $\left(y + \frac{2}{3}\right)^2 = \frac{8}{81}$

30. $\left(t - \frac{5}{6}\right)^2 = \frac{11}{25}$

31. $(a - 7)^2 + 5 = 55$

32. $(b - 1)^2 - 9 = 39$

33. $4(x + 3)^2 - 5 = 27$

34. $5(x + 3)^2 - 7 = 68$

35. $(5c + 1)^2 = -27$

36. $(8d - 6)^2 = -24$

37. $(4x - 3)^2 + 11 = -17$

38. $(2y + 1)^2 - 5 = -23$

39. $m^2 - 4m + 4 = 8$

40. $n^2 + 8n + 16 = 27$

41. $x^2 - 6x + 9 = 12$

42. $y^2 + 12y + 36 = 32$

43. $25x^2 - 30x + 9 = 36$

44. $9y^2 + 12y + 4 = 9$

45. $36x^2 - 24x + 4 = 81$

46. $64x^2 + 144x + 81 = 25$

Mixed Practice

In the following exercises, solve using the Square Root Property.

47. $2r^2 = 32$

48. $4t^2 = 16$

49. $(a-4)^2 = 28$

50. $(b+7)^2 = 8$

51. $9w^2 - 24w + 16 = 1$

52. $4z^2 + 4z + 1 = 49$

53. $a^2 - 18 = 0$

54. $b^2 - 108 = 0$

55. $\left(p - \frac{1}{3}\right)^2 = \frac{7}{9}$

56. $\left(q - \frac{3}{5}\right)^2 = \frac{3}{4}$

57. $m^2 + 12 = 0$

58. $n^2 + 48 = 0.$

59. $u^2 - 14u + 49 = 72$

60. $v^2 + 18v + 81 = 50$

61. $(m-4)^2 + 3 = 15$

62. $(n-7)^2 - 8 = 64$

63. $(x+5)^2 = 4$

64. $(y-4)^2 = 64$

65. $6c^2 + 4 = 29$

66. $2d^2 - 4 = 77$

67. $(x-6)^2 + 7 = 3$

68. $(y-4)^2 + 10 = 9$

Writing Exercises

69. In your own words, explain the Square Root Property.

70. In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)^2 = 16$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x-h)^2 = k$ using the square root property.			

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

ⓑ *If most of your checks were:*

…confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

…with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

…no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be* *overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

9.2 Solve Quadratic Equations by Completing the Square

Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square
- Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square

Be Prepared!

Before you get started, take this readiness quiz.

1. Expand: $(x + 9)^2$.
   If you missed this problem, review **Example 5.32**.
2. Factor $y^2 - 14y + 49$.
   If you missed this problem, review **Example 6.9**.
3. Factor $5n^2 + 40n + 80$.
   If you missed this problem, review **Example 6.14**.

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y - 7)^2 = 12$ because the left side was a perfect square.

$$\begin{aligned} (y-7)^2 &= 12 \\ y-7 &= \pm\sqrt{12} \\ y-7 &= \pm 2\sqrt{3} \\ y &= 7 \pm 2\sqrt{3} \end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x - k)^2$ in order to use the Square Root Property.

$$\begin{aligned} x^2 - 10x + 25 &= 18 \\ (x-5)^2 &= 18 \end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let’s look at two examples to help us recognize the patterns.

$$\begin{array}{cc} (x+9)^2 & (y-7)^2 \\ (x+9)(x+9) & (y-7)(y-7) \\ x^2+9x+9x+81 & y^2-7y-7y+49 \\ x^2+18x+81 & y^2-14y+49 \end{array}$$

We restate the patterns here for reference.

Binomial Squares Pattern

If *a* and *b* are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$$

We can use this pattern to "make" a perfect square.

We will start with the expression $x^2 + 6x$. Since there is a plus sign between the two terms, we will use the $(a + b)^2$ pattern, $a^2 + 2ab + b^2 = (a + b)^2$.

$$\overset{a^2 + 2ab + b^2}{x^2 + 6x + \_\_}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of $x^2 + 6x$ is a square, x^2. This tells us that $a = x$.

$$\overset{a^2 + \quad 2ab \quad + b^2}{x^2 + 2\cdot x\cdot b + b^2}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b = 3$.

$$\overset{a^2 + \quad 2ab \quad + b^2}{x^2 + 2\cdot 3\cdot x + \_\_}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2 = 9$.

$$\overset{a^2 + 2ab + b^2}{x^2 + 6x + 9}$$

We can now factor.

$$\overset{(a+b)^2}{(x+3)^2}$$

So we found that adding 9 to $x^2 + 6x$ 'completes the square', and we write it as $(x + 3)^2$.

HOW TO : : COMPLETE A SQUARE OF $x^2 + bx$.

Step 1. Identify b, the coefficient of x.

Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

Step 4. Factor the perfect square trinomial, writing it as a binomial squared.

EXAMPLE 9.11

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $x^2 - 26x$ ⓑ $y^2 - 9y$ ⓒ $n^2 + \frac{1}{2}n$

Solution

ⓐ

	$\begin{array}{c} x^2 - bx \\ x^2 - 26x \end{array}$
The coefficient of x is −26.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-26)\right)^2$ $(13)^2$ 169	
Add 169 to the binomial to complete the square.	$x^2 - 26x + 169$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x - 13)^2$

ⓑ

	$\begin{array}{c} x^2 - bx \\ y^2 - 9y \end{array}$
The coefficient of y is -9.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$y^2 - 9y + \frac{81}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(y - \frac{9}{2}\right)^2$

ⓒ

	$\begin{array}{c} x^2 + bx \\ n^2 + \frac{1}{2}n \end{array}$
The coefficient of n is $\frac{1}{2}$.	

Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)^2$ $\frac{1}{16}$	
Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2+\frac{1}{2}n+\frac{1}{16}$
Rewrite as a binomial square.	$\left(n+\frac{1}{4}\right)^2$

TRY IT : : 9.21

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $a^2 - 20a$ ⓑ $m^2 - 5m$ ⓒ $p^2 + \frac{1}{4}p$

TRY IT : : 9.22

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $b^2 - 4b$ ⓑ $n^2 + 13n$ ⓒ $q^2 - \frac{2}{3}q$

Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x = 40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

	$x^2+6x=40$
	$x^2+6x+\_\_ = 40+\_\_$
	$x^2+6x+9 = 40+9$
Add 9 to both sides to complete the square.	$(x+3)^2=49$

Now the equation is in the form to solve using the Square Root Property! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

EXAMPLE 9.12 HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $x^2+bx+c=0$ BY COMPLETING THE SQUARE

Solve by completing the square: $x^2+8x=48$.

Solution

Step 1. Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$\overset{x^2 + bx}{x^2 + 8x} = \overset{c}{48}$
Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2 = 16$ Add 16 to BOTH sides of the equation.	$x^2 + 8x + \underset{\left(\frac{1}{2} \cdot 8\right)^2}{\_\_\_} = 48$ $x^2 + 8x + 16 = 48 + 16$
Step 3. Factor the perfect square trinomial as a binomial square.	$x^2 + 8x + 16 = (x + 4)^2$ Add the terms on the right.	$(x + 4)^2 = 64$
Step 4. Use the Square Root Property.		$x + 4 = \pm\sqrt{64}$
Step 5. Simplify the radical and then solve the two resulting equations.		$x + 4 = \pm 8$ $x + 4 = 8 \quad x + 4 = -8$ $x = 4 \quad x = -12$
Step 6. Check the solutions.	Put each answer in the original equation to check. Substitute $x = 4$.	$x^2 + 8x = 48$ $(4)^2 + 8(4) \stackrel{?}{=} 48$ $16 + 32 \stackrel{?}{=} 48$ $48 = 48\ \checkmark$
	Substitute $x = -12$.	$x^2 + 8x = 48$ $(-12)^2 + 8(-12) \stackrel{?}{=} 48$ $144 - 96 \stackrel{?}{=} 48$ $48 = 48\ \checkmark$

TRY IT : : 9.23 Solve by completing the square: $x^2 + 4x = 5$.

TRY IT : : 9.24 Solve by completing the square: $y^2 - 10y = -9$.

The steps to solve a quadratic equation by completing the square are listed here.

HOW TO : : SOLVE A QUADRATIC EQUATION OF THE FORM $x^2 + bx + c = 0$ BY COMPLETING THE SQUARE.

Step 1. Isolate the variable terms on one side and the constant terms on the other.

Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

Step 3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right

Step 4. Use the Square Root Property.

Step 5. Simplify the radical and then solve the two resulting equations.

Step 6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

EXAMPLE 9.13

Solve by completing the square: $x^2 + 4x = -21$.

Solution

	$\overset{x^2\ +\ bx}{x^2 + 4x} = \overset{c}{-21}$
The variable terms are on the left side. Take half of 4 and square it.	$x^2 + 4x + \underset{\left(\frac{1}{2}\cdot 4\right)^2}{\underline{\qquad}} = -21$
$\left(\frac{1}{2}(4)\right)^2 = 4$	
Add 4 to both sides.	$x^2 + 4x + 4 = -21 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+2)^2 = -17$
Use the Square Root Property.	$x + 2 = \pm\sqrt{-17}$
Simplify using complex numbers.	$x + 2 = \pm\sqrt{17}\,i$
Subtract 2 from each side.	$x = -2 \pm \sqrt{17}\,i$
Rewrite to show two solutions.	$x = -2 + \sqrt{17}\,i,\quad x = -2 - \sqrt{17}\,i$
We leave the check to you.	

TRY IT : : 9.25 Solve by completing the square: $y^2 - 10y = -35$.

TRY IT : : 9.26 Solve by completing the square: $z^2 + 8z = -19$.

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

EXAMPLE 9.14

Solve by completing the square: $y^2 - 18y = -6$.

Solution

	$\overset{x^2\ -\ bx}{y^2 - 18y} = \overset{c}{-6}$
The variable terms are on the left side. Take half of -18 and square it.	
$\left(\frac{1}{2}(-18)\right)^2 = 81$	$y^2 - 18y + \underset{\left(\frac{1}{2}\cdot(-18)\right)^2}{\underline{\qquad}} = -6$
Add 81 to both sides.	$y^2 - 18y + 81 = -6 + 81$
Factor the perfect square trinomial, writing it as a binomial squared.	$(y-9)^2 = 75$

Use the Square Root Property.	$y-9=\pm\sqrt{75}$
Simplify the radical.	$y-9=\pm 5\sqrt{3}$
Solve for y.	$y=9\pm 5\sqrt{3}$
Check. $\begin{array}{r} y^2-18y=-6 \\ (9+5\sqrt{3})^2-18(9+5\sqrt{3})\stackrel{?}{=}-6 \\ 81+90\sqrt{3}+75-162+90\sqrt{3}\stackrel{?}{=}-6 \\ -6=-6\checkmark \end{array}$	$\begin{array}{r} y^2-18y=-6 \\ (9-5\sqrt{3})^2-18(9-5\sqrt{3})\stackrel{?}{=}-6 \\ 81+90\sqrt{3}+75-162+90\sqrt{3}\stackrel{?}{=}-6 \\ -6=-6\checkmark \end{array}$

Another way to check this would be to use a calculator. Evaluate y^2-18y for both of the solutions. The answer should be -6.

TRY IT : : 9.27 Solve by completing the square: $x^2-16x=-16$.

TRY IT : : 9.28 Solve by completing the square: $y^2+8y=11$.

We will start the next example by isolating the variable terms on the left side of the equation.

EXAMPLE 9.15

Solve by completing the square: $x^2+10x+4=15$.

Solution

	$x^2+10x+4=15$	
Isolate the variable terms on the left side. Subtract 4 to get the constant terms on the right side.	$x^2+10x=11$	
Take half of 10 and square it.		
$\left(\frac{1}{2}(10)\right)^2=25$	$x^2-10x+\underset{\left(\frac{1}{2}\cdot(10)\right)^2}{\underline{\qquad}}=11$	
Add 25 to both sides.	$x^2+10x+25=11+25$	
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+5)^2=36$	
Use the Square Root Property.	$x+5=\pm\sqrt{36}$	
Simplify the radical.	$x+5=\pm 6$	
Solve for x.	$x=-5\pm 6$	
Rewrite to show two solutions.	$x=-5+6,$	$x=-5-6$
Solve the equations.	$x=1,$	$x=-11$

Check:

$x^2+10x+4=15$	$x^2+10x+4=15$
$(1)^2+10(1)+4\stackrel{?}{=}15$	$(-11)^2+10(-11)+4\stackrel{?}{=}15$
$1+10+4\stackrel{?}{=}15$	$121+110+4\stackrel{?}{=}15$
$15=15\checkmark$	$15=15\checkmark$

TRY IT : : 9.29 Solve by completing the square: $a^2+4a+9=30$.

TRY IT : : 9.30 Solve by completing the square: $b^2+8b-4=16$.

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

EXAMPLE 9.16

Solve by completing the square: $n^2=3n+11$.

Solution

	$n^2=3n+11$
Subtract $3n$ to get the variable terms on the left side.	$n^2-3n=11$
Take half of -3 and square it.	
$\left(\frac{1}{2}(-3)\right)^2=\frac{9}{4}$	$n^2-3n+\underset{\left(\frac{1}{2}\cdot(-3)\right)^2}{\underline{\qquad}}=11$
Add $\frac{9}{4}$ to both sides.	$n^2-3n+\frac{9}{4}=11+\frac{9}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(n-\frac{3}{2}\right)^2=\frac{44}{4}+\frac{9}{4}$
Add the fractions on the right side.	$\left(n-\frac{3}{2}\right)^2=\frac{53}{4}$
Use the Square Root Property.	$n-\frac{3}{2}=\pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n-\frac{3}{2}=\pm\frac{\sqrt{53}}{2}$
Solve for n.	$n=\frac{3}{2}\pm\frac{\sqrt{53}}{2}$
Rewrite to show two solutions.	$n=\frac{3}{2}+\frac{\sqrt{53}}{2},\quad n=\frac{3}{2}-\frac{\sqrt{53}}{2}$
Check: We leave the check for you!	

TRY IT : : 9.31 Solve by completing the square: $p^2=5p+9$.

TRY IT : : 9.32 Solve by completing the square: $q^2 = 7q - 3$.

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the Zero Product Property since it says "If $a \cdot b = 0$, then *a* = 0 or *b* = 0." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

EXAMPLE 9.17

Solve by completing the square: $(x - 3)(x + 5) = 9$.

Solution

	$(x-3)(x+5)=9$
We multiply the binomials on the left.	$x^2+2x-15=9$
Add 15 to isolate the constant terms on the right.	$x^2+2x \quad = 24$
Take half of 2 and square it.	
$\left(\frac{1}{2} \cdot (2)\right)^2 = 1$	$x^2+2x+\underset{\left(\frac{1}{2}\cdot(2)\right)^2}{\underline{\quad\quad}}=24$
Add 1 to both sides.	$x^2+2x+1=24+1$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+1)^2=25$
Use the Square Root Property.	$x+1=\pm\sqrt{25}$
Solve for *x*.	$x=-1\pm5$
Rewrite to show two solutions.	$x=-1+5,\ x=-1-5$
Simplify.	$x=4, \quad x=-6$
Check: We leave the check for you!	

TRY IT : : 9.33 Solve by completing the square: $(c - 2)(c + 8) = 11$.

TRY IT : : 9.34 Solve by completing the square: $(d - 7)(d + 3) = 56$.

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of completing the square works best when the coefficient of x^2 is 1, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

EXAMPLE 9.18

Solve by completing the square: $3x^2 - 12x - 15 = 0$.

Solution

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2 - 12x - 15 = 0$
Factor out the greatest common factor.	$3(x^2 - 4x - 5) = 0$
Divide both sides by 3 to isolate the trinomial with coefficient 1.	$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$
Simplify.	$x^2 - 4x - 5 = 0$
Add 5 to get the constant terms on the right side.	$x^2 - 4x \quad = 5$
Take half of 4 and square it.	
$\left(\frac{1}{2}(-4)\right)^2 = 4$	$x^2 - 4x + \underset{\left(\frac{1}{2}\cdot(4)\right)^2}{\underline{\quad\quad}} = 5$
Add 4 to both sides.	$x^2 - 4x + 4 = 5 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x - 2)^2 = 9$
Use the Square Root Property.	$x - 2 = \pm\sqrt{9}$
Solve for x.	$x - 2 = \pm 3$
Rewrite to show two solutions.	$x = 2 + 3, \; x = 2 - 3$
Simplify.	$x = 5, \quad x = -1$
Check:	

$$\begin{array}{rr} x = 5 & x = -1 \\ 3x^2 - 12x - 15 = 0 & 3x^2 - 12x - 15 = 0 \\ 3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0 & 3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0 \\ 75 - 60 - 15 \stackrel{?}{=} 0 & 3 + 12 - 15 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$

TRY IT : : 9.35 Solve by completing the square: $2m^2 + 16m + 14 = 0.$

TRY IT : : 9.36 Solve by completing the square: $4n^2 - 24n - 56 = 8.$

To complete the square, the coefficient of the x^2 must be 1. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

EXAMPLE 9.19

Solve by completing the square: $2x^2 - 3x = 20.$

Solution

To complete the square we need the coefficient of x^2 to be one. We will divide both sides of the equation by the coefficient of x^2. Then we can continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it.	
$\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underset{\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2}{\underline{\qquad\qquad}} = 10$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for x.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show two solutions.	$x = \frac{3}{4} + \frac{13}{4},\quad x = \frac{3}{4} - \frac{13}{4}$
Simplify.	$x = 4,\qquad x = -\frac{5}{2}$
Check: We leave the check for you!	

TRY IT : : 9.37 Solve by completing the square: $3r^2 - 2r = 21$.

TRY IT : : 9.38 Solve by completing the square: $4t^2 + 2t = 20$.

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a quadratic equation by completing the square to include equations of the form $ax^2 + bx + c = 0$.

HOW TO : : SOLVE A QUADRATIC EQUATION OF THE FORM $ax^2 + bx + c = 0$ BY COMPLETING THE SQUARE.

Step 1. Divide by a to make the coefficient of x^2 term 1.

Step 2. Isolate the variable terms on one side and the constant terms on the other.

Step 3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

Step 4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right

Step 5. Use the Square Root Property.

Step 6. Simplify the radical and then solve the two resulting equations.

Step 7. Check the solutions.

EXAMPLE 9.20

Solve by completing the square: $3x^2 + 2x = 4$.

Solution

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2 + 2x = 4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2 + 2x}{3} = \frac{4}{3}$
Simplify.	$x^2 + \frac{2}{3}x = \frac{4}{3}$
Take half of $\frac{2}{3}$ and square it.	
$\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2 = \frac{1}{9}$	$x^2 + \frac{2}{3}x + \underline{\quad} = \frac{4}{3}$ $\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2$
Add $\frac{1}{9}$ to both sides.	$x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x + \frac{1}{3}\right)^2 = \frac{12}{9} + \frac{1}{9}$
Use the Square Root Property.	$x + \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x + \frac{1}{3} = \pm\frac{\sqrt{13}}{3}$
Solve for x .	$x = -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$
Rewrite to show two solutions.	$x = -\frac{1}{3} + \frac{\sqrt{13}}{3}, \quad x = -\frac{1}{3} - \frac{\sqrt{13}}{3}$
Check: We leave the check for you!	

TRY IT : : 9.39 Solve by completing the square: $4x^2 + 3x = 2$.

TRY IT : : 9.40 Solve by completing the square: $3y^2 - 10y = -5$.

MEDIA : :

Access these online resources for additional instruction and practice with completing the square.

- **Completing Perfect Square Trinomials (https://openstax.org/l/37CompTheSq1)**
- **Completing the Square 1 (https://openstax.org/l/37CompTheSq2)**
- **Completing the Square to Solve Quadratic Equations (https://openstax.org/l/37CompTheSq3)**
- **Completing the Square to Solve Quadratic Equations: More Examples (https://openstax.org/l/37CompTheSq4)**
- **Completing the Square 4 (https://openstax.org/l/37CompTheSq5)**

9.2 EXERCISES

Practice Makes Perfect

Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

71.
ⓐ $m^2 - 24m$
ⓑ $x^2 - 11x$
ⓒ $p^2 - \frac{1}{3}p$

72.
ⓐ $n^2 - 16n$
ⓑ $y^2 + 15y$
ⓒ $q^2 + \frac{3}{4}q$

73.
ⓐ $p^2 - 22p$
ⓑ $y^2 + 5y$
ⓒ $m^2 + \frac{2}{5}m$

74.
ⓐ $q^2 - 6q$
ⓑ $x^2 - 7x$
ⓒ $n^2 - \frac{2}{3}n$

Solve Quadratic Equations of the form $x^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

75. 5. $u^2 + 2u = 3$

76. $z^2 + 12z = -11$

77. $x^2 - 20x = 21$

78. $y^2 - 2y = 8$

79. $m^2 + 4m = -44$

80. $n^2 - 2n = -3$

81. $r^2 + 6r = -11$

82. $t^2 - 14t = -50$

83. $a^2 - 10a = -5$

84. $b^2 + 6b = 41$

85. $x^2 + 5x = 2$

86. $y^2 - 3y = 2$

87. $u^2 - 14u + 12 = 1$

88. $z^2 + 2z - 5 = 2$

89. $r^2 - 4r - 3 = 9$

90. $t^2 - 10t - 6 = 5$

91. $v^2 = 9v + 2$

92. $w^2 = 5w - 1$

93. $x^2 - 5 = 10x$

94. $y^2 - 14 = 6y$

95. $(x + 6)(x - 2) = 9$

96. $(y + 9)(y + 7) = 80$

97. $(x + 2)(x + 4) = 3$

98. $(x - 2)(x - 6) = 5$

Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

99. $3m^2 + 30m - 27 = 6$

100. $2x^2 - 14x + 12 = 0$

101. $2n^2 + 4n = 26$

102. $5x^2 + 20x = 15$

103. $2c^2 + c = 6$

104. $3d^2 - 4d = 15$

105. $2x^2 + 7x - 15 = 0$

106. $3x^2 - 14x + 8 = 0$

107. $2p^2 + 7p = 14$

108. $3q^2 - 5q = 9$

109. $5x^2 - 3x = -10$

110. $7x^2 + 4x = -3$

Writing Exercises

111. Solve the equation $x^2 + 10x = -25$

ⓐ by using the Square Root Property

ⓑ by Completing the Square

ⓒ Which method do you prefer? Why?

112. Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

9.3 Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $b^2 - 4ab$ when $a = 3$ and $b = -2$.
   If you missed this problem, review **Example 1.21**.
2. Simplify: $\sqrt{108}$.
   If you missed this problem, review **Example 8.13**.
3. Simplify: $\sqrt{50}$.
   If you missed this problem, review **Example 8.76**.

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering ‘isn’t there an easier way to do this?’ The answer is ‘yes’. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable ‘in general’, so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for *x*.

We start with the standard form of a quadratic equation and solve it for *x* by completing the square.

	$ax^2 + bx + c = 0 \qquad a \neq 0$
Isolate the variable terms on one side.	$ax^2 + bx = -c$
Make the coefficient of x^2 equal to 1, by dividing by *a*.	$\frac{ax^2}{a} + \frac{b}{a}x = -\frac{c}{a}$
Simplify.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$	$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + = -\frac{c}{a} + \frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c \cdot 4a}{a \cdot 4a}$

Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
	This equation is the Quadratic Formula.

Quadratic Formula

The solutions to a quadratic equation of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of *a*, *b*, and *c* from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the formula is an equation. Make sure you use both sides of the equation.

EXAMPLE 9.21 HOW TO SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution

Step 1. Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values.	This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$
Step 2. Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*.	Substitute in $a=2$, $b=9$, $c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$

Step 3. Simplify the fraction, and solve for x.		$x = \frac{-9 \pm \sqrt{81-(-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9+11}{4}$ $\quad x = \frac{-9-11}{4}$ $x = \frac{2}{4}$ $\quad x = \frac{-20}{4}$ $x = \frac{1}{2}$ $\quad x = -5$
Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0 ✓$
	Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0 ✓$

TRY IT : : 9.41 Solve by using the Quadratic Formula: $3y^2 - 5y + 2 = 0$.

TRY IT : : 9.42 Solve by using the Quadratic Formula: $4z^2 + 2z - 6 = 0$.

HOW TO : : SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA.

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, and c.

Step 2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.

Step 3. Simplify.

Step 4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic

Formula is an EQUATION. Be sure you start with “x =”.

EXAMPLE 9.22

Solve by using the Quadratic Formula: $x^2 - 6x = -5$.

Solution

	$x^2 - 6x = -5$
Write the equation in standard form by adding 5 to each side.	$x^2 - 6x + 5 = 0$
This equation is now in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the values of $a,\ b,\ c$.	$a = 1,\ b = -6,\ c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of $a,\ b,\ c$.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$ $x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6 + 4}{2},\quad x = \frac{6 - 4}{2}$
Simplify.	$x = \frac{10}{2},\quad x = \frac{2}{2}$
	$x = 5,\quad x = 1$

Check:

$$\begin{array}{rr} x^2 - 6x + 5 = 0 & x^2 - 6x + 5 = 0 \\ 5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0 & 1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0 \\ 25 - 30 + 5 \stackrel{?}{=} 0 & 1 - 6 + 5 \stackrel{?}{=} 0 \\ 0 = 0\ \checkmark & 0 = 0\ \checkmark \end{array}$$

TRY IT : : 9.43 Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

TRY IT : : 9.44 Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

EXAMPLE 9.23

Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

Solution

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$\begin{matrix} ax^2 & + & bx & + & c & = 0 \\ 2x^2 & + & 10x & + & 11 & = 0 \end{matrix}$
Identify the values of *a*, *b*, and *c*.	$a = 2,\ b = 10,\ c = 11$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, and *c*.	$x = \dfrac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \dfrac{-10 \pm \sqrt{100 - 88}}{4}$
	$x = \dfrac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \dfrac{-10 \pm 2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \dfrac{2(-5 \pm \sqrt{3})}{4}$
Remove the common factors.	$x = \dfrac{-5 \pm \sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \dfrac{-5 + \sqrt{3}}{2},\quad x = \dfrac{-5 - \sqrt{3}}{2}$
Check: We leave the check for you!	

TRY IT : : 9.45 Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

TRY IT : : 9.46 Solve by using the Quadratic Formula: $5n^2 + 4n - 4 = 0$.

When we substitute *a*, *b*, and *c* into the Quadratic Formula and the radicand is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

EXAMPLE 9.24

Solve by using the Quadratic Formula: $3p^2 + 2p + 9 = 0$.

Solution

	$3p^2 + 2p + 9 = 0$
This equation is in standard form	$\begin{matrix} ax^2 & + & bx & + & c & = 0 \\ 3p^2 & + & 2p & + & 9 & = 0 \end{matrix}$
Identify the values of a, b, c.	$a = 3,\ b = 2,\ c = 9$
Write the Quadratic Formula.	$p = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$p = \dfrac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$

Simplify.	$p = \frac{-2 \pm \sqrt{4 - 108}}{6}$
	$p = \frac{-2 \pm \sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p = \frac{-2 \pm \sqrt{104}\, i}{6}$
Simplify the radical.	$p = \frac{-2 \pm 2\sqrt{26}\, i}{6}$
Factor the common factor in the numerator.	$p = \frac{2(-1 \pm \sqrt{26}\, i)}{6}$
Remove the common factors.	$p = \frac{-1 \pm \sqrt{26}\, i}{3}$
Rewrite in standard $a + bi$ form.	$p = -\frac{1}{3} \pm \frac{\sqrt{26}\, i}{3}$
Write as two solutions.	$p = -\frac{1}{3} + \frac{\sqrt{26}\, i}{3}, \quad p = -\frac{1}{3} - \frac{\sqrt{26}\, i}{3}$

TRY IT : : 9.47 Solve by using the Quadratic Formula: $4a^2 - 2a + 8 = 0$.

TRY IT : : 9.48 Solve by using the Quadratic Formula: $5b^2 + 2b + 4 = 0$.

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

EXAMPLE 9.25

Solve by using the Quadratic Formula: $x(x + 6) + 4 = 0$.

Solution

Our first step is to get the equation in standard form.

	$x(x + 6) + 4 = 0$
Distribute to get the equation in standard form.	$x^2 + 6x + 4 = 0$
This equation is now in standard form	$\begin{array}{c} ax^2 + bx + c = 0 \\ x^2 + 6x + 4 = 0 \end{array}$
Identify the values of a, b, c.	$a = 1, b = 6, c = 4$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(6) \pm \sqrt{(6)^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$
Simplify.	$x = \frac{-6 \pm \sqrt{36 - 16}}{2}$
	$x = \frac{-6 \pm \sqrt{20}}{2}$
Simplify the radical.	$x = \frac{-6 \pm 2\sqrt{5}}{2}$

Factor the common factor in the numerator.	$x=\frac{2\left(-3\pm 2\sqrt{5}\right)}{2}$
Remove the common factors.	$x=-3\pm 2\sqrt{5}$
Write as two solutions.	$x=-3+2\sqrt{5},\quad x=-3-2\sqrt{5}$
Check: We leave the check for you!	

TRY IT : : 9.49 Solve by using the Quadratic Formula: $x(x+2)-5=0$.

TRY IT : : 9.50 Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

EXAMPLE 9.26

Solve by using the Quadratic Formula: $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$.

Solution

Our first step is to clear the fractions.

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$
Multiply.	$3u^2+4u=2$
Subtract 2 to get the equation in standard form.	$ax^2+bx+c=0$ $3u^2+4u-2=0$
Identify the values of *a*, *b*, and *c*.	$a=3,\ b=4,\ c=-2$
Write the Quadratic Formula.	$u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of *a*, *b*, and *c*.	$u=\frac{-(4)\pm\sqrt{(4)^2-4\cdot 3\cdot(-2)}}{2\cdot 3}$
Simplify.	$u=\frac{-4\pm\sqrt{16+24}}{6}$
	$u=\frac{-4\pm\sqrt{40}}{6}$
Simplify the radical.	$u=\frac{-4\pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u=\frac{2\left(-2\pm\sqrt{10}\right)}{6}$
Remove the common factors.	$u=\frac{-2\pm\sqrt{10}}{3}$
Rewrite to show two solutions.	$u=\frac{-2+\sqrt{10}}{3},\quad u=\frac{-2-\sqrt{10}}{3}$

Check:
We leave the check for you!

> **TRY IT : :** 9.51 Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

> **TRY IT : :** 9.52 Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Think about the equation $(x - 3)^2 = 0$. We know from the Zero Product Property that this equation has only one solution, $x = 3$.

We will see in the next example how using the Quadratic Formula to solve an equation whose standard form is a perfect square trinomial equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0 , which leads to only one solution.

EXAMPLE 9.27

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$\begin{matrix} ax^2 + bx + c = 0 \\ 4x^2 - 20x + 25 = 0 \end{matrix}$
Identify the values of *a*, *b*, and *c*.	$a = 4, b = -20, c = 25$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, and *c*.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$
	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x - 5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

> **TRY IT : :** 9.53 Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

> **TRY IT : :** 9.54 Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Discriminant

In the Quadratic Formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

the quantity $b^2 - 4ac$ is called the discriminant.

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Quadratic Equation (in standard form)	Discriminant $b^2 - 4ac$	Value of the Discriminant	Number and Type of solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5)$ 121	+	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25$ 0	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9$ -104	–	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions**. $x = \dfrac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x = \dfrac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x = \dfrac{-b \pm \sqrt{-}}{2a}$

Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0,\ a \neq 0,$

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

EXAMPLE 9.28

Determine the number of solutions to each quadratic equation.

ⓐ $3x^2 + 7x - 9 = 0$ ⓑ $5n^2 + n + 4 = 0$ ⓒ $9y^2 - 6y + 1 = 0$.

Solution

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

	$3x^2 + 7x - 9 = 0$
The equation is in standard form, identify $a, b,$ and c.	$a = 3, \quad b = 7, \quad c = -9$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of $a, b,$ and c.	$(7)^2 - 4 \cdot 3 \cdot (-9)$
Simplify.	$49 + 108$
	157

Since the discriminant is positive, there are 2 real solutions to the equation.

ⓑ

	$5n^2 + n + 4 = 0$
The equation is in standard form, identify $a, b,$ and c.	$a = 5, \quad b = 1, \quad c = 4$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of $a, b,$ and c.	$(1)^2 - 4 \cdot 5 \cdot 4$
Simplify.	$1 - 80$
	-79

Since the discriminant is negative, there are 2 complex solutions to the equation.

ⓒ

	$9y^2 - 6y + 1 = 0$
The equation is in standard form, identify $a, b,$ and c.	$a = 9, b = -6, c = 1$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of $a, b,$ and c.	$(-6)^2 - 4 \cdot 9 \cdot 1$
Simplify.	$36 - 36$
	0

Since the discriminant is 0, there is 1 real solution to the equation.

TRY IT : : 9.55 Determine the numberand type of solutions to each quadratic equation.

ⓐ $8m^2 - 3m + 6 = 0$ ⓑ $5z^2 + 6z - 2 = 0$ ⓒ $9w^2 + 24w + 16 = 0.$

TRY IT : : 9.56 Determine the number and type of solutions to each quadratic equation.

ⓐ $b^2 + 7b - 13 = 0$ ⓑ $5a^2 - 6a + 10 = 0$ ⓒ $4r^2 - 20r + 25 = 0.$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2 = k$ or $a(x - h)^2 = k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

HOW TO : : IDENTIFY THE MOST APPROPRIATE METHOD TO SOLVE A QUADRATIC EQUATION.

Step 1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.

Step 2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.

Step 3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

EXAMPLE 9.29

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $5z^2 = 17$ ⓑ $4x^2 - 12x + 9 = 0$ ⓒ $8u^2 + 6u = 11$.

Solution

ⓐ

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

ⓑ

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

ⓒ

	$8u^2 + 6u = 11$
Put the equation in standard form.	$8u^2 + 6u - 11 = 0$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

TRY IT : : 9.57 Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $x^2 + 6x + 8 = 0$ ⓑ $(n - 3)^2 = 16$ ⓒ $5p^2 - 6p = 9$.

TRY IT : : 9.58 Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $8a^2 + 3a - 9 = 0$ ⓑ $4b^2 + 4b + 1 = 0$ ⓒ $5c^2 = 125$.

MEDIA : :

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- **Using the Quadratic Formula (https://openstax.org/l/37QuadForm1)**
- **Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions (https://openstax.org/l/37QuadForm2)**
- **Discriminant in Quadratic Formula (https://openstax.org/l/37QuadForm3)**

9.3 EXERCISES

Practice Makes Perfect

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

113. $4m^2 + m - 3 = 0$

114. $4n^2 - 9n + 5 = 0$

115. $2p^2 - 7p + 3 = 0$

116. $3q^2 + 8q - 3 = 0$

117. $p^2 + 7p + 12 = 0$

118. $q^2 + 3q - 18 = 0$

119. $r^2 - 8r = 33$

120. $t^2 + 13t = -40$

121. $3u^2 + 7u - 2 = 0$

122. $2p^2 + 8p + 5 = 0$

123. $2a^2 - 6a + 3 = 0$

124. $5b^2 + 2b - 4 = 0$

125. $x^2 + 8x - 4 = 0$

126. $y^2 + 4y - 4 = 0$

127. $3y^2 + 5y - 2 = 0$

128. $6x^2 + 2x - 20 = 0$

129. $2x^2 + 3x + 3 = 0$

130. $2x^2 - x + 1 = 0$

131. $8x^2 - 6x + 2 = 0$

132. $8x^2 - 4x + 1 = 0$

133. $(v + 1)(v - 5) - 4 = 0$

134. $(x + 1)(x - 3) = 2$

135. $(y + 4)(y - 7) = 18$

136. $(x + 2)(x + 6) = 21$

137. $\frac{1}{3}m^2 + \frac{1}{12}m = \frac{1}{4}$

138. $\frac{1}{3}n^2 + n = -\frac{1}{2}$

139. $\frac{3}{4}b^2 + \frac{1}{2}b = \frac{3}{8}$

140. $\frac{1}{9}c^2 + \frac{2}{3}c = 3$

141. $16c^2 + 24c + 9 = 0$

142. $25d^2 - 60d + 36 = 0$

143. $25q^2 + 30q + 9 = 0$

144. $16y^2 + 8y + 1 = 0$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

145.
ⓐ $4x^2 - 5x + 16 = 0$
ⓑ $36y^2 + 36y + 9 = 0$
ⓒ $6m^2 + 3m - 5 = 0$

146.
ⓐ $9v^2 - 15v + 25 = 0$
ⓑ $100w^2 + 60w + 9 = 0$
ⓒ $5c^2 + 7c - 10 = 0$

147.
ⓐ $r^2 + 12r + 36 = 0$
ⓑ $8t^2 - 11t + 5 = 0$
ⓒ $3v^2 - 5v - 1 = 0$

148.
ⓐ $25p^2 + 10p + 1 = 0$
ⓑ $7q^2 - 3q - 6 = 0$
ⓒ $7y^2 + 2y + 8 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

149.
ⓐ $x^2 - 5x - 24 = 0$
ⓑ $(y + 5)^2 = 12$
ⓒ $14m^2 + 3m = 11$

150.
ⓐ $(8v + 3)^2 = 81$
ⓑ $w^2 - 9w - 22 = 0$
ⓒ $4n^2 - 10 = 6$

151.
ⓐ $6a^2 + 14 = 20$
ⓑ $\left(x - \frac{1}{4}\right)^2 = \frac{5}{16}$
ⓒ $y^2 - 2y = 8$

152.
ⓐ $8b^2 + 15b = 4$
ⓑ $\frac{5}{9}v^2 - \frac{2}{3}v = 1$
ⓒ $\left(w + \frac{4}{3}\right)^2 = \frac{2}{9}$

Writing Exercises

153. Solve the equation $x^2 + 10x = 120$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

154. Solve the equation $12y^2 + 23y = 24$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

9.4 Solve Quadratic Equations in Quadratic Form

Learning Objectives

By the end of this section, you will be able to:

- Solve equations in quadratic form

Be Prepared!

Before you get started, take this readiness quiz.

1. Factor by substitution: $y^4 - y^2 - 20$.
   If you missed this problem, review **Example 6.21**.
2. Factor by substitution: $(y-4)^2 + 8(y-4) + 15$.
   If you missed this problem, review **Example 6.22**.
3. Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$ ⓑ $\left(x^{\frac{1}{3}}\right)^2$ ⓒ $\left(x^{-1}\right)^2$.

   If you missed this problem, review **Example 8.33**.

Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard u for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$.) So we let $u = x^2$ and factored.

	$x^4 - 4x^2 - 5$
	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$u^2 - 4u - 5$
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$(x^2+1)(x^2-5)$

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard u to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax^2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

EXAMPLE 9.30 HOW TO SOLVE EQUATIONS IN QUADRATIC FORM

Solve: $6x^4 - 7x^2 + 2 = 0$

Solution

Step 1. Identify a substitution that will put the equation in quadratic form.	Since $(x^2)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$
Step 2. Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6(x^2)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3. Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u-1)(3u-2) = 0$ $2u - 1 = 0,\ 3u - 2 = 0$ $2u = 1,\ 3u = 2$ $u = \frac{1}{2} \quad u = \frac{2}{3}$
Step 4. Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2} \qquad x^2 = \frac{2}{3}$
Step 5. Solve for the original variable.	Solve for x, using the Square Root Property.	$x = \pm\sqrt{\frac{1}{2}} \qquad x = \pm\sqrt{\frac{2}{3}}$ $x = \pm\frac{\sqrt{2}}{2} \qquad x = \pm\frac{\sqrt{6}}{3}$ There are four solutions. $x = \frac{\sqrt{2}}{2} \qquad x = \frac{\sqrt{6}}{3}$ $x = -\frac{\sqrt{2}}{2} \qquad x = -\frac{\sqrt{6}}{3}$
Step 6. Check the solutions.	Check all four solutions. We will show one check here.	$x = \frac{\sqrt{2}}{2}$ $6x^4 - 7x^2 + 2 = 0$ $6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$ $6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right)^2 + 2 \stackrel{?}{=} 0$ $\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$ $0 = 0\ \checkmark$ We leave the other checks to you!

TRY IT : : 9.59 Solve: $x^4 - 6x^2 + 8 = 0$.

TRY IT : : 9.60 Solve: $x^4 - 11x^2 + 28 = 0$.

We summarize the steps to solve an equation in quadratic form.

HOW TO : : SOLVE EQUATIONS IN QUADRATIC FORM.

Step 1. Identify a substitution that will put the equation in quadratic form.
Step 2. Rewrite the equation with the substitution to put it in quadratic form.
Step 3. Solve the quadratic equation for u.
Step 4. Substitute the original variable back into the results, using the substitution.
Step 5. Solve for the original variable.
Step 6. Check the solutions.

In the next example, the binomial in the middle term, $(x - 2)$ is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

EXAMPLE 9.31

Solve: $(x-2)^2 + 7(x-2) + 12 = 0$.

Solution

	$(x-2)^2 + 7(x-2) + 12 = 0$
Prepare for the substitution.	$(x-2)^2 + 7(x-2) + 12 = 0$
Let $u = x - 2$ and substitute.	$u^2 + 7u + 12 = 0$
Solve by factoring.	$(u+3)(u+4) = 0$ $u+3=0,\quad u+4=0$ $u=-3,\quad u=-4$
Replace u with $x - 2$.	$x-2=-3,\quad x-2=-4$
Solve for x.	$x=-1,\quad x=-2$

Check:

$$\begin{aligned} x &= -1 \\ (x-2)^2 + 7(x-2) + 12 &= 0 \\ (-1-2)^2 + 7(-1-2) + 12 &\stackrel{?}{=} 0 \\ (-3)^2 + 7(-3) + 12 &\stackrel{?}{=} 0 \\ 9 - 21 + 12 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$

$$\begin{aligned} x &= -2 \\ (x-2)^2 + 7(x-2) + 12 &= 0 \\ (-2-2)^2 + 7(-2-2) + 12 &\stackrel{?}{=} 0 \\ (-4)^2 + 7(-4) + 12 &\stackrel{?}{=} 0 \\ 16 - 28 + 12 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$

TRY IT : : 9.61 Solve: $(x-5)^2 + 6(x-5) + 8 = 0$.

TRY IT : : 9.62 Solve: $(y-4)^2 + 8(y-4) + 15 = 0$.

In the next example, we notice that $(\sqrt{x})^2 = x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

EXAMPLE 9.32

Solve: $x - 3\sqrt{x} + 2 = 0$.

Solution

The $\sqrt{x}$ in the middle term, is squared in the first term $(\sqrt{x})^2 = x$. If we let $u = \sqrt{x}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$x - 3\sqrt{x} + 2 = 0$
Rewrite the trinomial to prepare for the substitution.	$(\sqrt{x})^2 - 3\sqrt{x} + 2 = 0$
Let $u = \sqrt{x}$ and substitute.	$u^2 - 3u + 2 = 0$
Solve by factoring.	$(u-2)(u-1) = 0$
	$u - 2 = 0, \quad u - 1 = 0$
	$u = 2, \quad u = 1$
Replace u with $\sqrt{x}$.	$\sqrt{x} = 2, \quad \sqrt{x} = 1$
Solve for x, by squaring both sides.	$x = 4, \quad x = 1$
Check:	

$$\begin{array}{cc} x = 4 & x = 1 \\ x - 3\sqrt{x} + 2 = 0 & x - 3\sqrt{x} + 2 = 0 \\ 4 - 3\sqrt{4} + 2 \stackrel{?}{=} 0 & 1 - 3\sqrt{1} + 2 \stackrel{?}{=} 0 \\ 4 - 6 + 2 \stackrel{?}{=} 0 & 1 - 3 + 2 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$

TRY IT : : 9.63 Solve: $x - 7\sqrt{x} + 12 = 0$.

TRY IT : : 9.64 Solve: $x - 6\sqrt{x} + 8 = 0$.

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

EXAMPLE 9.33

Solve: $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$.

Solution

The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$. If we let $u = x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$
Rewrite the trinomial to prepare for the substitution.	$\left(x^{\frac{1}{3}}\right)^2 - 2\left(x^{\frac{1}{3}}\right) - 24 = 0$
Let $u = x^{\frac{1}{3}}$ and substitute.	$u^2 - 2u - 24 = 0$

Solve by factoring.	$(u-6)(u+4)=0$ $u-6=0, \quad u+4=0$ $u=6, \quad u=-4$
Replace u with $x^{\frac{1}{3}}$.	$x^{\frac{1}{3}}=6, \quad x^{\frac{1}{3}}=-4$
Solve for x by cubing both sides.	$\left(x^{\frac{1}{3}}\right)^3=(6)^3, \quad \left(x^{\frac{1}{3}}\right)^3=(-4)^3$ $x=216, \quad x=-64$

Check:

$$\begin{array}{rr} x=216 & x=-64 \\ x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0 & x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0 \\ (216)^{\frac{2}{3}}-2(216)^{\frac{1}{3}}-24\stackrel{?}{=}0 & (-64)^{\frac{2}{3}}-2(-64)^{\frac{1}{3}}-24\stackrel{?}{=}0 \\ 36-12-24\stackrel{?}{=}0 & 16+8-24\stackrel{?}{=}0 \\ 0=0\checkmark & 0=0\checkmark \end{array}$$

TRY IT : : 9.65 Solve: $x^{\frac{2}{3}}-5x^{\frac{1}{3}}-14=0.$

TRY IT : : 9.66 Solve: $x^{\frac{1}{2}}+8x^{\frac{1}{4}}+15=0.$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

EXAMPLE 9.34

Solve: $3x^{-2}-7x^{-1}+2=0.$

Solution

The x^{-1} in the middle term is squared in the first term $\left(x^{-1}\right)^2=x^{-2}$. If we let $u=x^{-1}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$3x^{-2}-7x^{-1}+2=0$
Rewrite the trinomial to prepare for the substitution.	$3(x^{-1})^2-7(x^{-1})+2=0$
Let $u=x^{-1}$ and substitute.	$3u^2-7u+2=0$
Solve by factoring.	$(3u-1)(u-2)=0$
	$3u-1=0, \quad u-2=0$
	$u=\frac{1}{3}, \quad u=2$
Replace u with x^{-1}.	$x^{-1}=\frac{1}{3}, \quad x^{-1}=2$
Solve for x by taking the reciprocal since $x^{-1}=\frac{1}{x}$.	$x=3, \quad x=\frac{1}{2}$

Check:

$x = 3$	$x = \frac{1}{2}$
$3x^{-2} - 7x^{-1} + 2 = 0$	$3x^{-2} - 7x^{-1} + 2 = 0$
$3(3)^{-2} - 7(3)^{-1} + 2 \stackrel{?}{=} 0$	$3\left(\frac{1}{2}\right)^{-2} - 7\left(\frac{1}{2}\right)^{-1} + 2 \stackrel{?}{=} 0$
$3\left(\frac{1}{9}\right) - 7\left(\frac{1}{3}\right) + 2 \stackrel{?}{=} 0$	$3(4) - 7(2) + 2 \stackrel{?}{=} 0$
$\frac{1}{3} - \left(\frac{7}{3}\right) + \frac{6}{3} \stackrel{?}{=} 0$	$12 - 14 + 2 \stackrel{?}{=} 0$
$0 = 0 \checkmark$	$0 = 0 \checkmark$

TRY IT : : 9.67 Solve: $8x^{-2} - 10x^{-1} + 3 = 0$.

TRY IT : : 9.68 Solve: $6x^{-2} - 23x^{-1} + 20 = 0$.

MEDIA : :
Access this online resource for additional instruction and practice with solving quadratic equations.

- **Solving Equations in Quadratic Form (https://openstax.org/l/37QuadForm4)**

9.4 EXERCISES

Practice Makes Perfect

Solve Equations in Quadratic Form

In the following exercises, solve.

155. $x^4 - 7x^2 + 12 = 0$

156. $x^4 - 9x^2 + 18 = 0$

157. $x^4 - 13x^2 - 30 = 0$

158. $x^4 + 5x^2 - 36 = 0$

159. $2x^4 - 5x^2 + 3 = 0$

160. $4x^4 - 5x^2 + 1 = 0$

161. $2x^4 - 7x^2 + 3 = 0$

162. $3x^4 - 14x^2 + 8 = 0$

163. $(x-3)^2 - 5(x-3) - 36 = 0$

164. $(x+2)^2 - 3(x+2) - 54 = 0$

165. $(3y+2)^2 + (3y+2) - 6 = 0$

166. $(5y-1)^2 + 3(5y-1) - 28 = 0$

167. $\left(x^2+1\right)^2 - 5\left(x^2+1\right) + 4 = 0$

168. $\left(x^2-4\right)^2 - 4\left(x^2-4\right) + 3 = 0$

169. $2\left(x^2-5\right)^2 - 5\left(x^2-5\right) + 2 = 0$

170. $2\left(x^2-5\right)^2 - 7\left(x^2-5\right) + 6 = 0$

171. $x - \sqrt{x} - 20 = 0$

172. $x - 8\sqrt{x} + 15 = 0$

173. $x + 6\sqrt{x} - 16 = 0$

174. $x + 4\sqrt{x} - 21 = 0$

175. $6x + \sqrt{x} - 2 = 0$

176. $6x + \sqrt{x} - 1 = 0$

177. $10x - 17\sqrt{x} + 3 = 0$

178. $12x + 5\sqrt{x} - 3 = 0$

179. $x^{\frac{2}{3}} + 9x^{\frac{1}{3}} + 8 = 0$

180. $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} = 28$

181. $x^{\frac{2}{3}} + 4x^{\frac{1}{3}} = 12$

182. $x^{\frac{2}{3}} - 11x^{\frac{1}{3}} + 30 = 0$

183. $6x^{\frac{2}{3}} - x^{\frac{1}{3}} = 12$

184. $3x^{\frac{2}{3}} - 10x^{\frac{1}{3}} = 8$

185. $8x^{\frac{2}{3}} - 43x^{\frac{1}{3}} + 15 = 0$

186. $20x^{\frac{2}{3}} - 23x^{\frac{1}{3}} + 6 = 0$

187. $x + 8x^{\frac{1}{2}} + 7 = 0$

188. $2x - 7x^{\frac{1}{2}} = 15$

189. $6x^{-2} + 13x^{-1} + 5 = 0$

190. $15x^{-2} - 26x^{-1} + 8 = 0$

191. $8x^{-2} - 2x^{-1} - 3 = 0$

192. $15x^{-2} - 4x^{-1} - 4 = 0$

Writing Exercises

193. Explain how to recognize an equation in quadratic form.

194. Explain the procedure for solving an equation in quadratic form.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve equations in quadratic form.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

9.5 Solve Applications of Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by quadratic equations

Be Prepared!

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is −100. Find the numbers.
   If you missed this problem, review Example 2.18.
2. Solve: $\frac{2}{x+1} + \frac{1}{x-1} = \frac{1}{x^2 - 1}$.
   If you missed this problem, review Example 7.35.
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review Example 2.34.

Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let’s first summarize the methods we now have to solve quadratic equations.

Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As you solve each equation, choose the method that is most convenient for you to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

HOW TO : : USE A PROBLEM-SOLVING STRATEGY.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.

Step 5. **Solve** the equation using algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence

We have solved number applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n + 2$. The next one would be $n + 2 + 2$ or $n + 4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

	Consecutive even integers		**Consecutive odd integers**
	64, 66, 68		77, 79, 81
n	1^{st} even integer	n	1^{st} odd integer
$n+2$	2^{nd} consecutive even integer	$n+2$	2^{nd} consecutive odd integer
$n+4$	3^{rd} consecutive even integer	$n+4$	3^{rd} consecutive odd integer

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

EXAMPLE 9.35

The product of two consecutive odd integers is 195. Find the integers.

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. We are looking for two consecutive odd integers.

Step 3. Name what we are looking for. Let $n =$ the fir t odd integer.
$n + 2 =$ the next odd integer

Step 4. Translate into an equation. State the problem in one sentence. "The product of two consecutive odd integers is 195."

The product of the fir t odd integer and the second odd integer is 195.

Translate into an equation.

$$n(n+2) = 195$$

Step 5. Solve the equation. Distribute.

$$n^2 + 2n = 195$$

Write the equation in standard form.

$$n^2 + 2n - 195 = 0$$

Factor.

$$(n+15)(n-13) = 0$$

Use the Zero Product Property.

$$n + 15 = 0 \quad n - 13 = 0$$

Solve each equation.

$$n = -15, \quad n = 13$$

There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.

First odd integer $n = 13$	First odd integer $n = -15$
next odd integer $n + 2$	next odd integer $n + 2$
$13 + 2$	$-15 + 2$
15	-13

Step 6. Check the answer.

Do these pairs work?

Are they consecutive odd integers?

$13,\ 15$ yes
$-13,\ -15$ yes

Is their product 195?

$13 \cdot 15 = 195$ yes
$-13(-15) = 195$ yes

Step 7. Answer the question. Two consecutive odd integers whose product is 195 are 13, 15 and $-13,\ -15$.

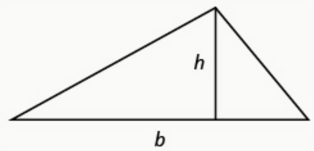
TRY IT : : 9.69 The product of two consecutive odd integers is 99. Find the integers.

TRY IT : : 9.70 The product of two consecutive even integers is 168. Find the integers.

We will use the formula for the area of a triangle to solve the next example.

Area of a Triangle

For a triangle with base, *b*, and height, *h*, the area, *A*, is given by the formula $A = \frac{1}{2}bh$.

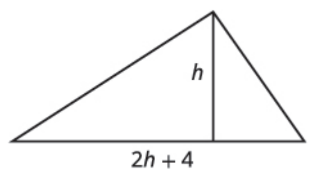


Recall that when we solve geometric applications, it is helpful to draw the figure.

EXAMPLE 9.36

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the base and height.
Step 3. Name what we are looking for.	Let *h* = the height of the triangle. 2*h* + 4 = the base of the triangle
Step 4. Translate into an equation. We know the area. Write the formula for the area of a triangle.	$A = \frac{1}{2}bh$
Step 5. Solve the equation. Substitute in the values.	$120 = \frac{1}{2}(2h + 4)h$
Distribute.	$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in standard form.	$h^2 + 2h - 120 = 0$
Factor.	$(h - 10)(h + 12) = 0$
Use the Zero Product Property.	$h - 10 = 0 \quad h + 12 = 0$
Simplify.	$h = 10, \quad \cancel{h = -12}$
Since *h* is the height of a window, a value of *h* = −12 does not make sense.	
The height of the triangle $h = 10$.	

	The base of the triangle $2h + 4.$ $2 \cdot 10 + 4$ 24
Step 6. Check the answer. Does a triangle with height 10 and base 24 have area 120? Yes.	
Step 7. Answer the question.	The height of the triangular window is 10 feet and the base is 24 feet.

TRY IT : : 9.71

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

TRY IT : : 9.72

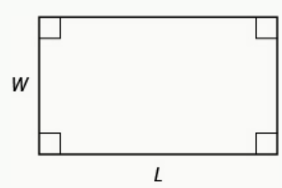
If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

Area of a Rectangle

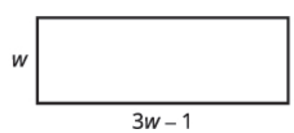
For a rectangle with length, L, and width, W, the area, A, is given by the formula $A = LW$.



EXAMPLE 9.37

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution

Step 1. **Read** the problem. Draw a picture.	
Step 2. **Identify** what we are looking for.	We are looking for the length and width.
Step 3. **Name** what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle

Step 4. **Translate** into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5. **Solve** the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$
This is a quadratic equation; rewrite it in standard form. Solve the equation using the Quadratic Formula.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 3w^2 - w - 150 = 0 \end{array}$
Identify the a, b, c values.	$a = 3, b = -1, c = -150$
Write the Quadratic Formula.	$w = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$w = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$w = \dfrac{1 \pm \sqrt{1 + 1800}}{6}$ $w = \dfrac{1 \pm \sqrt{1801}}{6}$
Rewrite to show two solutions.	$w = \dfrac{1 + \sqrt{1801}}{6}, \quad w = \dfrac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w \approx 7.2, \quad \cancel{w \approx -6.9}$ Width $w \approx 7.2$ Length $\approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Step 6. **Check** the answer. Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150.	
Step 7. **Answer** the question.	The width of the rectangle is approximately 7.2 feet and the length is approximately 20.6 feet.

TRY IT : : 9.73

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

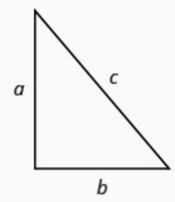
TRY IT : : 9.74

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot.?

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Pythagorean Theorem

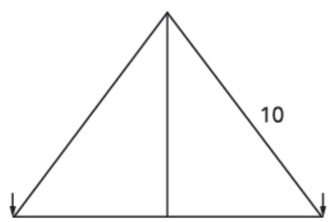
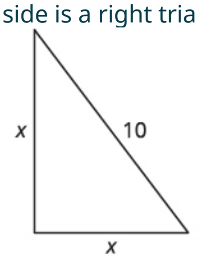
In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.



EXAMPLE 9.38

Rene is setting up a holiday light display. He wants to make a ‘tree’ in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height of the pole.
Step 3. Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. $x =$ the distance from pole to stake Each side is a right triangle. We draw a picture of one of them. 
Step 4. Translate into an equation. We can use the Pythagorean Theorem to solve for *x*. Write the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Step 5. Solve the equation. Substitute.	$x^2 + x^2 = 10^2$
Simplify.	$2x^2 = 100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2} = \frac{100}{2}$
Simplify.	$x^2 = 50$

Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}, \quad \cancel{x = -5\sqrt{2}}$
	If we approximate this number to the nearest tenth with a calculator, we find $x \approx 7.1$.
Step 6. Check the answer. Check on your own in the Pythagorean Theorem.	
Step 7. Answer the question.	The pole should be about 7.1 feet tall.

TRY IT : : 9.75

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

TRY IT : : 9.76

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Projectile motion

The height in feet, h , of an object shot upwards into the air with initial velocity, v_0 , after t seconds is given by the formula

$$h = -16t^2 + v_0 t$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

EXAMPLE 9.39

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of seconds, which is time.
Step 3. Name what we are looking for.	Let $t =$ the number of seconds.
Step 4. Translate into an equation. Use the formula.	$h = -16t^2 + v_0 t$
Step 5. Solve the equation. We know the velocity v_0 is 130 feet per second. The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$

This is a quadratic equation, rewrite it in standard form. Solve the equation using the Quadratic Formula.	$\underset{ax^2}{16t^2} \underset{+\;bx}{-\;130t} \underset{+\;c}{+\;260} = 0$
Identify the values of a, b, c.	$a = 16$, $b = -130$, $c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c .	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32}$, $t = \frac{130 - \sqrt{260}}{32}$
Approximate the answer with a calculator.	$t \approx 4.6$ seconds, $t \approx 3.6$ seconds
Step 6. Check the answer. The check is left to you.	
Step 7. Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

TRY IT : : 9.77

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h = -16t^2 + v_0t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

TRY IT : : 9.78

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h = -16t^2 + v_0t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate •	Time =	Distance

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

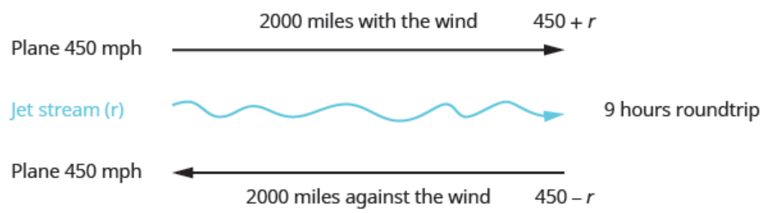
Some uniform motion problems are also modeled by quadratic equations.

EXAMPLE 9.40

Professor Smith just returned from a conference that was 2,000 miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is 450 + r.

When the plane flies against the wind, the wind decreases its speed and the rate is 450 − r.

Write in the rates.
Write in the distances.
Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$.
We divide the distance by the rate in each row, and place the expression in the time column.

Type	Rate ·	Time =	Distance
Headwind	$450 - r$	$\frac{2000}{450-r}$	2000
Tailwind	$450 + r$	$\frac{2000}{450+r}$	2000
		9	

We know the times add to 9 and so we write our equation.	$\frac{2000}{450-r} + \frac{2000}{450+r} = 9$
We multiply both sides by the LCD.	$(450-r)(450+r)\left(\frac{2000}{450-r} + \frac{2000}{450+r}\right) = 9(450-r)(450+r)$
Simplify.	$2000(450+r) + 2000(450-r) = 9(450-r)(450+r)$
Factor the 2,000.	$2000(450+r+450-r) = 9\left(450^2 - r^2\right)$
Solve.	$2000(900) = 9\left(450^2 - r^2\right)$
Divide by 9.	$2000(100) = 450^2 - r^2$
Simplify.	$200000 = 202500 - r^2$ $-2500 = -r^2$ $50 = r$ The speed of the jet stream.
Check: Is 50 mph a reasonable speed for the jet stream? Yes. If the plane is traveling 450 mph and the wind is 50 mph, Tailwind $450 + 50 = 500$ mph $\frac{2000}{500} = 4$ hours Headwind $450 - 50 = 400$ mph $\frac{2000}{400} = 5$ hours The times add to 9 hours, so it checks.	
The speed of the jet stream was 50 mph.	

TRY IT : : 9.79

MaryAnne just returned from a visit with her grandchildren back east . The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

TRY IT : : 9.80

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We'll use a similar scenario now.

EXAMPLE 9.41

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

Solution

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

Let $x =$ the number of hours for Press #2 to complete the job.
Enter the hours per job for Press #1, Press #2, and when they work together.

	Number of hours needed to complete the job.	Part of job completed/hour
Press #1	$x+12$	$\frac{1}{x+12}$
Press #2	x	$\frac{1}{x}$
Together	8	$\frac{1}{8}$

The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.
Translate to an equation.

$$\underbrace{\text{Press \#1} + \text{Press \#2} = \text{Together}}_{\text{Work completed by}}$$

$$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$$

Solve.

$$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$$

Multiply by the LCD, $8x(x+12)$.

$$8x(x+12)\left(\frac{1}{x+12} + \frac{1}{x}\right) = \left(\frac{1}{8}\right)8x(x+12)$$

Simplify.

$$8x + 8(x+12) = x(x+12)$$

$$8x + 8x + 96 = x^2 + 12x$$

$$0 = x^2 - 4x - 96$$

Solve.

$$0 = (x-12)(x+8)$$

$$x - 12 = 0,\ x + 8 = 0$$

$$x = 12,\ \cancel{x = -8} \text{ hours}$$

Since the idea of negative hours does not make sense, we use the value $x = 12$.

$12 + 12$	12
24 hours	12 hours

Write our sentence answer.

Press #1 would take 24 hours and
Press #2 would take 12 hours to do the job alone.

TRY IT : : 9.81

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

TRY IT : : 9.82

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

MEDIA : :

Access these online resources for additional instruction and practice with solving applications modeled by quadratic equations.

- **Word Problems Involving Quadratic Equations (https://openstax.org/l/37QuadForm5)**
- **Quadratic Equation Word Problems (https://openstax.org/l/37QuadForm6)**
- **Applying the Quadratic Formula (https://openstax.org/l/37QuadForm7)**

9.5 EXERCISES

Practice Makes Pefect

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method.

195. The product of two consecutive odd numbers is 255. Find the numbers.

196. The product of two consecutive even numbers is 360. Find the numbers.

197. The product of two consecutive even numbers is 624. Find the numbers.

198. The product of two consecutive odd numbers is 1,023. Find the numbers.

199. The product of two consecutive odd numbers is 483. Find the numbers.

200. The product of two consecutive even numbers is 528. Find the numbers.

In the following exercises, solve using any method. Round your answers to the nearest tenth, if needed.

201. A triangle with area 45 square inches has a height that is two less than four times the base Find the base and height of the triangle.

202. The base of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the base and height of the triangle.

203. The area of a triangular flower bed in the park has an area of 120 square feet. The base is 4 feet longer that twice the height. What are the base and height of the triangle?

204. A triangular banner for the basketball championship hangs in the gym. It has an area of 75 square feet. What is the length of the base and height , if the base is two-thirds of the height?

205. The length of a rectangular driveway is five feet more than three times the width. The area is 50 square feet. Find the length and width of the driveway.

206. A rectangular lawn has area 140 square yards. Its width that is six less than twice the length. What are the length and width of the lawn?

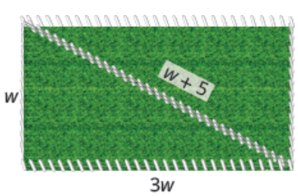
207. A rectangular table for the dining room has a surface area of 24 square feet. The length is two more feet than twice the width of the table. Find the length and width of the table.

208. The new computer has a surface area of 168 square inches. If the the width is 5.5 inches less that the length, what are the dimensions of the computer?

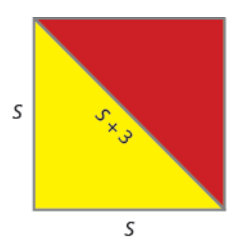
209. The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.

210. The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times as the length of the other leg . Round to the nearest tenth. Find the lengths of the three sides of the triangle.

211. A rectangular garden will be divided into two plots by fencing it on the diagonal. The diagonal distance from one corner of the garden to the opposite corner is five yards longer than the width of the garden. The length of the garden is three times the width. Find the length of the diagonal of the garden.



212. Nautical flags are used to represent letters of the alphabet. The flag for the letter, O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The hypotenuse of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.



213. Gerry plans to place a 25-foot ladder against the side of his house to clean his gutters. The bottom of the ladder will be 5 feet from the house.How for up the side of the house will the ladder reach?

214. John has a 10-foot piece of rope that he wants to use to support his 8-foot tree. How far from the base of the tree should he secure the rope?

215. A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when the height of the firework rocket will be 1200 feet.

216. An arrow is shot vertically upward at a rate of 220 feet per second. Use the projectile formula $h = -16t^2 + v_0 t$, to determine when height of the arrow will be 400 feet.

217. A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0 t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

218. A stone is dropped from a 196-foot platform. Use the formula $h = -16t^2 + v_0 t + 196$ to determine how many seconds it will take for the stone to hit the ground. (Since the stone is dropped, $v_0 = 0$.)

219. The businessman took a small airplane for a quick flight up the coast for a lunch meeting and then returned home. The plane flew a total of 4 hours and each way the trip was 200 miles. What was the speed of the wind that affected the plane which was flying at a speed of 120 mph?

220. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 300 miles. If the plane was flying at 125 mph, what was the speed of the wind that affected the plane?

221. Roy kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

222. Rick paddled up the river, spent the night camping, and and then paddled back. He spent 10 hours paddling and the campground was 24 miles away. If Rick kayaked at a speed of 5 mph, what was the speed of the current?

223. Two painters can paint a room in 2 hours if they work together. The less experienced painter takes 3 hours more than the more experienced painter to finish the job. How long does it take for each painter to paint the room individually?

224. Two gardeners can do the weekly yard maintenance in 8 minutes if they work together. The older gardener takes 12 minutes more than the younger gardener to finish the job by himself. How long does it take for each gardener to do the weekly yard maintainence individually?

225. It takes two hours for two machines to manufacture 10,000 parts. If Machine #1 can do the job alone in one hour less than Machine #2 can do the job, how long does it take for each machine to manufacture 10,000 parts alone?

226. Sully is having a party and wants to fill his swimming pool. If he only uses his hose it takes 2 hours more than if he only uses his neighbor's hose. If he uses both hoses together, the pool fills in 4 hours. How long does it take for each hose to fill the hot tub?

Writing Exercises

227. Make up a problem involving the product of two consecutive odd integers.

ⓐ Start by choosing two consecutive odd integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where p is the product you found in part (b).

ⓓ Did you get the numbers you started with?

228. Make up a problem involving the product of two consecutive even integers.

ⓐ Start by choosing two consecutive even integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where p is the product you found in part (b).

ⓓ Did you get the numbers you started with?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

9.6 Graph Quadratic Functions Using Properties

Learning Objectives

By the end of this section, you will be able to:

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
   If you missed this problem, review **Example 3.54**.
2. Solve: $2x^2 + 3x - 2 = 0.$
   If you missed this problem, review **Example 6.45**.
3. Evaluate $-\frac{b}{2a}$ when $a = 3$ and $b = -6$.
   If you missed this problem, review **Example 1.21**.

Recognize the Graph of a Quadratic Function

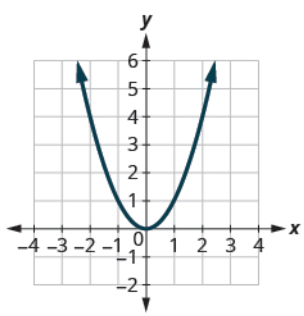
Previously we very briefly looked at the function $f(x) = x^2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f(x) = ax^2 + bx + c$ if $a \neq 0.$ We call this kind of function a quadratic function.

Quadratic Function

A **quadratic function**, where *a*, *b*, and *c* are real numbers and $a \neq 0,$ is a function of the form

$$f(x) = ax^2 + bx + c$$

We graphed the quadratic function $f(x) = x^2$ by plotting points.



x	$f(x) = x^2$	$(x, f(x))$
–3	9	(–3, 9)
–2	4	(–2, 4)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Every quadratic function has a graph that looks like this. We call this figure a **parabola**.

Let's practice graphing a parabola by plotting a few points.

EXAMPLE 9.42

Graph $f(x) = x^2 - 1.$

Solution

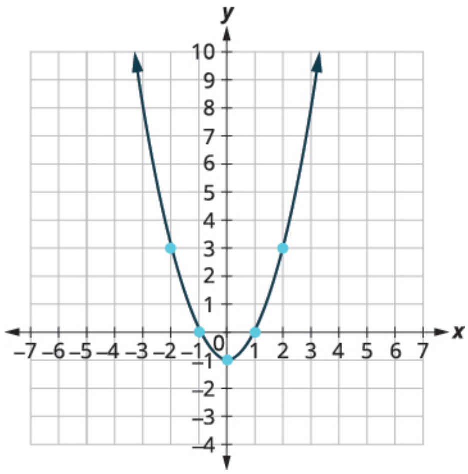
We will graph the function by plotting points.

Choose integer values for x, substitute them into the equation and simplify to find $f(x)$.

Record the values of the ordered pairs in the chart.

$f(x) = x^2 - 1$	
x	$f(x)$
0	–1
1	0
–1	0
2	3
–2	3

Plot the points, and then connect them with a smooth curve. The result will be the graph of the function $f(x) = x^2 - 1$.



TRY IT : : 9.83 Graph $f(x) = -x^2$..

TRY IT : : 9.84 Graph $f(x) = x^2 + 1$.

All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward. See **Figure 9.2**.

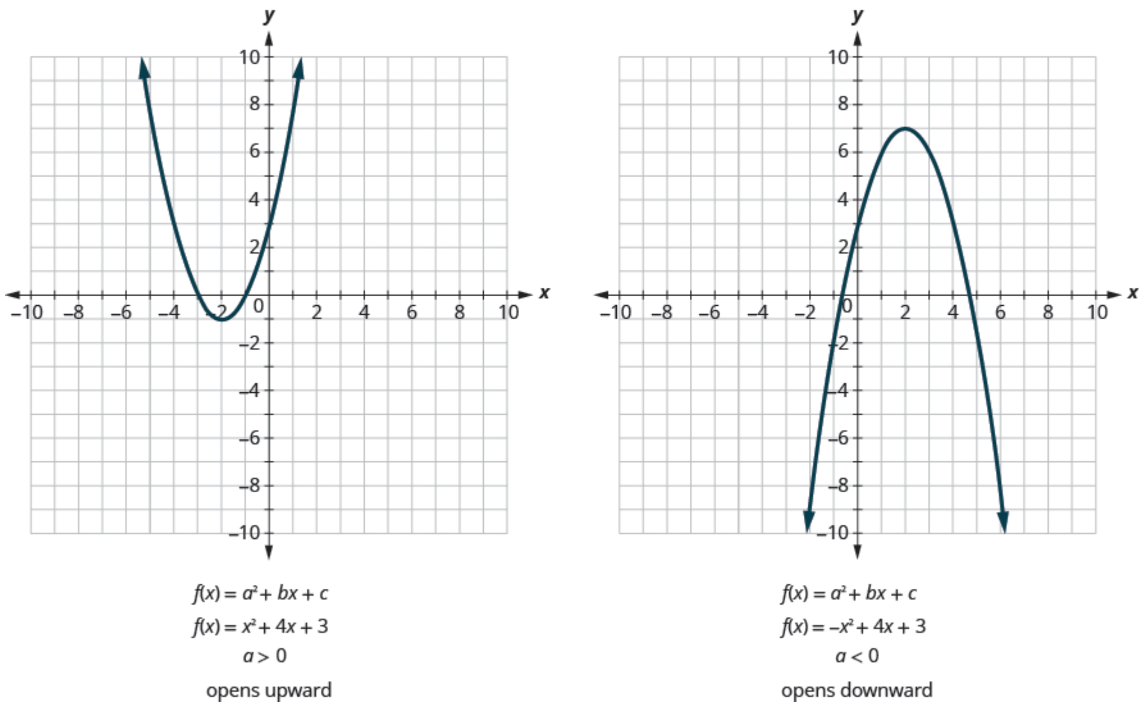


Figure 9.2

Notice that the only difference in the two functions is the negative sign before the quadratic term (x^2 in the equation of the graph in Figure 9.2). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Parabola Orientation

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward
- $a < 0$, the parabola opens downward

EXAMPLE 9.43

Determine whether each parabola opens upward or downward:

ⓐ $f(x) = -3x^2 + 2x - 4$ ⓑ $f(x) = 6x^2 + 7x - 9$.

Solution

ⓐ

Find the value of "a".	$f(x) = ax^2 + bx + c$ $f(x) = -3x^2 + 2x - 4$ $a = -3$
	Since the "a" is negative, the parabola will open downward.

ⓑ

Find the value of "*a*".	$f(x) = ax^2 + bx + c$ $f(x) = 6x^2 + 7x - 9$ $a = 6$
	Since the "*a*" is positive, the parabola will open upward.

TRY IT : : 9.85

Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ $f(x) = 2x^2 + 5x - 2$ ⓑ $f(x) = -3x^2 - 4x + 7.$

TRY IT : : 9.86

Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ $f(x) = -2x^2 - 2x - 3$ ⓑ $f(x) = 5x^2 - 2x - 1.$

Find the Axis of Symmetry and Vertex of a Parabola

Look again at Figure 9.2. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry. See Figure 9.3.

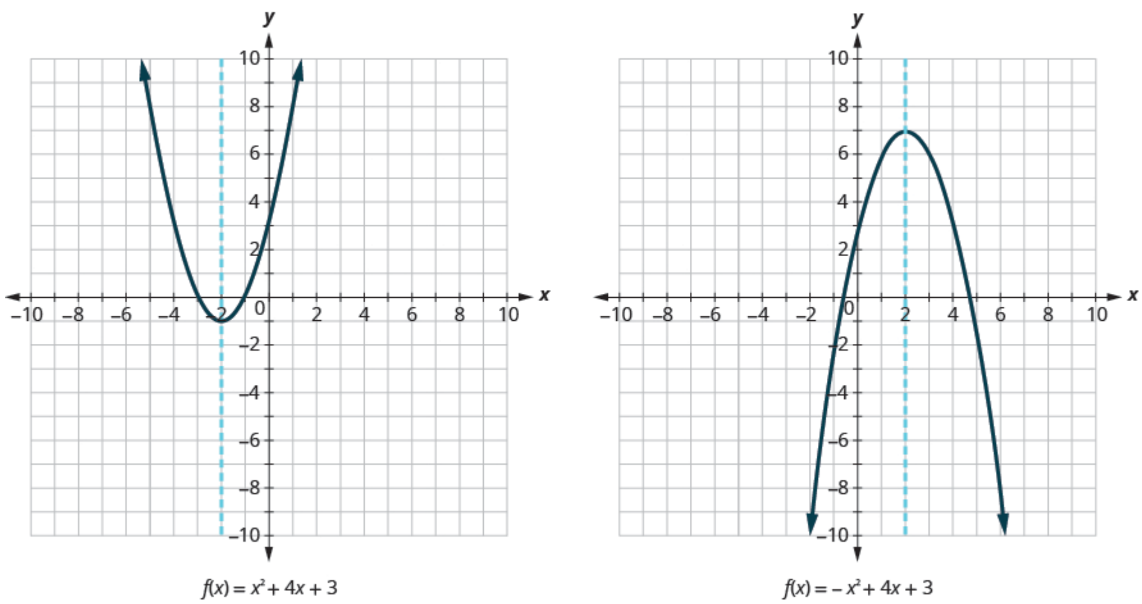


Figure 9.3

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$f(x) = ax^2 + bx + c$	$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its *x*-coordinate is $-\frac{b}{2a}$. To find the *y*-coordinate of the vertex we substitute the value of the *x*-coordinate into the quadratic function.

$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry is $x = -2$	axis of symmetry is $x = 2$
vertex is $(-2, \_\_\_)$	vertex is $(2, \_\_\_)$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
$f(x) = (-2)^2 + 4(-2) + 3$	$f(x) = -(2)^2 + 4(2) + 3$
$f(x) = -1$	$f(x) = 7$
vertex is $(-2, -1)$	vertex is $(2, 7)$

Axis of Symmetry and Vertex of a Parabola

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
- the *y*-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

EXAMPLE 9.44

For the graph of $f(x) = 3x^2 - 6x + 2$ find:

ⓐ the axis of symmetry ⓑ the vertex.

Solution

ⓐ

$$f(x) = ax^2 + bx + c$$
$$f(x) = 3x^2 - 6x + 2$$

The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.	
Substitute the values of a, b into the equation.	$x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$

The axis of symmetry is the line $x = 1$.

ⓑ

	$f(x) = 3x^2 - 6x + 2$
The vertex is a point on the line of symmetry, so its *x*-coordinate will be $x = 1$. Find $f(1)$.	$f(1) = 3(1)^2 - 6(1) + 2$
Simplify.	$f(1) = 3 \cdot 1 - 6 + 2$
The result is the *y*-coordinate.	$f(1) = -1$
	The vertex is $(1, -1)$.

TRY IT : : 9.87 For the graph of $f(x) = 2x^2 - 8x + 1$ find:

ⓐ the axis of symmetry ⓑ the vertex.

TRY IT : : 9.88 For the graph of $f(x) = 2x^2 - 4x - 3$ find:

ⓐ the axis of symmetry ⓑ the vertex.

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the *x*- and *y*-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the *y*-intercept the value of *x* is zero. So to find the *y*-intercept, we substitute *x* = 0 into the function.

Let's find the *y*-intercepts of the two parabolas shown in **Figure 9.4**.

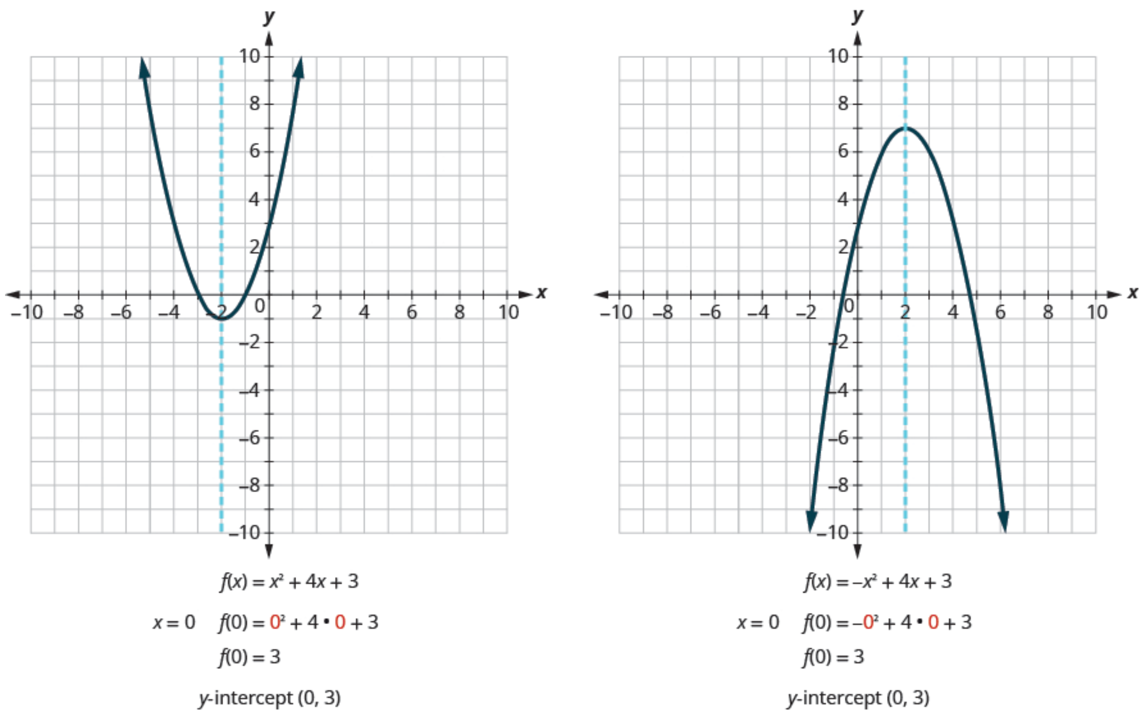


Figure 9.4

An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x) = 0$. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for x.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ 0 &= ax^2 + bx + c \end{aligned}$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the x-intercepts of the two parabolas we looked at. First we will find the x-intercepts of the parabola whose function is $f(x) = x^2 + 4x + 3$.

	$f(x) = x^2 + 4x + 3$
Let $f(x) = 0$.	$0 = x^2 + 4x + 3$
Factor.	$0 = (x + 1)(x + 3)$
Use the Zero Product Property.	$x + 1 = 0 \quad x + 3 = 0$
Solve.	$x = -1 \qquad x = -3$
	The x-intercepts are $(-1, 0)$ and $(-3, 0)$.

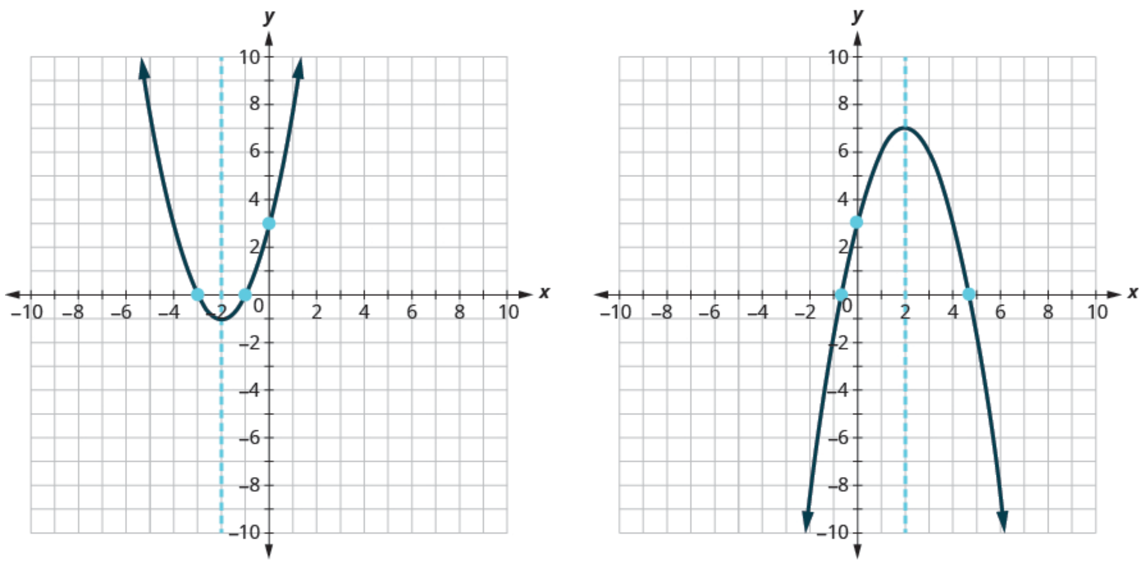
Now we will find the x-intercepts of the parabola whose function is $f(x) = -x^2 + 4x + 3$.

	$f(x) = -x^2 + 4x + 3$
Let $f(x) = 0$.	$0 = -x^2 + 4x + 3$
This quadratic does not factor, so we use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$a = -1,\ b = 4,\ c = 3$	$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(3)}}{2(-1)}$
Simplify.	$x = \frac{-4 \pm \sqrt{28}}{-2}$
	$x = \frac{-4 \pm 2\sqrt{7}}{-2}$
	$x = \frac{-2\left(2 \pm \sqrt{7}\right)}{-2}$
	$x = 2 \pm \sqrt{7}$
	The *x*-intercepts are $\left(2 + \sqrt{7},\ 0\right)$ and $\left(2 - \sqrt{7},\ 0\right)$.

We will use the decimal approximations of the *x*-intercepts, so that we can locate these points on the graph,

$$\left(2 + \sqrt{7},\ 0\right) \approx (4.6,\ 0) \qquad \left(2 - \sqrt{7},\ 0\right) \approx (-0.6,\ 0)$$

Do these results agree with our graphs? See **Figure 9.5**.



$f(x) = x^2 + 4x + 3$
y-intercept (0, 3)
x-intercepts (–1, 0) and (–3, 0)

$f(x) = -x^2 + 4x + 3$
y-intercept (0, 3)
x-intercepts $(2 + \sqrt{7}, 0) \approx (4.6, 0)$
$(2 - \sqrt{7}, 0) \approx (-0.6, 0)$

Figure 9.5

Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

y-intercept	x-intercepts
Let $x = 0$ and solve for $f(x)$.	Let $f(x) = 0$ and solve for x.

EXAMPLE 9.45

Find the intercepts of the parabola whose function is $f(x) = x^2 - 2x - 8$.

Solution

To find the ***y***-intercept, let $x = 0$ and solve for $f(x)$.	$f(x) = x^2 - 2x - 8$
	$f(0) = 0^2 - 2 \cdot 0 - 8$
	$f(0) = -8$
	When $x = 0$, then $f(0) = -8$. The y-intercept is the point $(0, -8)$.
To find the ***x***-intercept, let $f(x) = 0$ and solve for x.	$f(x) = x^2 - 2x - 8$
	$0 = x^2 - 2x - 8$
Solve by factoring.	$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$

$$4 = x \qquad -2 = x$$

When $f(x) = 0$, then $x = 4$ or $x = -2$.
The *x*-intercepts are the points $(4, 0)$ and $(-2, 0)$.

TRY IT : : 9.89 Find the intercepts of the parabola whose function is $f(x) = x^2 + 2x - 8$.

TRY IT : : 9.90 Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 12$.

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for *x* and the results were the solutions to the equation.

We are now looking at quadratic functions of the form $f(x) = ax^2 + bx + c$. The graphs of these functions are parabolas. The *x*-intercepts of the parabolas occur where $f(x) = 0$.

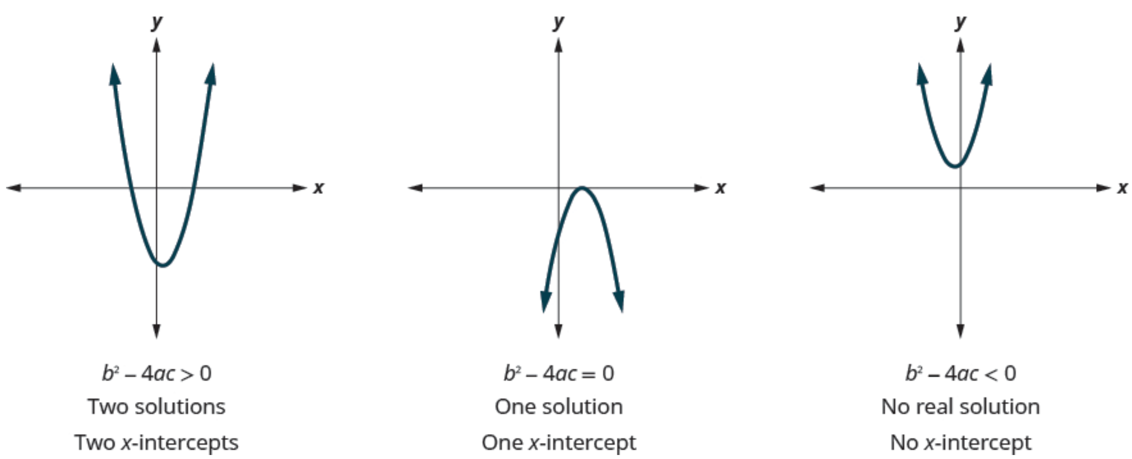
For example:

Quadratic equation		Quadratic function
$\begin{aligned} x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \\ x - 5 = 0 \quad x + 3 &= 0 \\ x = 5 \qquad x &= -3 \end{aligned}$	Let $f(x) = 0$.	$\begin{aligned} f(x) &= x^2 - 2x - 15 \\ 0 &= x^2 - 2x - 15 \\ 0 &= (x-5)(x+3) \\ x - 5 &= 0 \quad x + 3 = 0 \\ x &= 5 \qquad x = -3 \end{aligned}$ $(5, 0)$ and $(-3, 0)$ *x*-intercepts

The solutions of the quadratic function are the *x* values of the *x*-intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the *x*-intercepts of the graphs, the number of *x*-intercepts is the same as the number of solutions.

Previously, we used the discriminant to determine the number of solutions of a quadratic function of the form $ax^2 + bx + c = 0$. Now we can use the discriminant to tell us how many *x*-intercepts there are on the graph.



Before you to find the values of the *x*-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

EXAMPLE 9.46

Find the intercepts of the parabola for the function $f(x) = 5x^2 + x + 4$.

Solution

	$f(x) = 5x^2 + x + 4$
To find the *y*-intercept, let $x = 0$ and solve for $f(x)$.	$f(0) = 5 \cdot 0^2 + 0 + 4$
	$f(0) = 4$
	When $x = 0$, then $f(0) = 4$. The *y*-intercept is the point $(0, 4)$.
To find the *x*-intercept, let $f(x) = 0$ and solve for x.	$f(x) = 5x^2 + x + 4$
	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions which is also the number of *x*-intercepts.	
$b^2 - 4ac$ $1^2 - 4 \cdot 5 \cdot 4$ $1 - 80$ -79	
	Since the value of the discriminant is negative, there is no real solution to the equation. There are no *x*-intercepts.

TRY IT : : 9.91 Find the intercepts of the parabola whose function is $f(x) = 3x^2 + 4x + 4$.

TRY IT : : 9.92 Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 5$.

Graph Quadratic Functions Using Properties

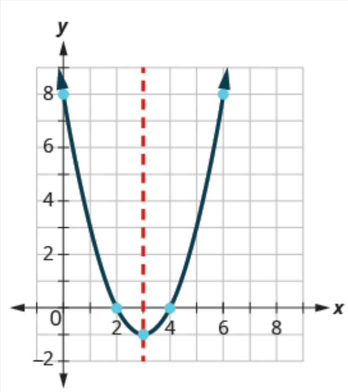
Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

EXAMPLE 9.47 HOW TO GRAPH A QUADRATIC FUNCTION USING PROPERTIES

Graph $f(x) = x^2 - 6x + 8$ by using its properties.

Solution

Step 1. Determine whether the parabola opens upward or downward.	Look at *a* in the equation. $f(x) = x^2 - 6x + 8$ Since *a* is positive, the parabola opens upward. 	$f(x) = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.**

Step 2. Find the axis of symmetry.	$f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
Step 3. Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is (3, −1).**
Step 4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find $f(0)$. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	y-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The y-intercept is (0, 8).** Point symmetric to y-intercept: **The point is (6, 8).**
Step 5. Find the x-intercepts. Find additional points if needed.	We solve $f(x) = 0$. We can solve this quadratic equation by factoring.	x-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The x-intercepts are (2, 0) and (4, 0).**
Step 6. Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

TRY IT : : 9.93 Graph $f(x) = x^2 + 2x - 8$ by using its properties.

TRY IT : : 9.94 Graph $f(x) = x^2 - 8x + 12$ by using its properties.

We list the steps to take in order to graph a quadratic function here.

HOW TO : : TO GRAPH A QUADRATIC FUNCTION USING PROPERTIES.

Step 1. Determine whether the parabola opens upward or downward.
Step 2. Find the equation of the axis of symmetry.
Step 3. Find the vertex.
Step 4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
Step 5. Find the x-intercepts. Find additional points if needed.
Step 6. Graph the parabola.

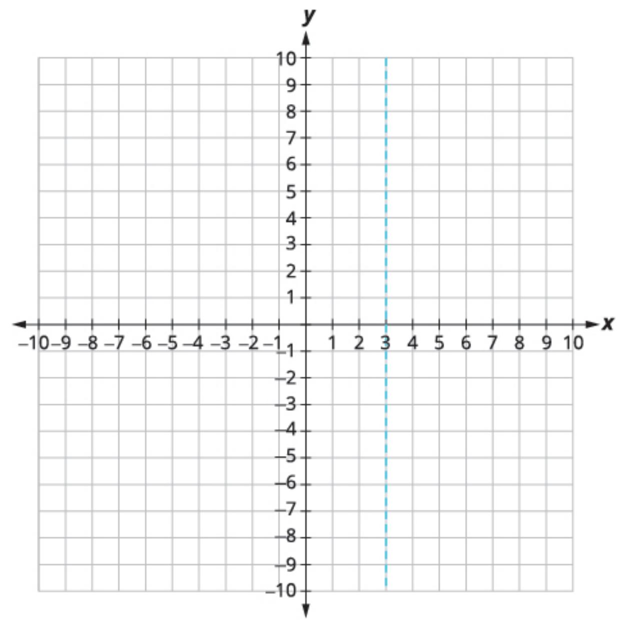
We were able to find the x-intercepts in the last example by factoring. We find the x-intercepts in the next example by factoring, too.

EXAMPLE 9.48

Graph $f(x) = x^2 + 6x - 9$ by using its properties.

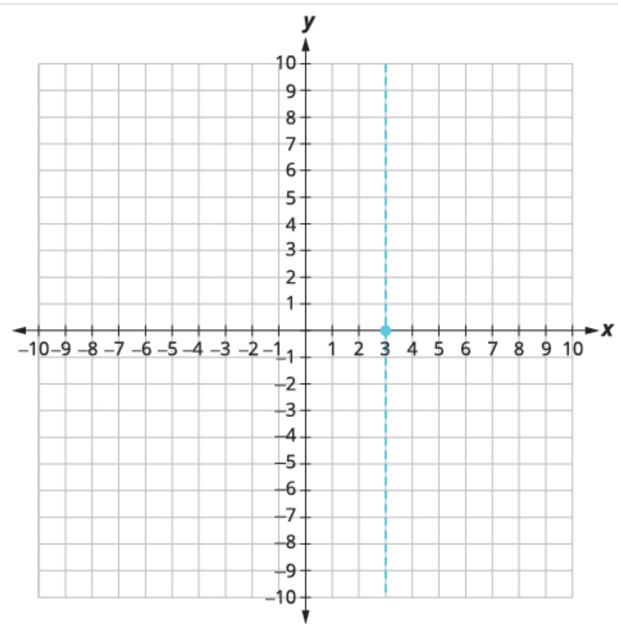
Solution

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{6}{2(-1)}$
	$x = 3$
	The axis of symmetry is $x = 3$. The vertex is on the line $x = 3$.

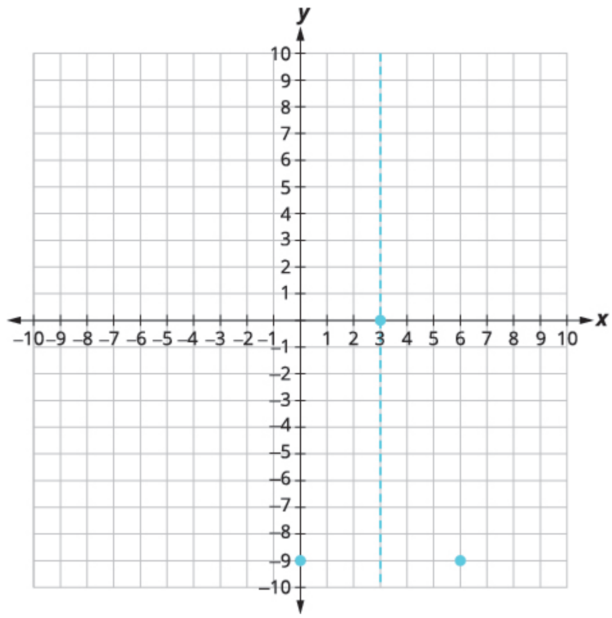
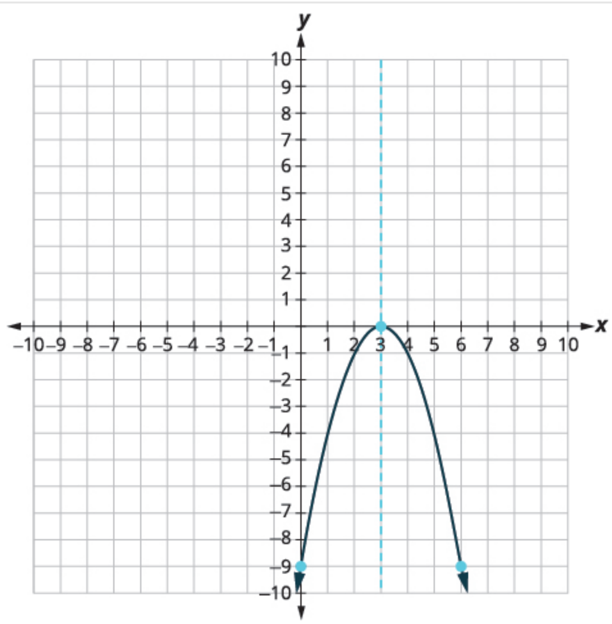


Find $f(3)$.	$f(x) = -x^2 + 6x - 9$
	$f(3) = -3^2 + 6 \cdot 3 - 9$
	$f(3) = -9 + 18 - 9$
	$f(3) = 0$

The vertex is $(3, 0)$.



The *y*-intercept occurs when $x = 0$. Find $f(0)$.	$f(x) = -x^2 + 6x - 9$
Substitute $x = 0$.	$f(0) = -0^2 + 6 \cdot 0 - 9$
Simplify.	$f(0) = -9$

	The y-intercept is $(0, -9)$.
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.	
	Point symmetric to the y-intercept is $(6, -9)$
The x-intercept occurs when $f(x) = 0$.	$f(x) = -x^2 + 6x - 9$
Find $f(x) = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for x.	$0 = 3$
Connect the points to graph the parabola.	

TRY IT : : 9.95 Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.

TRY IT : : 9.96 Graph $f(x) = 4x^2 + 24x + 36$ by using its properties.

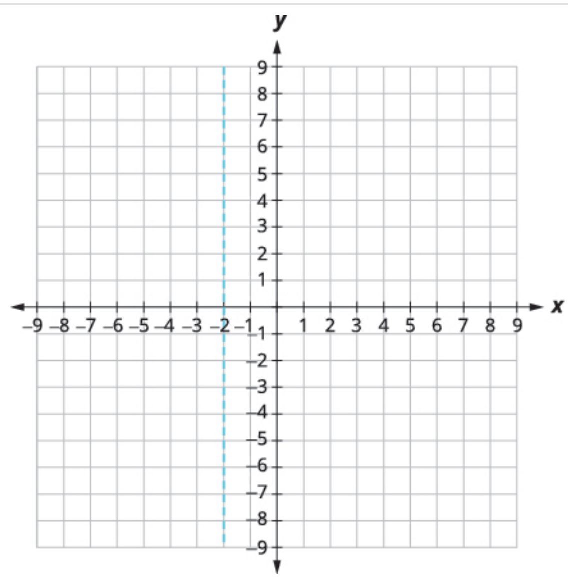
For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

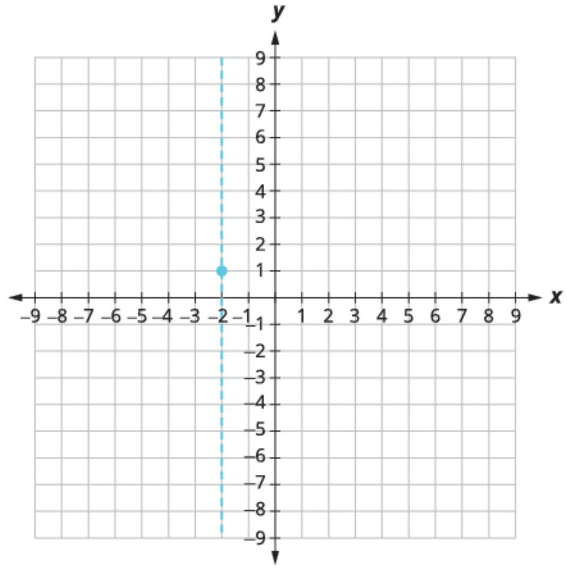
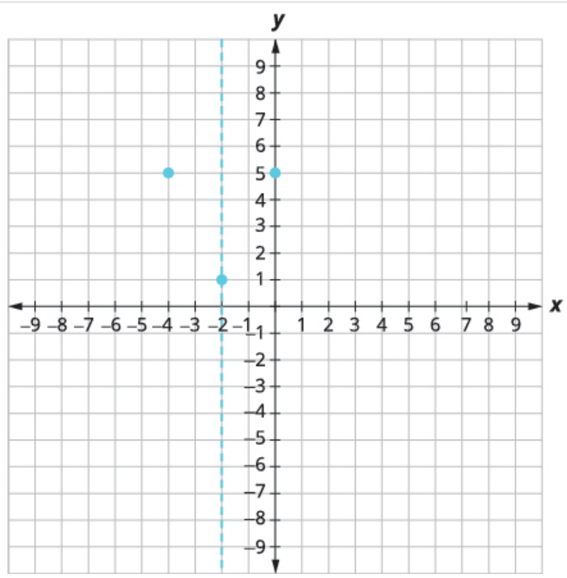
How many x-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

EXAMPLE 9.49

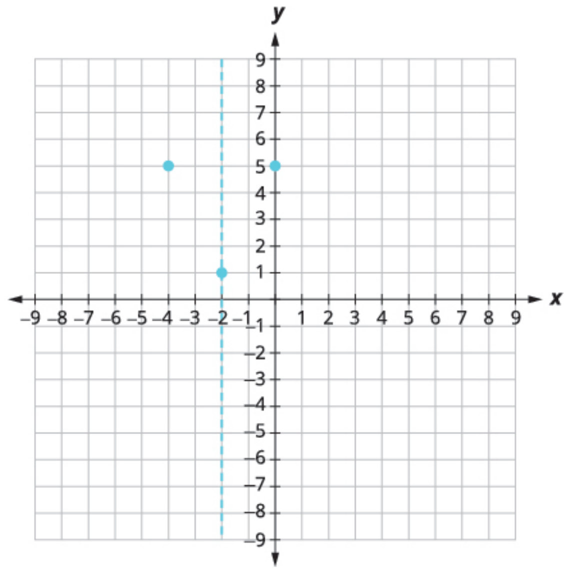
Graph $f(x) = x^2 + 4x + 5$ by using its properties.

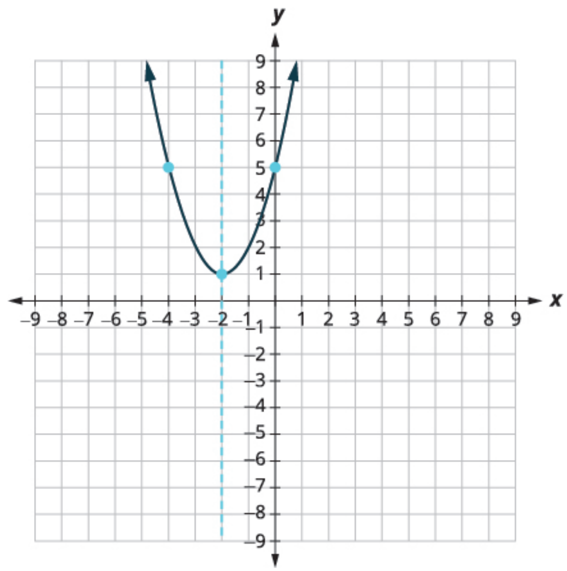
Solution

	$f(x) = ax^2 + bx + c$ $f(x) = x^2 + 4x + 5$
Since a is 1, the parabola opens upward.	
	
To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{4}{(2)1}$
	$x = -2$
	The equation of the axis of symmetry is $x = -2$.
	
The vertex is on the line $x = -2$.	
Find $f(x)$ when $x = -2$.	$f(x) = x^2 + 4x + 5$
	$f(-2) = (-2)^2 + 4(-2) + 5$
	$f(-2) = 4 - 8 + 5$

	$f(-2) = 1$
	The vertex is $(-2, 1)$.
	
The *y*-intercept occurs when $x = 0$.	$f(x) = x^2 + 4x - 5$
Find $f(0)$.	$f(0) = 5$
Simplify.	$f(0) = 5$
	The *y*-intercept is $(0, 5)$.
The point $(-4, 5)$ is two units to the left of the line of symmetry. The point two units to the right of the line of symmetry is $(0, 5)$.	
	Point symmetric to the *y*-intercept is $(-4, 5)$.

The *x*-intercept occurs when $f(x) = 0$.



Find $f(x) = 0$.	$0 = x^2 + 4x + 5$
Test the discriminant.	
	$b^2 - 4ac$
	$4^2 - 4 \cdot 1 \cdot 5$
	$16 - 20$
	-4
Since the value of the discriminant is negative, there is no real solution and so no *x*-intercept.	
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.	

TRY IT : : 9.97 Graph $f(x) = x^2 - 2x + 3$ by using its properties.

TRY IT : : 9.98 Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.

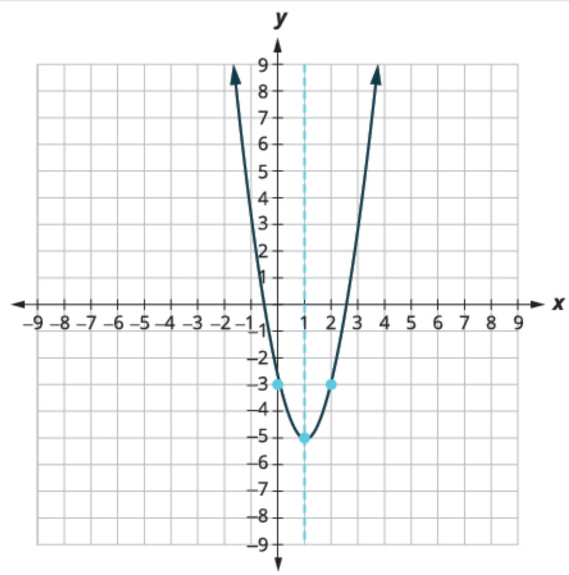
Finding the y-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the Quadratic Formula to find the x-intercepts.

EXAMPLE 9.50

Graph $f(x) = 2x^2 - 4x - 3$ by using its properties.

Solution

	$f(x) = ax^2 + bx + c$ $f(x) = 2x^2 - 4x - 3$
Since a is 2, the parabola opens upward.	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{-4}{2 \cdot 2}$
	$x = 1$
	The equation of the axis of symmetry is $x = 1$.
The vertex is on the line $x = 1$.	$f(x) = 2x^2 - 4x - 3$
Find $f(1)$.	$f(x) = 2(1)^2 - 4(1) - 3$
	$f(1) = 2 - 4 - 3$
	$f(1) = -5$
	The vertex is $(1, -5)$.
The y-intercept occurs when $x = 0$.	$f(x) = 2x^2 - 4x - 3$
Find $f(0)$.	$f(0) = 2(0)^2 - 4(0) - 3$
Simplify.	$f(0) = -3$
	The y-intercept is $(0, -3)$.
The point $(0, -3)$ is one unit to the left of the line of symmetry.	**Point symmetric to the y-intercept is $(2, -3)$**
The point one unit to the right of the line of symmetry is $(2, -3)$.	
The x-intercept occurs when $y = 0$.	$f(x) = 2x^2 - 4x - 3$
Find $f(x) = 0$.	$0 = 2x^2 - 4x - 3$
Use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute in the values of $a, b,$ and c.	$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(3)}}{2(2)}$
Simplify.	$x = \frac{-4 \pm \sqrt{16 + 24}}{4}$

Simplify inside the radical.	$x = \dfrac{4 \pm \sqrt{40}}{4}$
Simplify the radical.	$x = \dfrac{4 \pm 2\sqrt{10}}{4}$
Factor the GCF.	$x = \dfrac{2\left(2 \pm \sqrt{10}\right)}{4}$
Remove common factors.	$x = \dfrac{2 \pm \sqrt{10}}{2}$
Write as two equations.	$x = \dfrac{2 + \sqrt{10}}{2}, \quad x = \dfrac{2 - \sqrt{10}}{2}$
Approximate the values.	$x \approx 2.5, \quad x \approx -0.6$
	The approximate values of the *x*-intercepts are $(2.5, 0)$ and $(-0.6, 0)$.
Graph the parabola using the points found.	

TRY IT : : 9.99 Graph $f(x) = 5x^2 + 10x + 3$ by using its properties.

TRY IT : : 9.100 Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.

Solve Maximum and Minimum Applications

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The *y*-coordinate of the vertex is the minimum value of a parabola that opens upward. It is the maximum value of a parabola that opens downward. See Figure 9.6.

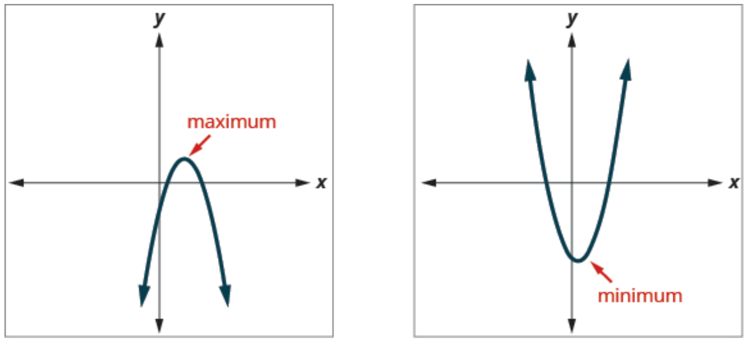


Figure 9.6

Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

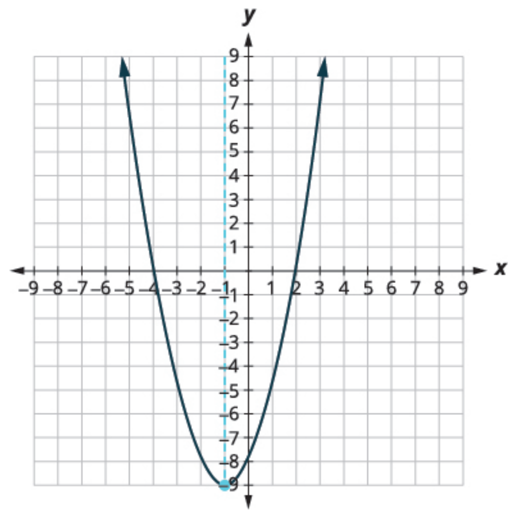
- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

EXAMPLE 9.51

Find the minimum or maximum value of the quadratic function $f(x) = x^2 + 2x - 8$.

Solution

	$f(x) = x^2 + 2x - 8$
Since *a* is positive, the parabola opens upward. The quadratic equation has a minimum.	
Find the equation of the axis of symmetry.	$x = -\frac{b}{2a}$
	$x = -\frac{2}{2 \times 1}$
	$x = -1$
	The equation of the axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$f(x) = x^2 + 2x - 8$
Find $f(-1)$.	$f(-1) = (-1)^2 + 2(-1) - 8$
	$f(-1) = 1 - 2 - 8$
	$f(-1) = -9$
	The vertex is $(-1, -9)$.
Since the parabola has a minimum, the *y*-coordinate of the vertex is the minimum *y*-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x = -1$.	



Show the graph to verify the result.

TRY IT : : 9.101 Find the maximum or minimum value of the quadratic function $f(x) = x^2 - 8x + 12$.

TRY IT : : 9.102 Find the maximum or minimum value of the quadratic function $f(x) = -4x^2 + 16x - 11$.

We have used the formula

$$h(t) = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, h , of an object shot upwards into the air with initial velocity, v_0, after t seconds .

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex (t, h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

EXAMPLE 9.52

The quadratic equation $h(x) = -16t^2 + 176t + 4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

ⓐ How many seconds will it take the volleyball to reach its maximum height? ⓑ Find the maximum height of the volleyball.

Solution

$$h(t) = -16t^2 + 176t + 4$$

Since a is negative, the parabola opens downward.
The quadratic function has a maximum.

ⓐ

Find the equation of the axis of symmetry.	$t = -\frac{b}{2a}$
	$t = -\frac{176}{2(-16)}$
	$t = 5.5$
	The equation of the axis of symmetry is $t = 5.5$.
The vertex is on the line $t = 5.5$.	The maximum occurs when $t = 5.5$ seconds.

ⓑ

Find $h(5.5)$.	$h(t) = -16t^2 + 176t + 4$
	$h(t) = -16(5.5)^2 + 176(5.5) + 4$
Use a calculator to simplify.	$h(t) = 488$
	The vertex is $(5.5,\ 488)$.

Since the parabola has a maximum, the *h*-coordinate of the vertex is the maximum value of the quadratic function.

The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

TRY IT : : 9.103

Solve, rounding answers to the nearest tenth.

The quadratic function $h(x) = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

TRY IT : : 9.104

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of. $h(x) = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height?

MEDIA : :

Access these online resources for additional instruction and practice with graphing quadratic functions using properties.

- **Quadratic Functions: Axis of Symmetry and Vertex (https://openstax.org/l/37QuadFunct1)**
- **Finding x- and y-intercepts of a Quadratic Function (https://openstax.org/l/37QuadFunct2)**
- **Graphing Quadratic Functions (https://openstax.org/l/37QuadFunct3)**
- **Solve Maxiumum or Minimum Applications (https://openstax.org/l/37QuadFunct4)**
- **Quadratic Applications: Minimum and Maximum (https://openstax.org/l/37QuadFunct5)**

9.6 EXERCISES

Practice Makes Perfect

Recognize the Graph of a Quadratic Function

In the following exercises, graph the functions by plotting points.

229. $f(x) = x^2 + 3$

230. $f(x) = x^2 - 3$

231. $y = -x^2 + 1$

232. $f(x) = -x^2 - 1$

For each of the following exercises, determine if the parabola opens up or down.

233.
ⓐ $f(x) = -2x^2 - 6x - 7$
ⓑ $f(x) = 6x^2 + 2x + 3$

234.
ⓐ $f(x) = 4x^2 + x - 4$
ⓑ $f(x) = -9x^2 - 24x - 16$

235.
ⓐ $f(x) = -3x^2 + 5x - 1$
ⓑ $f(x) = 2x^2 - 4x + 5$

236.
ⓐ $f(x) = x^2 + 3x - 4$
ⓑ $f(x) = -4x^2 - 12x - 9$

Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find ⓐ the equation of the axis of symmetry and ⓑ the vertex of its graph.

237. $f(x) = x^2 + 8x - 1$

238. $f(x) = x^2 + 10x + 25$

239. $f(x) = -x^2 + 2x + 5$

240. $f(x) = -2x^2 - 8x - 3$

Find the Intercepts of a Parabola

In the following exercises, find the intercepts of the parabola whose function is given.

241. $f(x) = x^2 + 7x + 6$

242. $f(x) = x^2 + 10x - 11$

243. $f(x) = x^2 + 8x + 12$

244. $f(x) = x^2 + 5x + 6$

245. $f(x) = -x^2 + 8x - 19$

246. $f(x) = -3x^2 + x - 1$

247. $f(x) = x^2 + 6x + 13$

248. $f(x) = x^2 + 8x + 12$

249. $f(x) = 4x^2 - 20x + 25$

250. $f(x) = -x^2 - 14x - 49$

251. $f(x) = -x^2 - 6x - 9$

252. $f(x) = 4x^2 + 4x + 1$

Graph Quadratic Functions Using Properties

In the following exercises, graph the function by using its properties.

253. $f(x) = x^2 + 6x + 5$

254. $f(x) = x^2 + 4x - 12$

255. $f(x) = x^2 + 4x + 3$

256. $f(x) = x^2 - 6x + 8$

257. $f(x) = 9x^2 + 12x + 4$

258. $f(x) = -x^2 + 8x - 16$

259. $f(x) = -x^2 + 2x - 7$

260. $f(x) = 5x^2 + 2$

261. $f(x) = 2x^2 - 4x + 1$

262. $f(x) = 3x^2 - 6x - 1$

263. $f(x) = 2x^2 - 4x + 2$

264. $f(x) = -4x^2 - 6x - 2$

265. $f(x) = -x^2 - 4x + 2$

266. $f(x) = x^2 + 6x + 8$

267. $f(x) = 5x^2 - 10x + 8$

268. $f(x) = -16x^2 + 24x - 9$

269. $f(x) = 3x^2 + 18x + 20$

270. $f(x) = -2x^2 + 8x - 10$

Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value of each function.

271. $f(x) = 2x^2 + x - 1$

272. $y = -4x^2 + 12x - 5$

273. $y = x^2 - 6x + 15$

274. $y = -x^2 + 4x - 5$

275. $y = -9x^2 + 16$

276. $y = 4x^2 - 49$

In the following exercises, solve. Round answers to the nearest tenth.

277. An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

278. A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic function $h(t) = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

279. A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

280. A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic function $h(t) = -16t^2 + 122t + 0$ to find how long it will take for the ball to reach its maxiumum height, and then find the maximum height.

281. A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

282. A retailer who sells backpacks estimates that by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic function $R(x) = -x^2 + 100x$ is used to find the R, received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

283. A retailer who sells fashion boots estimates that by selling them for x dollars each, he will be able to sell $70 - x$ boots a week. Use the quadratic function $R(x) = -x^2 + 70x$ to find the revenue received when the average selling price of a pair of fashion boots is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

284. A cell phone company estimates that by charging x dollars each for a certain cell phone, they can sell $8 - x$ cell phones per day. Use the quadratic function $R(x) = -x^2 + 8x$ to find the revenue received when the selling price of a cell phone is x. Find the selling price that will give them the maximum revenue, and then find the amount of the maximum revenue.

285. A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A(x) = x(240 - 2x)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

286. A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic function $A(x) = x(100 - 2x)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

287. A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the "walls." He wants to maximize the area using 80 feet of fencing. The quadratic function $A(x) = x(80 - 2x)$ gives the area of the patio, where x is the width of one side. Find the maximum area of the patio.

288. A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic function $A(x) = x(300 - 2x)$ to determine the maximum area of the fenced in yard.

Writing Exercise

289. How do the graphs of the functions $f(x) = x^2$ and $f(x) = x^2 - 1$ differ? We graphed them at the start of this section. What is the difference between their graphs? How are their graphs the same?

290. Explain the process of finding the vertex of a parabola.

291. Explain how to find the intercepts of a parabola.

292. How can you use the discriminant when you are graphing a quadratic function?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

9.7 Graph Quadratic Functions Using Transformations

Learning Objectives

By the end of this section, you will be able to:

- Graph quadratic functions of the form $f(x) = x^2 + k$
- Graph quadratic functions of the form $f(x) = (x - h)^2$
- Graph quadratic functions of the form $f(x) = ax^2$
- Graph quadratic functions using transformations
- Find a quadratic function from its graph

Be Prepared!

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
   If you missed this problem, review **Example 3.54**.
2. Factor completely: $y^2 - 14y + 49$.
   If you missed this problem, review **Example 6.24**.
3. Factor completely: $2x^2 - 16x + 32$.
   If you missed this problem, review **Example 6.26**.

Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the last section, we learned how to graph quadratic functions using their properties. Another method involves starting with the basic graph of $f(x) = x^2$ and 'moving' it according to information given in the function equation. We call this graphing quadratic functions using transformations.

In the first example, we will graph the quadratic function $f(x) = x^2$ by plotting points. Then we will see what effect adding a constant, *k*, to the equation will have on the graph of the new function $f(x) = x^2 + k$.

EXAMPLE 9.53

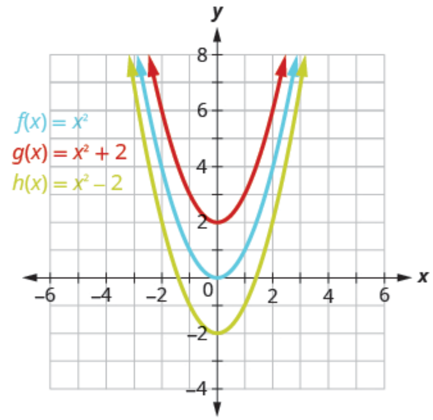
Graph $f(x) = x^2$, $g(x) = x^2 + 2$, and $h(x) = x^2 - 2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

Solution

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = x^2 + 2$	$(x, g(x))$	$h(x) = x^2 - 2$	$(x, h(x))$
–3	9	(–3, 9)	9 + 2	(–3, 11)	9 – 2	(–3, 7)
–2	4	(–2, 4)	4 + 2	(–2, 6)	4 – 2	(–2, 2)
–1	1	(–1, 1)	1 + 2	(–1, 3)	1 – 2	(–1, 1)
0	0	(0, 0)	0 + 2	(0, 2)	0 – 2	(0, –2)
1	1	(1, 1)	1 + 2	(1, 3)	1 – 2	(1, –1)
2	4	(2, 4)	4 + 2	(2, 6)	4 – 2	(2, 2)
3	9	(3, 9)	9 + 2	(3, 11)	9 – 2	(3, 7)

The $g(x)$ values are two more than the $f(x)$ values. Also, the $h(x)$ values are two less than the $f(x)$ values. Now we will graph all three functions on the same rectangular coordinate system.



The graph of $g(x) = x^2 + 2$ is the same as the graph of $f(x) = x^2$ but shifted up 2 units.

The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ but shifted down 2 units.

The graph of $g(x) = x^2 + 2$ is the same as the graph of $f(x) = x^2$ but shifted up 2 units.

The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ but shifted down 2 units.

TRY IT : : 9.105

ⓐ Graph $f(x) = x^2$, $g(x) = x^2 + 1$, and $h(x) = x^2 - 1$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

TRY IT : : 9.106

ⓐ Graph $f(x) = x^2$, $g(x) = x^2 + 6$, and $h(x) = x^2 - 6$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

The last example shows us that to graph a quadratic function of the form $f(x) = x^2 + k$, we take the basic parabola graph of $f(x) = x^2$ and vertically shift it up $(k > 0)$ or shift it down $(k < 0)$.

This transformation is called a vertical shift.

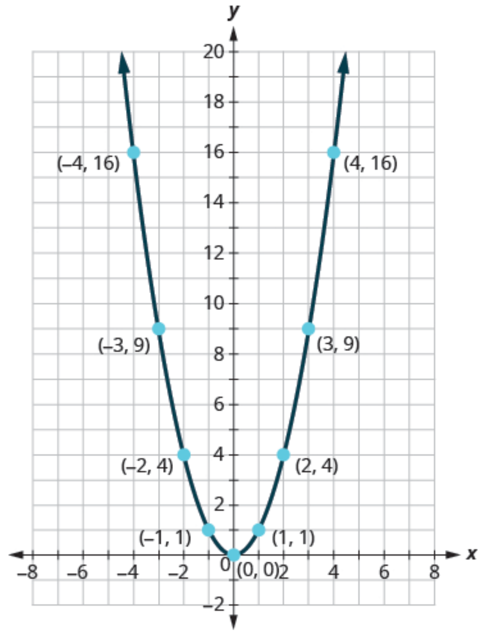
Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift

The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.

- If $k > 0$, shift the parabola vertically up k units.
- If $k < 0$, shift the parabola vertically down $|k|$ units.

Now that we have seen the effect of the constant, k, it is easy to graph functions of the form $f(x) = x^2 + k$. We just start with the basic parabola of $f(x) = x^2$ and then shift it up or down.

It may be helpful to practice sketching $f(x) = x^2$ quickly. We know the values and can sketch the graph from there.



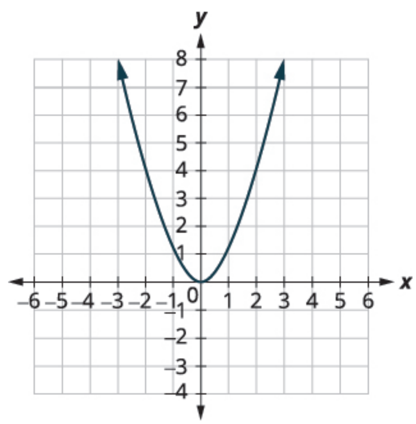
Once we know this parabola, it will be easy to apply the transformations. The next example will require a vertical shift.

EXAMPLE 9.54

Graph $f(x) = x^2 - 3$ using a vertical shift.

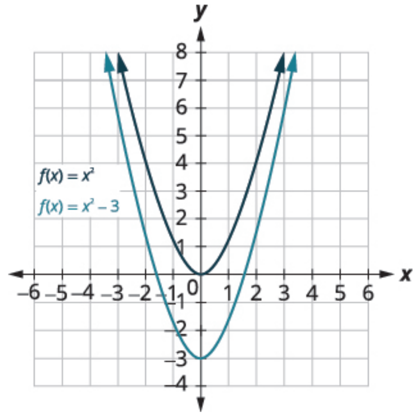
Solution

We first draw the graph of $f(x) = x^2$ on the grid.



Determine k.	$f(x) = x^2 + k$ $f(x) = x^2 - 3$
	$k = -3$

Shift the graph $f(x) = x^2$ down 3.



TRY IT : : 9.107 Graph $f(x) = x^2 - 5$ using a vertical shift.

TRY IT : : 9.108 Graph $f(x) = x^2 + 7$ using a vertical shift.

Graph Quadratic Functions of the form $f(x) = (x - h)^2$

In the first example, we graphed the quadratic function $f(x) = x^2$ by plotting points and then saw the effect of adding a constant *k* to the function had on the resulting graph of the new function $f(x) = x^2 + k$.

We will now explore the effect of subtracting a constant, *h*, from *x* has on the resulting graph of the new function $f(x) = (x - h)^2$.

EXAMPLE 9.55

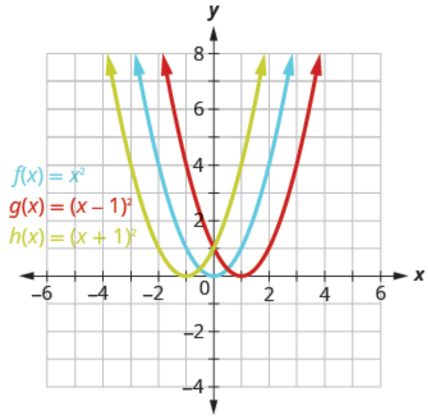
Graph $f(x) = x^2$, $g(x) = (x - 1)^2$, and $h(x) = (x + 1)^2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

Solution

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = (x - 1)^2$	$(x, g(x))$	$h(x) = (x + 1)^2$	$(x, h(x))$
–3	9	(–3, 9)	16	(–3, 16)	4	(–3, 4)
–2	4	(–2, 4)	9	(–2, 9)	1	(–2, 1)
–1	1	(–1, 1)	4	(–1, 4)	0	(–1, 0)
0	0	(0, 0)	1	(0, 1)	1	(0, 1)
1	1	(1, 1)	0	(1, 0)	4	(1, 4)
2	4	(2, 4)	1	(2, 1)	9	(2, 9)
3	9	(3, 9)	4	(3, 4)	16	(3, 16)

The $g(x)$ values and the $h(x)$ values share the common numbers 0, 1, 4, 9, and 16, but are shifted.



The graph of $g(x) = (x - 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 1 unit.

The graph of $h(x) = (x + 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 1 unit.

$g(x) = (x - 1)^2$	$h(x) = (x + 1)^2$
$\longrightarrow$ 1 unit	$\longleftarrow$ 1 unit

> **TRY IT : :** 9.109
>
> ⓐ Graph $f(x) = x^2$, $g(x) = (x + 2)^2$, and $h(x) = (x - 2)^2$ on the same rectangular coordinate system.
>
> ⓑ Describe what effect adding a constant to the function has on the basic parabola.

> **TRY IT : :** 9.110
>
> ⓐ Graph $f(x) = x^2$, $g(x) = x^2 + 5$, and $h(x) = x^2 - 5$ on the same rectangular coordinate system.
>
> ⓑ Describe what effect adding a constant to the function has on the basic parabola.

The last example shows us that to graph a quadratic function of the form $f(x) = (x - h)^2$, we take the basic parabola graph of $f(x) = x^2$ and shift it left ($h > 0$) or shift it right ($h < 0$).

This transformation is called a horizontal shift.

Graph a Quadratic Function of the form $f(x) = (x - h)^2$ Using a Horizontal Shift

The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.

- If $h > 0$, shift the parabola horizontally left h units.
- If $h < 0$, shift the parabola horizontally right $|h|$ units.

Now that we have seen the effect of the constant, h, it is easy to graph functions of the form $f(x) = (x - h)^2$. We just start with the basic parabola of $f(x) = x^2$ and then shift it left or right.

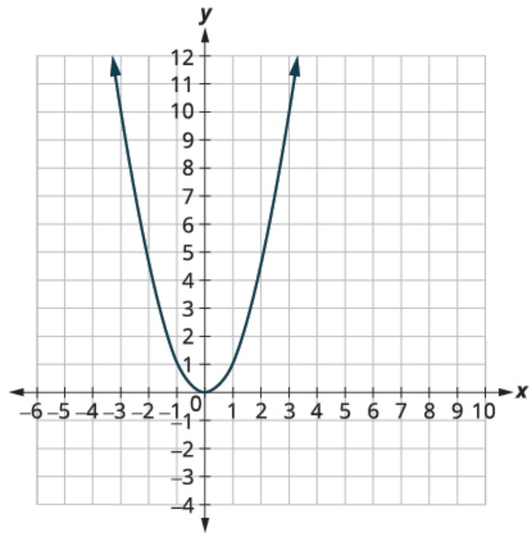
The next example will require a horizontal shift.

EXAMPLE 9.56

Graph $f(x) = (x - 6)^2$ using a horizontal shift.

Solution

We first draw the graph of $f(x)=x^2$ on the grid.

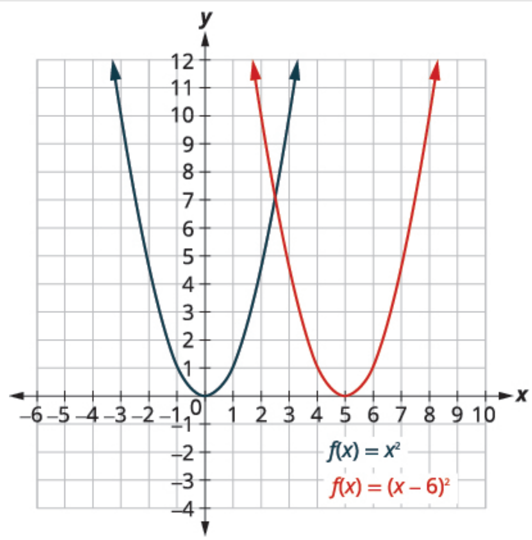


Determine *h*.

$f(x) = (x-h)^2$
$f(x) = (x-6)^2$

$h=6$

Shift the graph $f(x)=x^2$ to the right 6 units.



TRY IT : : 9.111 Graph $f(x)=(x-4)^2$ using a horizontal shift.

TRY IT : : 9.112 Graph $f(x)=(x+6)^2$ using a horizontal shift.

Now that we know the effect of the constants *h* and *k*, we will graph a quadratic function of the form $f(x)=(x-h)^2+k$ by first drawing the basic parabola and then making a horizontal shift followed by a vertical shift. We could do the vertical shift followed by the horizontal shift, but most students prefer the horizontal shift followed by the vertical.

EXAMPLE 9.57

Graph $f(x) = (x + 1)^2 - 2$ using transformations.

Solution

This function will involve two transformations and we need a plan.

Let's first identify the constants *h*, *k*.

$$f(x) = (x + 1)^2 - 2$$

$$f(x) = \quad (x - h)^2 \quad + \quad k$$

$$f(x) = (x - (-1))^2 + (-2)$$

$$h = -1 \quad k = -2$$

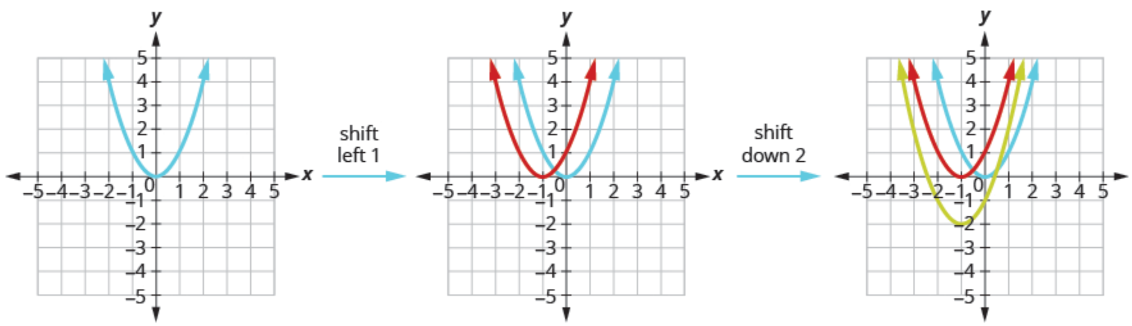
The *h* constant gives us a horizontal shift and the *k* gives us a vertical shift.

$$f(x) = x^2 \xrightarrow[\substack{h=-1 \\ \text{Shift left 1 unit}}]{} f(x) = (x + 1)^2 \xrightarrow[\substack{k=-2 \\ \text{Shift down 2 units}}]{} f(x) = (x + 1)^2 - 2$$

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = (x + 1)^2$, shift the graph $f(x) = x^2$ to the left 1 unit.

To graph $f(x) = (x + 1)^2 - 2$, shift the graph $f(x) = (x + 1)^2$ down 2 units.



$f(x) = x^2$ $\qquad$ $f(x) = (x + 1)^2$ $\qquad$ $f(x) = (x + 1)^2 - 2$

TRY IT : : 9.113 Graph $f(x) = (x + 2)^2 - 3$ using transformations.

TRY IT : : 9.114 Graph $f(x) = (x - 3)^2 + 1$ using transformations.

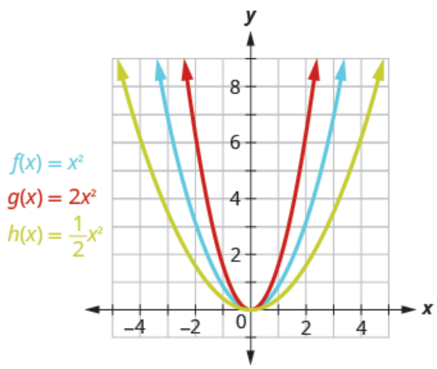
Graph Quadratic Functions of the Form $f(x) = ax^2$

So far we graphed the quadratic function $f(x) = x^2$ and then saw the effect of including a constant *h* or *k* in the equation had on the resulting graph of the new function. We will now explore the effect of the coefficient *a* on the resulting graph of the new function $f(x) = ax^2$.

Let's look at the quadratic functions $f(x) = x^2$, $g(x) = 2x^2$ and $h(x) = \frac{1}{2}x^2$.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = 2x^2$	$(x, g(x))$	$h(x) = \frac{1}{2}x^2$	$(x, h(x))$
-2	4	$(-2, 4)$	$2 \cdot 4$	$(-2, 8)$	$\frac{1}{2} \cdot 4$	$(-2, 2)$
-1	1	$(-1, 1)$	$2 \cdot 1$	$(-1, 2)$	$\frac{1}{2} \cdot 1$	$\left(-1, \frac{1}{2}\right)$
0	0	$(0, 0)$	$2 \cdot 0$	$(0, 0)$	$\frac{1}{2} \cdot 0$	$(0, 0)$
1	1	$(1, 1)$	$2 \cdot 1$	$(1, 2)$	$\frac{1}{2} \cdot 1$	$\left(1, \frac{1}{2}\right)$
2	4	$(2, 4)$	$2 \cdot 4$	$(2, 8)$	$\frac{1}{2} \cdot 4$	$(2, 2)$

If we graph these functions, we can see the effect of the constant a, assuming $a > 0$.



The graph of the function $g(x) = 2x^2$ is "skinnier" than the graph of $f(x) = x^2$.

The graph of the function $h(x) = \frac{1}{2}x^2$ is "wider" than the graph of $f(x) = x^2$.

To graph a function with constant a it is easiest to choose a few points on $f(x) = x^2$ and multiply the y-values by a.

Graph of a Quadratic Function of the form $f(x) = ax^2$

The coefficient a in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.

- If $0 < |a| < 1,$ the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
- If $|a| > 1$, the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.

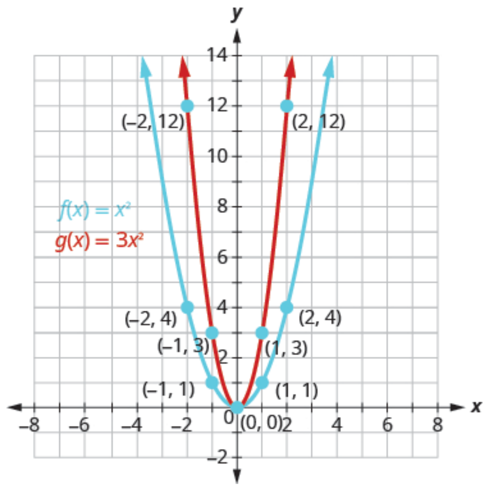
EXAMPLE 9.58

Graph $f(x) = 3x^2$.

Solution

We will graph the functions $f(x) = x^2$ and $g(x) = 3x^2$ on the same grid. We will choose a few points on $f(x) = x^2$ and then multiply the y-values by 3 to get the points for $g(x) = 3x^2$.

x	$f(x) = x^2$ $(x, f(x))$	$g(x) = 3x^2$ $(x, g(x))$	
–2	(–2, 4)	(–2, 12)	$3 \cdot 4 = 12$
–1	(–1, 1)	(–1, 3)	$3 \cdot 1 = 3$
0	(0, 0)	(0, 0)	$3 \cdot 0 = 0$
1	(1, 1)	(1, 3)	$3 \cdot 1 = 3$
2	(2, 4)	(2, 12)	$3 \cdot 4 = 12$



TRY IT : : 9.115 Graph $f(x) = -3x^2$.

TRY IT : : 9.116 Graph $f(x) = 2x^2$.

Graph Quadratic Functions Using Transformations

We have learned how the constants *a*, *h*, and *k* in the functions, $f(x) = x^2 + k$, $f(x) = (x - h)^2$, and $f(x) = ax^2$ affect their graphs. We can now put this together and graph quadratic functions $f(x) = ax^2 + bx + c$ by first putting them into the form $f(x) = a(x - h)^2 + k$ by completing the square. This form is sometimes known as the vertex form or standard form.

We must be careful to both add and subtract the number to the SAME side of the function to complete the square. We cannot add the number to both sides as we did when we completed the square with quadratic equations.

Quadratic Equation	Quadratic Function
$x^2 + 8x + 6 = 0$	$f(x) = x^2 + 8x + 6$
$x^2 + 8x = -6$	$f(x) = x^2 + 8x + 6$
$x^2 + 8x + 16 = -6 + 16$	$f(x) = x^2 + 8x + 16 + 6 - 16$
$(x + 4)^2 = 10$	$f(x) = (x + 4)^2 - 10$
Add 16 to both sides	Add and subtract 16 from the same side

When we complete the square in a function with a coefficient of x^2 that is not one, we have to factor that coefficient from just the *x*-terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to focus only on the *x*-terms.

Once we get the constant we want to complete the square, we must remember to multiply it by that coefficient before we then subtract it.

EXAMPLE 9.59

Rewrite $f(x) = -3x^2 - 6x - 1$ in the $f(x) = a(x - h)^2 + k$ form by completing the square.

Solution

	$f(x) = -3x^2 - 6x - 1$
Separate the *x* terms from the constant.	$f(x) = -3x^2 - 6x \quad - 1$
Factor the coefficient of x^2, -3.	$f(x) = -3(x^2 + 2x) \quad - 1$
Prepare to complete the square.	$f(x) = -3(x^2 + 2x \quad\quad) - 1$
Take half of 2 and then square it to complete the square. $\left(\frac{1}{2} \cdot 2\right)^2 = 1$	
The constant 1 completes the square in the parentheses, but the parentheses is multiplied by -3. So we are really adding -3 We must then add 3 to not change the value of the function.	$f(x) = -3(x^2 + 2x + 1) - 1 + 3$ $-3 \cdot 1 = -3$ so add 3
Rewrite the trinomial as a square and subtract the constants.	$f(x) = -3(x + 1) + 2$
The function is now in the $f(x) = a(x - h)^2 + k$ form.	$f(x) = \quad a(x-h)^2 \quad + k$ $f(x) = -3(x + 1)^2 + 2$

TRY IT : : 9.117

Rewrite $f(x) = -4x^2 - 8x + 1$ in the $f(x) = a(x - h)^2 + k$ form by completing the square.

TRY IT : : 9.118

Rewrite $f(x) = 2x^2 - 8x + 3$ in the $f(x) = a(x - h)^2 + k$ form by completing the square.

Once we put the function into the $f(x) = (x - h)^2 + k$ form, we can then use the transformations as we did in the last few problems. The next example will show us how to do this.

EXAMPLE 9.60

Graph $f(x) = x^2 + 6x + 5$ by using transformations.

Solution

Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ vertex form by completing the square.

	$f(x) = x^2 + 6x + 5$
Separate the x terms from the constant.	$f(x) = x^2 + 6x + 5$
Take half of 6 and then square it to complete the square. $\left(\frac{1}{2}\cdot 6\right)^2 = 9$	
We both add 9 and subtract 9 to not change the value of the function.	$f(x) = x^2 + 6x + 9 + 5 - 9$
Rewrite the trinomial as a square and subtract the constants.	$f(x) = (x + 3)^2 - 4$
The function is now in the $f(x) = (x - h)^2 + k$ form.	$f(x) = (x - h)^2 + k$ $f(x) = (x + 3)^2 - 4$

Step 2: Graph the function using transformations.

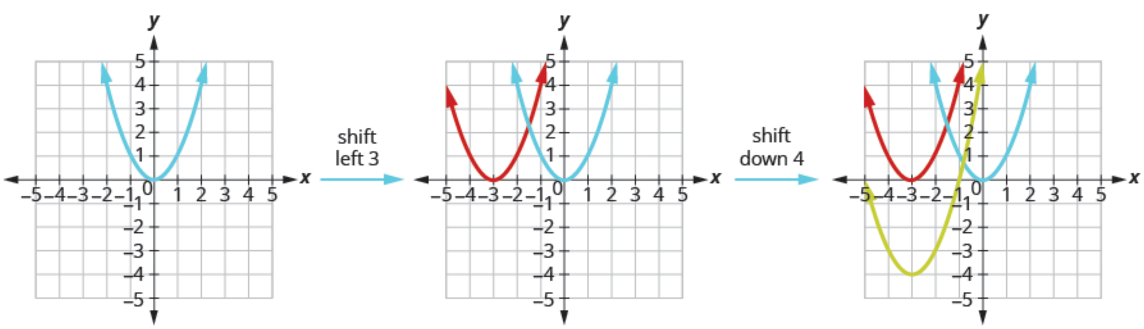
Looking at the *h*, *k* values, we see the graph will take the graph of $f(x) = x^2$ and shift it to the left 3 units and down 4 units.

$$f(x) = x^2 \xrightarrow[\text{Shift left 3 units}]{h = -3} f(x) = (x + 3)^2 \xrightarrow[\text{Shift down 4 units}]{k = -4} f(x) = (x + 3)^2 - 4$$

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = (x + 3)^2$, shift the graph $f(x) = x^2$ to the left 3 units.

To graph $f(x) = (x + 3)^2 - 4$, shift the graph $f(x) = (x + 3)^2$ down 4 units.



$f(x) = x^2$ $\quad$ $f(x) = (x + 3)^2$ $\quad$ $f(x) = (x + 3)^2 - 4$

TRY IT : : 9.119 Graph $f(x) = x^2 + 2x - 3$ by using transformations.

TRY IT : : 9.120 Graph $f(x) = x^2 - 8x + 12$ by using transformations.

We list the steps to take to graph a quadratic function using transformations here.

HOW TO : : GRAPH A QUADRATIC FUNCTION USING TRANSFORMATIONS.

Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form by completing the square.

Step 2. Graph the function using transformations.

EXAMPLE 9.61

Graph $f(x) = -2x^2 - 4x + 2$ by using transformations.

Solution

Step 1. Rewrite the function in $f(x) = a(x-h)^2 + k$ vertex form by completing the square.

	$f(x) = -2x^2 - 4x + 2$
Separate the x terms from the constant.	$f(x) = -2x^2 - 4x + 2$
We need the coefficient of x^2 to be one. We factor -2 from the x-terms.	$f(x) = -2(x^2 + 2x) + 2$
Take half of 2 and then square it to complete the square. $\left(\frac{1}{2} \cdot 2\right)^2 = 1$	
We add 1 to complete the square in the parentheses, but the parentheses is multiplied by -2. Se we are really adding -2. To not change the value of the function we add 2.	$f(x) = -2(x^2 + 2x + 1) + 2 + 2$
Rewrite the trinomial as a square and subtract the constants.	$f(x) = -2(x+1)^2 + 4$
The function is now in the $f(x) = a(x-h)^2 + k$ form.	$f(x) = a(x-h)^2 + k$ $f(x) = -2(x+1)^2 + 4$

Step 2. Graph the function using transformations.

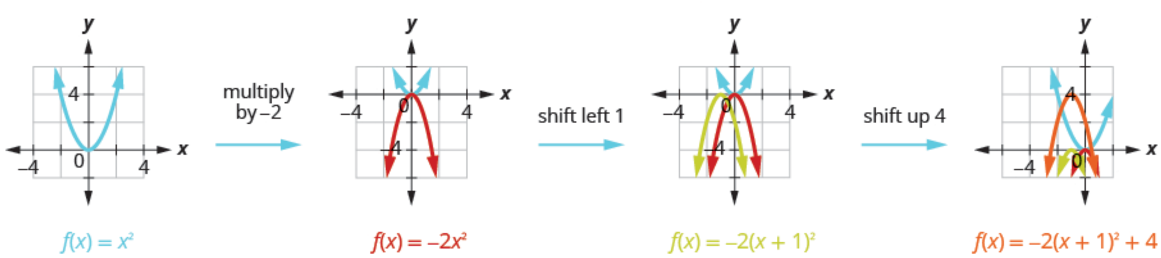
$$f(x) = x^2 \xrightarrow[\text{Multiply } y\text{-values by } -2]{a=-2} f(x) = -2x^2 \xrightarrow[\text{Shift left 1 unit}]{h=-1} f(x) = -2(x+1)^2 \xrightarrow[\text{Shift up 4 units}]{k=4} f(x) = -2(x+1)^2 + 4$$

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = -2x^2$, multiply the y-values in parabola of $f(x) = x^2$ by -2.

To graph $f(x) = -2(x+1)^2$, shift the graph $f(x) = -2x^2$ to the left 1 unit.

To graph $f(x) = -2(x+1)^2 + 4$, shift the graph $f(x) = (x+1)^2$ up 4 units.

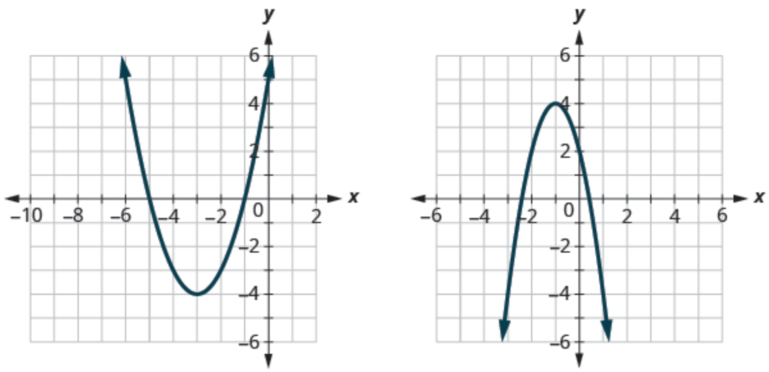


TRY IT : : 9.121 Graph $f(x) = -3x^2 + 12x - 4$ by using transformations.

TRY IT : : 9.122 Graph $f(x) = -2x^2 + 12x - 9$ by using transformations.

Now that we have completed the square to put a quadratic function into $f(x) = a(x-h)^2 + k$ form, we can also use this technique to graph the function using its properties as in the previous section.

If we look back at the last few examples, we see that the vertex is related to the constants *h* and *k*.



$f(x) = (x-h)^2 + k$	$f(x) = a(x-h)^2 + k$
$f(x) = (x+3)^2 - 4$	$f(x) = -2(x+1)^2 + 4$
$h = -3 \quad k = -4$	$h = -1 \quad k = 4$
Vertex = $(-3, -4)$	Vertex = $(-1, 4)$

In each case, the vertex is (h, k). Also the axis of symmetry is the line $x = h$.

We rewrite our steps for graphing a quadratic function using properties for when the function is in $f(x) = a(x-h)^2 + k$ form.

HOW TO : : GRAPH A QUADRATIC FUNCTION IN THE FORM $f(x) = a(x-h)^2 + k$ USING PROPERTIES.

Step 1. Rewrite the function in $f(x) = a(x-h)^2 + k$ form.

Step 2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.

Step 3. Find the axis of symmetry, $x = h$.

Step 4. Find the vertex, (h, k).

Step 5. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.

Step 6. Find the *x*-intercepts.

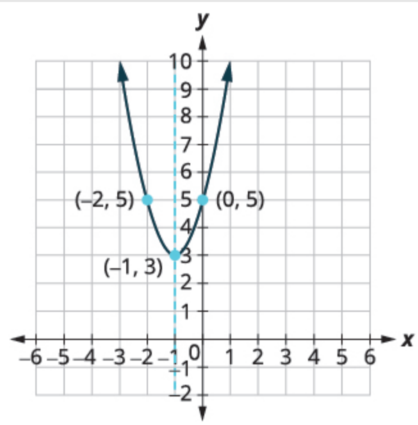
Step 7. Graph the parabola.

EXAMPLE 9.62

ⓐ Rewrite $f(x) = 2x^2 + 4x + 5$ in $f(x) = a(x-h)^2 + k$ form and ⓑ graph the function using properties.

Solution

Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square.	$f(x) = 2x^2 + 4x + 5$
	$f(x) = 2(x^2 + 2x) + 5$
	$f(x) = 2(x^2 + 2x + 1) + 5 - 2$
	$f(x) = 2(x+1)^2 + 3$
Identify the constants a, h, k.	$a = 2 \quad h = -1 \quad k = 3$

Since $a = 2$, the parabola opens upward.	
The axis of symmetry is $x = h$.	The axis of symmetry is $x = -1$.
The vertex is (h, k).	The vertex is $(-1, 3)$.
Find the *y*-intercept by finding $f(0)$.	$f(0) = 2 \cdot 0^2 + 4 \cdot 0 + 5$
	$f(0) = 5$
	y-intercept $(0, 5)$
Find the point symmetric to $(0, 5)$ across the axis of symmetry.	$(-2, \ 5)$
Find the *x*-intercepts.	The discriminant negative, so there are no *x*-intercepts. Graph the parabola.
	

> **TRY IT : :** 9.123
>
> ⓐ Rewrite $f(x) = 3x^2 - 6x + 5$ in $f(x) = a(x - h)^2 + k$ form and ⓑ graph the function using properties.

> **TRY IT : :** 9.124
>
> ⓐ Rewrite $f(x) = -2x^2 + 8x - 7$ in $f(x) = a(x - h)^2 + k$ form and ⓑ graph the function using properties.

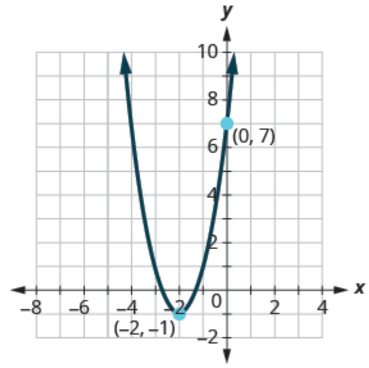
Find a Quadratic Function from its Graph

So far we have started with a function and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the function.

EXAMPLE 9.63

Determine the quadratic function whose graph is shown.

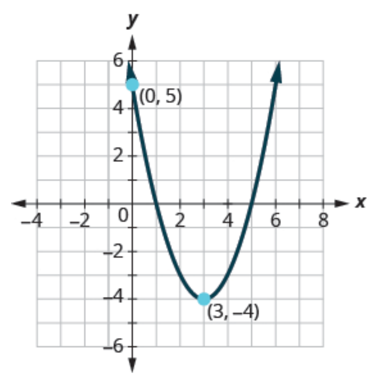


Solution

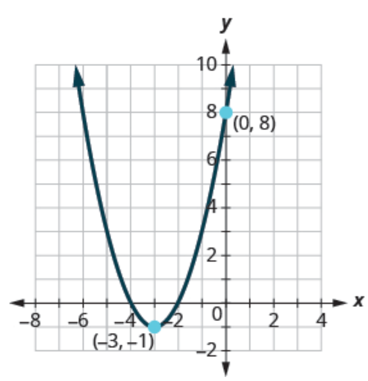
Since it is quadratic, we start with the $f(x) = a(x-h)^2 + k$ form.

The vertex, (h, k), is $(-2, -1)$ so $h = -2$ and $k = -1$.	$f(x) = a(x-(-2))^2 - 1$
To fin a, we use the y-intercept, $(0, 7)$.	
So $f(0) = 7$.	$7 = a(0+2)^2 - 1$
Solve for a.	$7 = 4a - 1$
	$8 = 4a$
	$2 = a$
Write the function.	$f(x) = a(x-h)^2 + k$
Substitute in $h = -2$, $k = -1$ and $a = 2$.	$f(x) = 2(x+2)^2 - 1$

TRY IT : : 9.125 Write the quadratic function in $f(x) = a(x-h)^2 + k$ form whose graph is shown.



TRY IT : : 9.126 Determine the quadratic function whose graph is shown.



MEDIA : :

Access these online resources for additional instruction and practice with graphing quadratic functions using transformations.

- **Function Shift Rules Applied to Quadratic Functions (https://openstax.org/l/37QuadFuncTran1)**
- **Changing a Quadratic from Standard Form to Vertex Form (https://openstax.org/l/37QuadFuncTran2)**
- **Using Transformations to Graph Quadratic Functions (https://openstax.org/l/37QuadFuncTran3)**
- **Finding Quadratic Equation in Vertex Form from Graph (https://openstax.org/l/37QuadFuncTran4)**

9.7 EXERCISES

Practice Makes Perfect

Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the following exercises, ⓐ graph the quadratic functions on the same rectangular coordinate system and ⓑ describe what effect adding a constant, k, to the function has on the basic parabola.

293. $f(x) = x^2,\ g(x) = x^2 + 4,$ and $h(x) = x^2 - 4.$

294. $f(x) = x^2,\ g(x) = x^2 + 7,$ and $h(x) = x^2 - 7.$

In the following exercises, graph each function using a vertical shift.

295. $f(x) = x^2 + 3$

296. $f(x) = x^2 - 7$

297. $g(x) = x^2 + 2$

298. $g(x) = x^2 + 5$

299. $h(x) = x^2 - 4$

300. $h(x) = x^2 - 5$

Graph Quadratic Functions of the form $f(x) = (x - h)^2$

In the following exercises, ⓐ graph the quadratic functions on the same rectangular coordinate system and ⓑ describe what effect adding a constant, h, to the function has on the basic parabola.

301. $f(x) = x^2,\ g(x) = (x - 3)^2,$ and $h(x) = (x + 3)^2.$

302. $f(x) = x^2,\ g(x) = (x + 4)^2,$ and $h(x) = (x - 4)^2.$

In the following exercises, graph each function using a horizontal shift.

303. $f(x) = (x - 2)^2$

304. $f(x) = (x - 1)^2$

305. $f(x) = (x + 5)^2$

306. $f(x) = (x + 3)^2$

307. $f(x) = (x - 5)^2$

308. $f(x) = (x + 2)^2$

In the following exercises, graph each function using transformations.

309. $f(x) = (x + 2)^2 + 1$

310. $f(x) = (x + 4)^2 + 2$

311. $f(x) = (x - 1)^2 + 5$

312. $f(x) = (x - 3)^2 + 4$

313. $f(x) = (x + 3)^2 - 1$

314. $f(x) = (x + 5)^2 - 2$

315. $f(x) = (x - 4)^2 - 3$

316. $f(x) = (x - 6)^2 - 2$

Graph Quadratic Functions of the form $f(x) = ax^2$

In the following exercises, graph each function.

317. $f(x) = -2x^2$

318. $f(x) = 4x^2$

319. $f(x) = -4x^2$

320. $f(x) = -x^2$

321. $f(x) = \frac{1}{2}x^2$

322. $f(x) = \frac{1}{3}x^2$

323. $f(x) = \frac{1}{4}x^2$

324. $f(x) = -\frac{1}{2}x^2$

Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x - h)^2 + k$ form by completing the square.

325. $f(x) = -3x^2 - 12x - 5$

326. $f(x) = 2x^2 - 12x + 7$

327. $f(x) = 3x^2 + 6x - 1$

328. $f(x) = -4x^2 - 16x - 9$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x - h)^2 + k$ form and ⓑ graph it by using transformations.

329. $f(x) = x^2 + 6x + 5$

330. $f(x) = x^2 + 4x - 12$

331. $f(x) = x^2 + 4x - 12$

332. $f(x) = x^2 - 6x + 8$

333. $f(x) = x^2 - 6x + 15$

334. $f(x) = x^2 + 8x + 10$

335. $f(x) = -x^2 + 8x - 16$

336. $f(x) = -x^2 + 2x - 7$

337. $f(x) = -x^2 - 4x + 2$

338. $f(x) = -x^2 + 4x - 5$

339. $f(x) = 5x^2 - 10x + 8$

340. $f(x) = 3x^2 + 18x + 20$

341. $f(x) = 2x^2 - 4x + 1$

342. $f(x) = 3x^2 - 6x - 1$

343. $f(x) = -2x^2 + 8x - 10$

344. $f(x) = -3x^2 + 6x + 1$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x - h)^2 + k$ form and ⓑ graph it using properties.

345. $f(x) = 2x^2 + 4x + 6$

346. $f(x) = 3x^2 - 12x + 7$

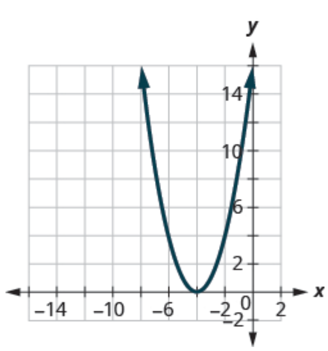
347. $f(x) = -x^2 + 2x - 4$

348. $f(x) = -2x^2 - 4x - 5$

Matching

In the following exercises, match the graphs to one of the following functions: ⓐ $f(x) = x^2 + 4$ ⓑ $f(x) = x^2 - 4$ ⓒ $f(x) = (x + 4)^2$ ⓓ $f(x) = (x - 4)^2$ ⓔ $f(x) = (x + 4)^2 - 4$ ⓕ $f(x) = (x + 4)^2 + 4$ ⓖ $f(x) = (x - 4)^2 - 4$ ⓗ $f(x) = (x - 4)^2 + 4$

349.



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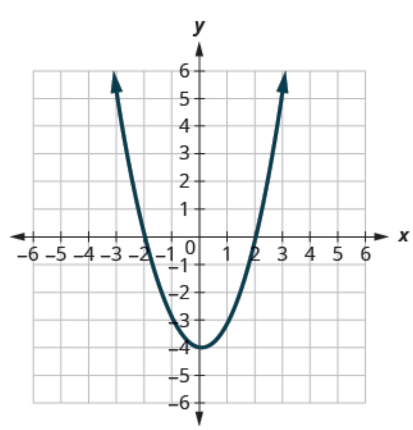

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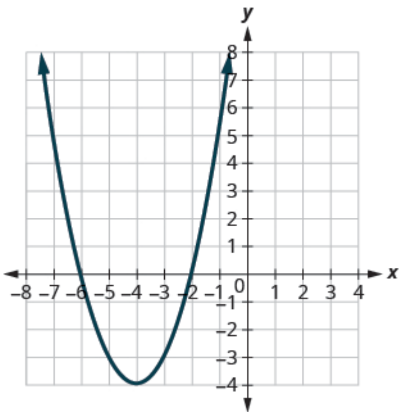

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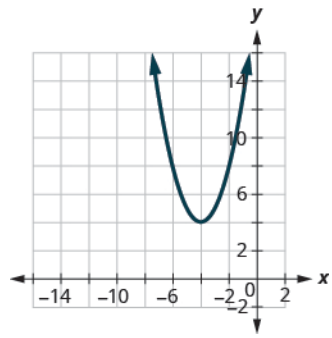

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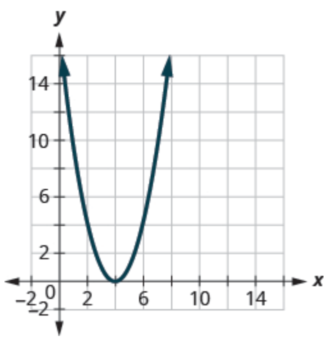

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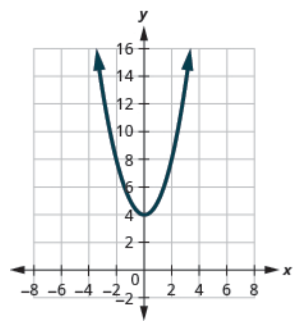

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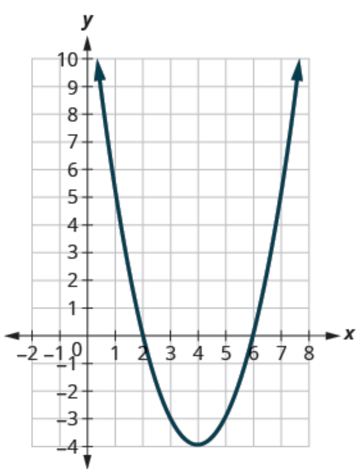

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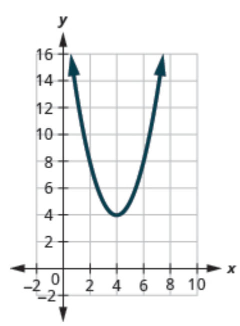
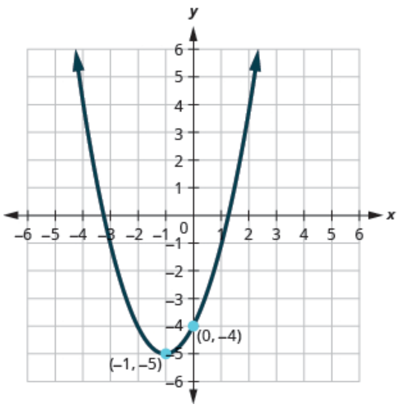


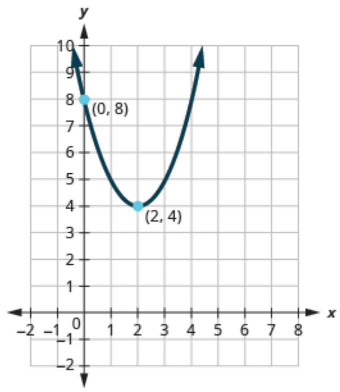
Find a Quadratic Function from its Graph

In the following exercises, write the quadratic function in $f(x) = a(x-h)^2 + k$ *form whose graph is shown.*

357.



358.



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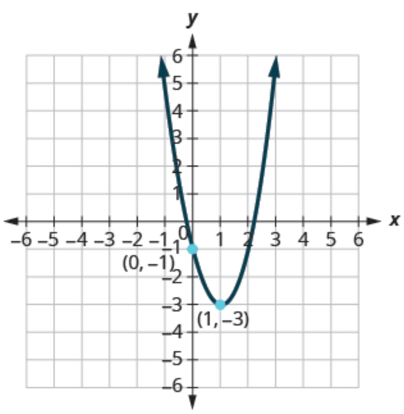
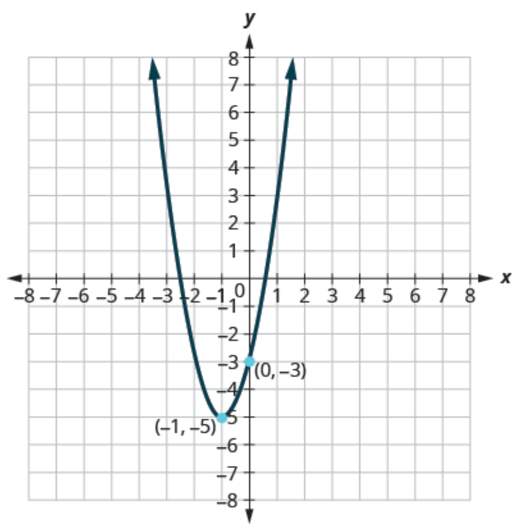


360.



Writing Exercise

361. Graph the quadratic function $f(x) = x^2 + 4x + 5$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

362. Graph the quadratic function $f(x) = 2x^2 - 4x - 3$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
graph Quadratic Functions of the form $f(x)=x^2+k$.			
graph Quadratic Functions of the form $f(x)=(x-h)^2$.			
graph Quadratic Functions of the form $f(x)=ax^2$.			
graph Quadratic Functions Using Transformations.			
find a Quadratic Function from its Graph.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

9.8 Solve Quadratic Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic inequalities graphically
- Solve quadratic inequalities algebraically

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $2x - 3 = 0.$
   If you missed this problem, review Example 2.2.
2. Solve: $2y^2 + y = 15\,.$
   If you missed this problem, review Example 6.45.
3. Solve $\dfrac{1}{x^2 + 2x - 8} > 0$
   If you missed this problem, review Example 7.56.

We have learned how to solve linear inequalities and rational inequalities previously. Some of the techniques we used to solve them were the same and some were different.

We will now learn to solve inequalities that have a quadratic expression. We will use some of the techniques from solving linear and rational inequalities as well as quadratic equations.

We will solve quadratic inequalities two ways—both graphically and algebraically.

Solve Quadratic Inequalities Graphically

A quadratic equation is in standard form when written as $ax^2 + bx + c = 0$. If we replace the equal sign with an inequality sign, we have a **quadratic inequality** in standard form.

Quadratic Inequality

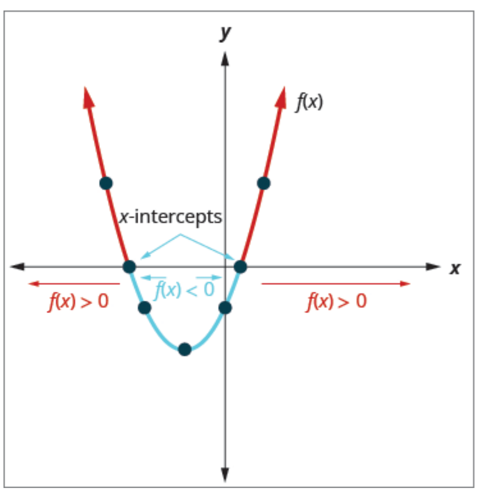
A **quadratic inequality** is an inequality that contains a quadratic expression.

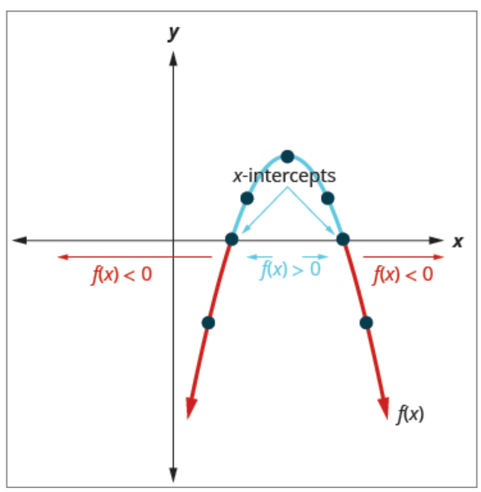
The standard form of a quadratic inequality is written:

$$ax^2 + bx + c < 0 \qquad ax^2 + bx + c \le 0$$
$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c \ge 0$$

The graph of a quadratic function $f(x) = ax^2 + bx + c = 0$ is a parabola. When we ask when is $ax^2 + bx + c < 0$, we are asking when is $f(x) < 0$. We want to know when the parabola is below the x-axis.

When we ask when is $ax^2 + bx + c > 0$, we are asking when is $f(x) > 0$. We want to know when the parabola is above the y-axis.



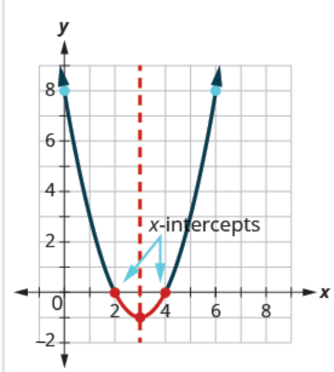


EXAMPLE 9.64 HOW TO SOLVE A QUADRATIC INEQUALITY GRAPHICALLY

Solve $x^2 - 6x + 8 < 0$ graphically. Write the solution in interval notation.

Solution

Step 1. Write the quadratic inequality in standard form.	The inequality is in standard form	$x^2 - 6x + 8 < 0$

Step 2. Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations.	We will graph using the properties. Look at a in the equation. $f(x) = x^2 - 6x + 8$ Since a is positive, the parabola opens upward. 	$f(x) = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.**
	$f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is (3, –1).**
	We find $f(0)$	y-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The y-intercept is (0, 8).**
	We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	Point symmetric to y-intercept **The point is (6, 8).**
	We solve $f(x) = 0$. We can solve this quadratic equation by factoring.	x-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The x-intercepts are (2, 0) and (4, 0).**
	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Step 3. Determine the solution from the graph.	$x^2 - 6x + 8 < 0$ The inequality asks for the values of x which make the function less than 0. Which values of x make the parabola below the x-axis. We do not include the values 2, 4 as the inequality is less than only.	The solution, in interval notation, is (2, 4).

TRY IT : : 9.127 ⓐ Solve $x^2 + 2x - 8 < 0$ graphically and ⓑ write the solution in interval notation.

TRY IT : : 9.128 ⓐ Solve $x^2 - 8x + 12 \geq 0$ graphically and ⓑ write the solution in interval notation.

We list the steps to take to solve a quadratic inequality graphically.

HOW TO : : SOLVE A QUADRATIC INEQUALITY GRAPHICALLY.

Step 1. Write the quadratic inequality in standard form.

Step 2. Graph the function $f(x) = ax^2 + bx + c$.

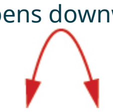
Step 3. Determine the solution from the graph.

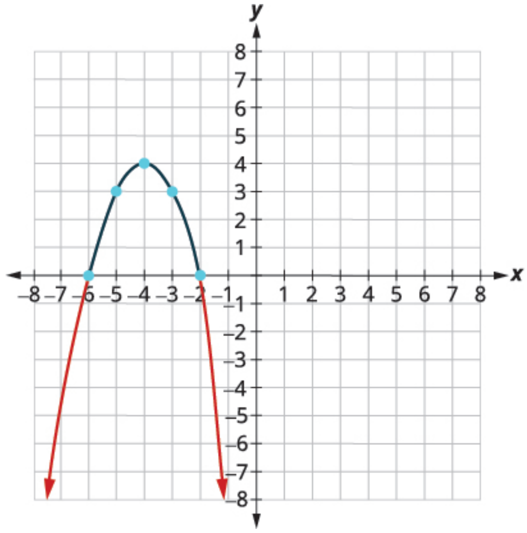
In the last example, the parabola opened upward and in the next example, it opens downward. In both cases, we are looking for the part of the parabola that is below the *x*-axis but note how the position of the parabola affects the solution.

EXAMPLE 9.65

Solve $-x^2 - 8x - 12 \leq 0$ graphically. Write the solution in interval notation.

Solution

The quadratic inequality in standard form.	$-x^2 - 8x - 12 \leq 0$
Graph the function $f(x) = -x^2 - 8x - 12$.	The parabola opens downward. 
Find the line of symmetry.	$x = -\frac{b}{2a}$ $x = -\frac{-8}{2(-1)}$ $x = -4$
Find the vertex.	$f(x) = -x^2 - 8x - 12$ $f(-4) = -(-4)^2 - 8(-4) - 12$ $f(-4) = -16 + 32 - 12$ $f(-4) = 4$ Vertex $(-4, 4)$

Find the *x*-intercepts. Let $f(x) = 0$.	$f(x) = -x^2 - 8x - 12$ $0 = -x^2 - 8x - 12$
Factor. Use the Zero Product Property.	$0 = -1(x + 6)(x + 2)$ $x = -6 \quad x = -2$
Graph the parabola.	*x*-intercepts $(-6, 0), (-2, 0)$ 
Determine the solution from the graph. We include the *x*-intercepts as the inequality is "less than or equal to."	$(-\infty, \ -6] \cup [-2, \ \infty)$

> **TRY IT : :** 9.129 ⓐ Solve $-x^2 - 6x - 5 > 0$ graphically and ⓑ write the solution in interval notation.

> **TRY IT : :** 9.130 ⓐ Solve $-x^2 + 10x - 16 \leq 0$ graphically and ⓑ write the solution in interval notation.

Solve Quadratic Inequalities Algebraically

The algebraic method we will use is very similar to the method we used to solve rational inequalities. We will find the critical points for the inequality, which will be the solutions to the related quadratic equation. Remember a polynomial expression can change signs only where the expression is zero.

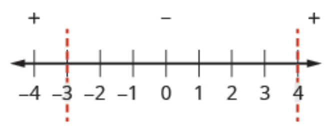
We will use the critical points to divide the number line into intervals and then determine whether the quadratic expression willl be postive or negative in the interval. We then determine the solution for the inequality.

EXAMPLE 9.66 HOW TO SOLVE QUADRATIC INEQUALITIES ALGEBRAICALLY

Solve $x^2 - x - 12 \geq 0$ algebraically. Write the solution in interval notation.

Solution

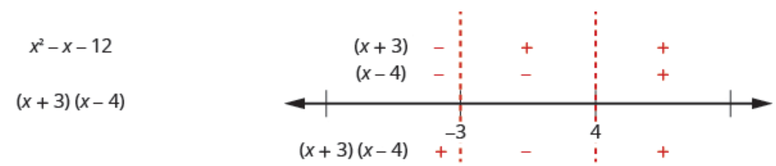
Step 1. Write the quadratic inequality in standard form.	The inequality is in standard form	$x^2 - x - 12 \geq 0$

Step 2. Determine the critical points—the solutions to the related quadratic equation.	Change the inequality sign to an equal sign and then solve the equation.	$x^2 - x - 12 = 0$ $(x+3)(x-4) = 0$ $x+3=0 \quad x-4=0$ $x=-3 \quad x=4$
Step 3. Use the critical points to divide the number line into intervals.	Use −3 and 4 to divide the number line into intervals	
Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.	Test: $x=-5$ $x=0$ $x=5$	$x^2 - x - 12$ $\quad$ $x^2 - x - 12$ $\quad$ $x^2 - x - 12$ $(-5)^2 - (-5) - 12$ $\quad$ $0^2 - 0 - 12$ $\quad$ $5^2 - 5 - 12$ 18 $\quad$ -12 $\quad$ 8 
Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.	$x^2 - x - 12 \geq 0$ The inequality is positive in the first and last intervals and equals 0 at the points −4, 3.	The solution, in interval notation, is $(-\infty, -3] \cup [4, \infty)$.

TRY IT : : 9.131 Solve $x^2 + 2x - 8 \geq 0$ algebraically. Write the solution in interval notation.

TRY IT : : 9.132 Solve $x^2 - 2x - 15 \leq 0$ algebraically. Write the solution in interval notation.

In this example, since the expression $x^2 - x - 12$ factors nicely, we can also find the sign in each interval much like we did when we solved rational inequalities. We find the sign of each of the factors, and then the sign of the product. Our number line would like this:



The result is the same as we found using the other method.

We summarize the steps here.

HOW TO : : SOLVE A QUADRATIC INEQUALITY ALGEBRAICALLY.

Step 1. Write the quadratic inequality in standard form.

Step 2. Determine the critical points—the solutions to the related quadratic equation.

Step 3. Use the critical points to divide the number line into intervals.

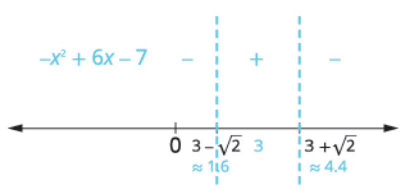
Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.

Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

EXAMPLE 9.67

Solve $x^2 + 6x - 7 \geq 0$ algebraically. Write the solution in interval notation.

Solution

Write the quadratic inequality in standard form.	$-x^2 + 6x - 7 \geq 0$
Multiply both sides of the inequality by -1. Remember to reverse the inequality sign.	$x^2 - 6x + 7 \leq 0$
Determine the critical points by solving the related quadratic equation.	$x^2 - 6x + 7 = 0$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (7)}}{2 \cdot 1}$
Simplify.	$x = \dfrac{6 \pm \sqrt{8}}{2}$
Simplify the radical.	$x = \dfrac{6 \pm 2\sqrt{2}}{2}$
Remove the common factor, 2.	$x = \dfrac{2(3 \pm \sqrt{2})}{2}$ $x = 3 \pm \sqrt{2}$ $x = 3 + \sqrt{2} \qquad x = 3 - \sqrt{2}$ $x \approx 1.6 \qquad x \approx 4.4$
Use the critical points to divide the number line into intervals. Test numbers from each interval in the original inequality.	
Determine the intervals where the inequality is correct. Write the solution in interval notation.	$-x^2 + 6x - 7 \geq 0$ in the middle interval $\left[3 - \sqrt{2},\ \ 3 + \sqrt{2}\right]$

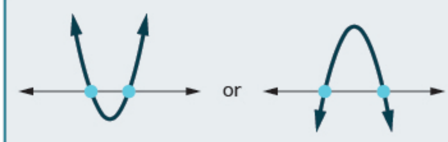
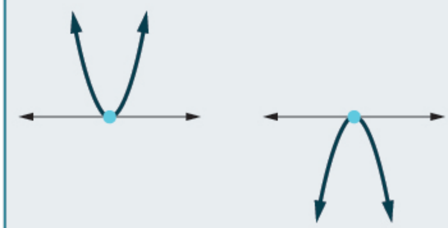
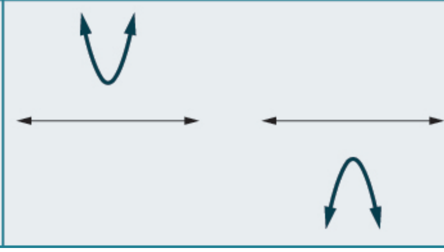
TRY IT : : 9.133 Solve $-x^2 + 2x + 1 \geq 0$ algebraically. Write the solution in interval notation.

TRY IT : : 9.134 Solve $-x^2 + 8x - 14 < 0$ algebraically. Write the solution in interval notation.

The solutions of the quadratic inequalities in each of the previous examples, were either an interval or the union of two intervals. This resulted from the fact that, in each case we found two solutions to the corresponding quadratic equation $ax^2 + bx + c = 0$. These two solutions then gave us either the two x-intercepts for the graph or the two critical points to divide the number line into intervals.

This correlates to our previous discussion of the number and type of solutions to a quadratic equation using the discriminant.

For a quadratic equation of the form $ax^2 + bx + c = 0,\ a \neq 0$.

Discriminant	Number/Type of solution	Typical Graph
$b^2 - 4ac > 0$	2 real solutions 2 x-intercepts on graph	
$b^2 - 4ac = 0$	1 real solution 1 x-intercept on graph	
$b^2 - 4ac < 0$	2 complex solutions No x-intercept	

The last row of the table shows us when the parabolas never intersect the x-axis. Using the Quadratic Formula to solve the quadratic equation, the radicand is a negative. We get two complex solutions.

In the next example, the quadratic inequality solutions will result from the solution of the quadratic equation being complex.

EXAMPLE 9.68

Solve, writing any solution in interval notation:

ⓐ $x^2 - 3x + 4 > 0$ ⓑ $x^2 - 3x + 4 \leq 0$

Solution

ⓐ

Write the quadratic inequality in standard form.	$-x^2 - 3x + 4 > 0$
Determine the critical points by solving the related quadratic equation.	$x^2 - 3x + 4 = 0$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$
Simplify.	$x = \dfrac{3 \pm \sqrt{-7}}{2}$
Simplify the radicand.	$x = \dfrac{3 \pm \sqrt{7}i}{2}$

The complex solutions tell us the parabola does not intercept the *x*-axis. Also, the parabola opens upward. This tells us that the parabola is completely above the *x*-axis.

Complex solutions

We are to find the solution to $x^2 - 3x + 4 > 0.$ Since for all values of x the graph is above the *x*-axis, all values of *x* make the inequality true. In interval notation we write $(-\infty, \infty)$.

ⓑ

Write the quadratic inequality in standard form.	$x^2 - 3x + 4 \leq 0$
Determine the critical points by solving the related quadratic equation	$x^2 - 3x + 4 = 0$

Since the corresponding quadratic equation is the same as in part (a), the parabola will be the same. The parabola opens upward and is completely above the *x*-axis—no part of it is below the *x*-axis.

We are to find the solution to $x^2 - 3x + 4 \leq 0.$ Since for all values of *x* the graph is never below the *x*-axis, no values of *x* make the inequality true. There is no solution to the inequality.

TRY IT : : 9.135 Solve and write any solution in interval notation:
ⓐ $-x^2 + 2x - 4 \leq 0$ ⓑ $-x^2 + 2x - 4 \geq 0$

TRY IT : : 9.136 Solve and write any solution in interval notation:
ⓐ $x^2 + 3x + 3 < 0$ ⓑ $x^2 + 3x + 3 > 0$

9.8 EXERCISES

Practice Makes Perfect

Solve Quadratic Inequalities Graphically

In the following exercises, ⓐ solve graphically and ⓑ write the solution in interval notation.

363. $x^2 + 6x + 5 > 0$

364. $x^2 + 4x - 12 < 0$

365. $x^2 + 4x + 3 \leq 0$

366. $x^2 - 6x + 8 \geq 0$

367. $-x^2 - 3x + 18 \leq 0$

368. $-x^2 + 2x + 24 < 0$

369. $-x^2 + x + 12 \geq 0$

370. $-x^2 + 2x + 15 > 0$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

371. $x^2 + 3x - 4 \geq 0$

372. $x^2 + x - 6 \leq 0$

373. $x^2 - 7x + 10 < 0$

374. $x^2 - 4x + 3 > 0$

375. $x^2 + 8x > -15$

376. $x^2 + 8x < -12$

377. $x^2 - 4x + 2 \leq 0$

378. $-x^2 + 8x - 11 < 0$

379. $x^2 - 10x > -19$

380. $x^2 + 6x < -3$

381. $-6x^2 + 19x - 10 \geq 0$

382. $-3x^2 - 4x + 4 \leq 0$

383. $-2x^2 + 7x + 4 \geq 0$

384. $2x^2 + 5x - 12 > 0$

385. $x^2 + 3x + 5 > 0$

386. $x^2 - 3x + 6 \leq 0$

387. $-x^2 + x - 7 > 0$

388. $-x^2 - 4x - 5 < 0$

389. $-2x^2 + 8x - 10 < 0$

390. $-x^2 + 2x - 7 \geq 0$

Writing Exercises

391. Explain critical points and how they are used to solve quadratic inequalities algebraically.

392. Solve $x^2 + 2x \geq 8$ both graphically and algebraically. Which method do you prefer, and why?

393. Describe the steps needed to solve a quadratic inequality graphically.

394. Describe the steps needed to solve a quadratic inequality algebraically.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic inequalities graphically.			
solve quadratic inequalities algebraically.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

CHAPTER 9 REVIEW

KEY TERMS

discriminant In the Quadratic Formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

quadratic function A quadratic function, where a, b, and c are real numbers and $a \neq 0$, is a function of the form $f(x) = ax^2 + bx + c$.

quadratic inequality A quadratic inequality is an inequality that contains a quadratic expression.

KEY CONCEPTS

9.1 Solve Quadratic Equations Using the Square Root Property

- Square Root Property
  - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$ or $x = \pm\sqrt{k}$

  How to solve a quadratic equation using the square root property.

  Step 1. Isolate the quadratic term and make its coefficient one.

  Step 2. Use Square Root Property.

  Step 3. Simplify the radical.

  Step 4. Check the solutions.

9.2 Solve Quadratic Equations by Completing the Square

- Binomial Squares Pattern
  If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\,\cdot\,(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\,\cdot\,(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- How to Complete a Square

  Step 1. Identify b, the coefficient of x.

  Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

  Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$

  Step 4. Rewrite the trinomial as a binomial square
- How to solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square.

  Step 1. Divide by a to make the coefficient of x^2 term 1.

  Step 2. Isolate the variable terms on one side and the constant terms on the other.

  Step 3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

  Step 4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.

  Step 5. Use the Square Root Property.

  Step 6. Simplify the radical and then solve the two resulting equations.

  Step 7.

Check the solutions.

9.3 Solve Quadratic Equations Using the Quadratic Formula

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0,\ a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.

  Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, c.

  Step 2. Write the Quadratic Formula. Then substitute in the values of a, b, c.

  Step 3. Simplify.

  Step 4. Check the solutions.
- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form $ax^2 + bx + c = 0,\ a \neq 0,$
    - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
    - if $b^2 - 4ac = 0$, the equation has 1 real solution.
    - if $b^2 - 4ac < 0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.

  Step 1. Try Factoring first. If the quadratic factors easily, this method is very quick.

  Step 2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.

  Step 3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

9.4 Solve Quadratic Equations in Quadratic Form

- How to solve equations in quadratic form.

  Step 1. Identify a substitution that will put the equation in quadratic form.

  Step 2. Rewrite the equation with the substitution to put it in quadratic form.

  Step 3. Solve the quadratic equation for u.

  Step 4. Substitute the original variable back into the results, using the substitution.

  Step 5. Solve for the original variable.

  Step 6. Check the solutions.

9.5 Solve Applications of Quadratic Equations

- Methods to Solve Quadratic Equations
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to use a Problem-Solving Strategy.

  Step 1. **Read** the problem. Make sure all the words and ideas are understood.

  Step 2. **Identify** what we are looking for.

  Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

  Step 4.

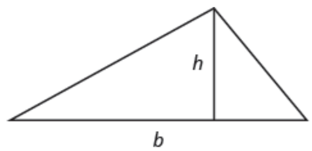
Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5. **Solve** the equation using good algebra techniques.

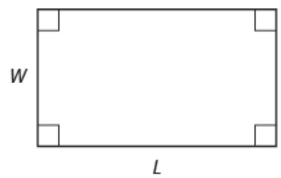
Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

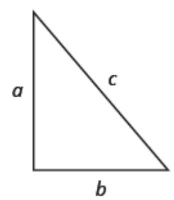
- Area of a Triangle
  - For a triangle with base, *b*, and height, *h*, the area, *A*, is given by the formula $A = \frac{1}{2}bh$.



- Area of a Rectangle
  - For a rectangle with length, *L*, and width, *W*, the area, *A*, is given by the formula $A = LW$.



- Pythagorean Theorem
  - In any right triangle, where *a* and *b* are the lengths of the legs, and *c* is the length of the hypotenuse, $a^2 + b^2 = c^2$.



- Projectile motion
  - The height in feet, *h*, of an object shot upwards into the air with initial velocity, v_0, after *t* seconds is given by the formula $h = -16t^2 + v_0t$.

9.6 Graph Quadratic Functions Using Properties

- Parabola Orientation
  - For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if
    - $a > 0$, the parabola opens upward.
    - $a < 0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:
  - the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
  - the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
  - the *y*-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

y-intercept	**x-intercepts**
Let $x = 0$ and solve for $f(x)$.	Let $f(x) = 0$ and solve for x.

- How to graph a quadratic function using properties.
  Step 1. Determine whether the parabola opens upward or downward.
  Step 2. Find the equation of the axis of symmetry.
  Step 3. Find the vertex.
  Step 4. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  Step 5. Find the *x*-intercepts. Find additional points if needed.
  Step 6. Graph the parabola.
- Minimum or Maximum Values of a Quadratic Equation
  - The *y*-coordinate of the vertex of the graph of a quadratic equation is the
  - *minimum* value of the quadratic equation if the parabola opens *upward*.
  - *maximum* value of the quadratic equation if the parabola opens *downward*.

9.7 Graph Quadratic Functions Using Transformations

- Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift
  - The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.
    - If *k* > 0, shift the parabola vertically up *k* units.
    - If *k* < 0, shift the parabola vertically down $|k|$ units.
- Graph a Quadratic Function of the form $f(x) = (x - h)^2$ Using a Horizontal Shift
  - The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.
    - If *h* > 0, shift the parabola horizontally left *h* units.
    - If *h* < 0, shift the parabola horizontally right $|h|$ units.
- Graph of a Quadratic Function of the form $f(x) = ax^2$
  - The coefficient *a* in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.
    If $0 < |a| < 1,$ then the graph of $f(x) = ax^2$ will be “wider” than the graph of $f(x) = x^2$.
    If $|a| > 1,$ then the graph of $f(x) = ax^2$ will be “skinnier” than the graph of $f(x) = x^2$.
- How to graph a quadratic function using transformations
  Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form by completing the square.
  Step 2. Graph the function using transformations.
- Graph a quadratic function in the vertex form $f(x) = a(x - h)^2 + k$ using properties
  Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form.
  Step 2. Determine whether the parabola opens upward, *a* > 0, or downward, a < 0.
  Step 3. Find the axis of symmetry, *x* = *h*.
  Step 4. Find the vertex, (*h*, *k*).
  Step 5. Find the*y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  Step 6. Find the *x*-intercepts, if possible.
  Step 7. Graph the parabola.

9.8 Solve Quadratic Inequalities

- Solve a Quadratic Inequality Graphically
  Step 1. Write the quadratic inequality in standard form.
  Step 2. Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations.
  Step 3. Determine the solution from the graph.
- How to Solve a Quadratic Inequality Algebraically
  Step 1. Write the quadratic inequality in standard form.
  Step 2. Determine the critical points -- the solutions to the related quadratic equation.
  Step 3. Use the critical points to divide the number line into intervals.
  Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.
  Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

REVIEW EXERCISES

9.1 Section 9.1 Solve Quadratic Equations Using the Square Root Property

Solve Quadratic Equations of the form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

395. $y^2 = 144$

396. $n^2 - 80 = 0$

397. $4a^2 = 100$

398. $2b^2 = 72$

399. $r^2 + 32 = 0$

400. $t^2 + 18 = 0$

401. $\frac{2}{3}w^2 - 20 = 30$

402. 11. $5c^2 + 3 = 19$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

403. $(p - 5)^2 + 3 = 19$

404. $(u + 1)^2 = 45$

405. $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$

406. $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$

407. $(n - 4)^2 - 50 = 150$

408. $(4c - 1)^2 = -18$

409. $n^2 + 10n + 25 = 12$

410. $64a^2 + 48a + 9 = 81$

9.2 Section 9.2 Solve Quadratic Equations by Completing the Square

Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

411. $x^2 + 22x$

412. $m^2 - 8m$

413. $a^2 - 3a$

414. $b^2 + 13b$

In the following exercises, solve by completing the square.

415. $d^2 + 14d = -13$

416. $y^2 - 6y = 36$

417. $m^2 + 6m = -109$

418. $t^2 - 12t = -40$

419. $v^2 - 14v = -31$

420. $w^2 - 20w = 100$

421. $m^2 + 10m - 4 = -13$

422. $n^2 - 6n + 11 = 34$

423. $a^2 = 3a + 8$

424. $b^2 = 11b - 5$

425. $(u + 8)(u + 4) = 14$

426. $(z - 10)(z + 2) = 28$

Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

427. $3p^2 - 18p + 15 = 15$

428. $5q^2 + 70q + 20 = 0$

429. $4y^2 - 6y = 4$

430. $2x^2 + 2x = 4$

431. $3c^2 + 2c = 9$

432. $4d^2 - 2d = 8$

433. $2x^2 + 6x = -5$

434. $2x^2 + 4x = -5$

9.3 Section 9.3 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

435. $4x^2 - 5x + 1 = 0$

436. $7y^2 + 4y - 3 = 0$

437. $r^2 - r - 42 = 0$

438. $t^2 + 13t + 22 = 0$

439. $4v^2 + v - 5 = 0$

440. $2w^2 + 9w + 2 = 0$

441. $3m^2 + 8m + 2 = 0$

442. $5n^2 + 2n - 1 = 0$

443. $6a^2 - 5a + 2 = 0$

444. $4b^2 - b + 8 = 0$

445. $u(u - 10) + 3 = 0$

446. $5z(z - 2) = 3$

447. $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$

448. $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$

449. $4c^2 + 4c + 1 = 0$

450. $9d^2 - 12d = -4$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

451.

ⓐ $9x^2 - 6x + 1 = 0$
ⓑ $3y^2 - 8y + 1 = 0$
ⓒ $7m^2 + 12m + 4 = 0$
ⓓ $5n^2 - n + 1 = 0$

452.

ⓐ $5x^2 - 7x - 8 = 0$
ⓑ $7x^2 - 10x + 5 = 0$
ⓒ $25x^2 - 90x + 81 = 0$
ⓓ $15x^2 - 8x + 4 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

453.

ⓐ $16r^2 - 8r + 1 = 0$
ⓑ $5t^2 - 8t + 3 = 9$
ⓒ $3(c + 2)^2 = 15$

454.

ⓐ $4d^2 + 10d - 5 = 21$
ⓑ $25x^2 - 60x + 36 = 0$
ⓒ $6(5v - 7)^2 = 150$

9.4 Section 9.4 Solve Equations in Quadratic Form

Solve Equations in Quadratic Form

In the following exercises, solve.

455. $x^4 - 14x^2 + 24 = 0$

456. $x^4 + 4x^2 - 32 = 0$

457. $4x^4 - 5x^2 + 1 = 0$

458. $(2y + 3)^2 + 3(2y + 3) - 28 = 0$

459. $x + 3\sqrt{x} - 28 = 0$

460. $6x + 5\sqrt{x} - 6 = 0$

461. $x^{\frac{2}{3}} - 10x^{\frac{1}{3}} + 24 = 0$

462. $x + 7x^{\frac{1}{2}} + 6 = 0$

463. $8x^{-2} - 2x^{-1} - 3 = 0$

9.5 Section 9.5 Solve Applications Modeled by Quadratic Equations

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.

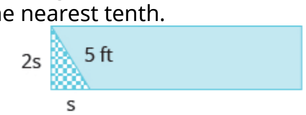
464. Find two consecutive odd numbers whose product is 323.

465. Find two consecutive even numbers whose product is 624.

466. A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.

467. Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.

468. A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.



469. A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

470. The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

471. For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

472. A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

473. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

474. Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

475. Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

9.6 Section 9.6 Graphing Quadratic Functions Using Properties

Recognize the Graph of a Quadratic Function

In the following exercises, graph by plotting point.

476. Graph $y = x^2 - 2$

477. Graph $y = -x^2 + 3$

In the following exercises, determine if the following parabolas open up or down.

478.

ⓐ $y = -3x^2 + 3x - 1$

ⓑ $y = 5x^2 + 6x + 3$

479.

ⓐ $y = x^2 + 8x - 1$

ⓑ $y = -4x^2 - 7x + 1$

Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find ⓐ the equation of the axis of symmetry and ⓑ the vertex.

480. $y = -x^2 + 6x + 8$

481. $y = 2x^2 - 8x + 1$

Find the Intercepts of a Parabola

In the following exercises, find the x- and y-intercepts.

482. $y = x^2 - 4x + 5$

483. $y = x^2 - 8x + 15$

484. $y = x^2 - 4x + 10$

485. $y = -5x^2 - 30x - 46$

486. $y = 16x^2 - 8x + 1$

487. $y = x^2 + 16x + 64$

Graph Quadratic Functions Using Properties

In the following exercises, graph by using its properties.

488. $y = x^2 + 8x + 15$

489. $y = x^2 - 2x - 3$

490. $y = -x^2 + 8x - 16$

491. $y = 4x^2 - 4x + 1$

492. $y = x^2 + 6x + 13$

493. $y = -2x^2 - 8x - 12$

Solve Maximum and Minimum Applications

In the following exercises, find the minimum or maximum value.

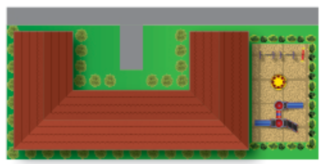
494. $y = 7x^2 + 14x + 6$

495. $y = -3x^2 + 12x - 10$

In the following exercises, solve. Rounding answers to the nearest tenth.

496. A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.

497. A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.



9.7 Section 9.7 Graphing Quadratic Functions Using Transformations

Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the following exercises, graph each function using a vertical shift.

498. $g(x) = x^2 + 4$

499. $h(x) = x^2 - 3$

In the following exercises, graph each function using a horizontal shift.

500. $f(x) = (x + 1)^2$

501. $g(x) = (x - 3)^2$

In the following exercises, graph each function using transformations.

502. $f(x) = (x + 2)^2 + 3$

503. $f(x) = (x + 3)^2 - 2$

504. $f(x) = (x - 1)^2 + 4$

505. $f(x) = (x - 4)^2 - 3$

Graph Quadratic Functions of the form $f(x) = ax^2$

In the following exercises, graph each function.

506. $f(x) = 2x^2$

507. $f(x) = -x^2$

508. $f(x) = \frac{1}{2}x^2$

Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x - h)^2 + k$ form by completing the square.

509. $f(x) = 2x^2 - 4x - 4$

510. $f(x) = 3x^2 + 12x + 8$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)^2 + k$ form and ⓑ graph it by using transformations.

511. $f(x) = 3x^2 - 6x - 1$

512. $f(x) = -2x^2 - 12x - 5$

513. $f(x) = 2x^2 + 4x + 6$

514. $f(x) = 3x^2 - 12x + 7$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)^2 + k$ form and ⓑ graph it using properties.

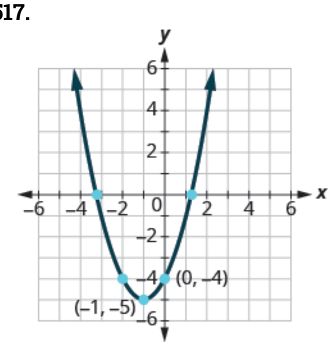
515. $f(x) = -3x^2 - 12x - 5$

516. $f(x) = 2x^2 - 12x + 7$

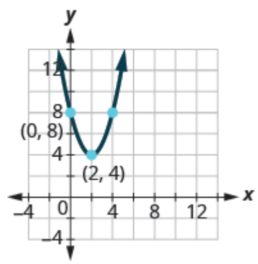
Find a Quadratic Function from its Graph

In the following exercises, write the quadratic function in $f(x) = a(x-h)^2 + k$ form.

517.



518.



9.8 Section 9.8 Solve Quadratic Inequalities

Solve Quadratic Inequalities Graphically

In the following exercises, solve graphically and write the solution in interval notation.

519. $x^2 - x - 6 > 0$

520. $x^2 + 4x + 3 \leq 0$

521. $-x^2 - x + 2 \geq 0$

522. $-x^2 + 2x + 3 < 0$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

523. $x^2 - 6x + 8 < 0$

524. $x^2 + x > 12$

525. $x^2 - 6x + 4 \leq 0$

526. $2x^2 + 7x - 4 > 0$

527. $-x^2 + x - 6 > 0$

528. $x^2 - 2x + 4 \geq 0$

PRACTICE TEST

529. Use the Square Root Property to solve the quadratic equation $3(w+5)^2 = 27$.

530. Use Completing the Square to solve the quadratic equation $a^2 - 8a + 7 = 23$.

531. Use the Quadratic Formula to solve the quadratic equation $2m^2 - 5m + 3 = 0$.

Solve the following quadratic equations. Use any method.

532. $2x(3x-2) - 1 = 0$

533. $\frac{9}{4}y^2 - 3y + 1 = 0$

Use the discriminant to determine the number and type of solutions of each quadratic equation.

534. $6p^2 - 13p + 7 = 0$

535. $3q^2 - 10q + 12 = 0$

Solve each equation.

536. $4x^4 - 17x^2 + 4 = 0$

537. $y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - 3 = 0$

For each parabola, find ⓐ which direction it opens, ⓑ the equation of the axis of symmetry, ⓒ the vertex, ⓓ the x- and y-intercepts, and e) the maximum or minimum value.

538. $y = 3x^2 + 6x + 8$

539. $y = -x^2 - 8x + 16$

Graph each quadratic function using intercepts, the vertex, and the equation of the axis of symmetry.

540. $f(x) = x^2 + 6x + 9$

541. $f(x) = -2x^2 + 8x + 4$

In the following exercises, graph each function using transformations.

542. $f(x) = (x+3)^2 + 2$

543. $f(x) = x^2 - 4x - 1$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

544. $x^2 - 6x - 8 \leq 0$

545. $2x^2 + x - 10 > 0$

Model the situation with a quadratic equation and solve by any method.

546. Find two consecutive even numbers whose product is 360.

547. The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

548. A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.