MTH95

# Intermediate Algebra

SENIOR CONTRIBUTING AUTHORS

LYNN MARECEK, SANTA ANA COLLEGE

**OpenStax**
Rice University
6100 Main Street MS-375
Houston, Texas 77005

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# PREFACE

Welcome to *Intermediate Algebra,* an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

## About OpenStax

OpenStax is a nonprofit based at Rice University, and it's our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 25 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

## About OpenStax Resources

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### Format

You can access this textbook for free in web view or PDF through openstax.org, and for a low cost in print.

## About *Intermediate Algebra*

*Intermediate Algebra* is designed to meet the scope and sequence requirements of a one-semester Intermediate algebra course. The book's organization makes it easy to adapt to a variety of course syllabi. The text expands on the fundamental concepts of algebra while addressing the needs of students with diverse backgrounds and learning styles. Each topic builds upon previously developed material to demonstrate the cohesiveness and structure of mathematics.

### Coverage and Scope

*Intermediate Algebra* continues the philosophies and pedagogical features of *Prealgebra* and *Elementary Algebra*, by Lynn Marecek and MaryAnne Anthony-Smith. By introducing the concepts and vocabulary of algebra in a nurturing, non-threatening environment while also addressing the needs of students with diverse backgrounds and learning styles, the book helps students gain confidence in their ability to succeed in the course and become successful college students.

The material is presented as a sequence of small, and clear steps to conceptual understanding. The order of topics was carefully planned to emphasize the logical progression throughout the course and to facilitate a thorough understanding of each concept. As new ideas are presented, they are explicitly related to previous topics.

**Chapter 1: Foundations**
Chapter 1 reviews arithmetic operations with whole numbers, integers, fractions, decimals and real numbers, to give the student a solid base that will support their study of algebra.

**Chapter 2: Solving Linear Equations and Inequalities**
In Chapter 2, students learn to solve linear equations using the Properties of Equality and a general strategy. They use a problem-solving strategy to solve number, percent, mixture and uniform motion applications. Solving a formula for a specific variable, and also solving both linear and compound inequalities is presented.

**Chapter 3: Graphs and Functions**
Chapter 3 covers the rectangular coordinate system where students learn to plot graph linear equations in two variables, graph with intercepts, understand slope of a line, use the slope-intercept form of an equation of a line, find the equation of a line, and create graphs of linear inequalities. The chapter also introduces relations and

functions as well as graphing of functions.

**Chapter 4: Systems of Linear Equations**
Chapter 4 covers solving systems of equations by graphing, substitution, and elimination; solving applications with systems of equations, solving mixture applications with systems of equations, and graphing systems of linear inequalities. Systems of equations are also solved using matrices and determinants.

**Chapter 5: Polynomials and Polynomial Functions**
In Chapter 5, students learn how to add and subtract polynomials, use multiplication properties of exponents, multiply polynomials, use special products, divide monomials and polynomials, and understand integer exponents and scientific notation.

**Chapter 6: Factoring**
In Chapter 6, students learn the process of factoring expressions and see how factoring is used to solve quadratic equations.

**Chapter 7: Rational Expressions and Functions**
In Chapter 7, students work with rational expressions, solve rational equations and use them to solve problems in a variety of applications, and solve rational inequalities.

**Chapter 8: Roots and Radical**
In Chapter 8, students simplify radical expressions, rational exponents, perform operations on radical expressions, and solve radical equations. Radical functions and the complex number system are introduced

**Chapter 9: Quadratic Equations**
In Chapter 9, students use various methods to solve quadratic equations and equations in quadratic form and learn how to use them in applications. Students will graph quadratic functions using their properties and by transformations.

**Chapter 10: Exponential and Logarithmic Functions**
In Chapter 10, students find composite and inverse functions, evaluate, graph, and solve both exponential and logarithmic functions.

**Chapter 11: Conics**
In Chapter 11, the properties and graphs of circles, parabolas, ellipses and hyperbolas are presented. Students also solve applications using the conics and solve systems of nonlinear equations.

**Chapter 12: Sequences, Series and the Binomial Theorem**
In Chapter 12, students are introduced to sequences, arithmetic sequences, geometric sequences and series and the binomial theorem.

All chapters are broken down into multiple sections, the titles of which can be viewed in the **Table of Contents**.

## Key Features and Boxes

**Examples** Each learning objective is supported by one or more worked examples, which demonstrate the problem-solving approaches that students must master. Typically, we include multiple examples for each learning objective to model different approaches to the same type of problem, or to introduce similar problems of increasing complexity.

All examples follow a simple two- or three-part format. First, we pose a problem or question. Next, we demonstrate the solution, spelling out the steps along the way. Finally (for select examples), we show students how to check the solution. Most examples are written in a two-column format, with explanation on the left and math on the right to mimic the way that instructors "talk through" examples as they write on the board in class.

**Be Prepared!** Each section, beginning with Section 2.1, starts with a few "Be Prepared!" exercises so that students can determine if they have mastered the prerequisite skills for the section. Reference is made to specific Examples from previous sections so students who need further review can easily find explanations. Answers to these exercises can be found in the supplemental resources that accompany this title.

**Try It**

**Try it** The Try It feature includes a pair of exercises that immediately follow an Example, providing the student with an immediate opportunity to solve a similar problem with an easy reference to the example. In the Web View version of the text, students can click an Answer link directly below the question to check their understanding. In the PDF, answers to the Try It exercises are located in the Answer Key.

**How To**

**How To Examples** use a three column format to demonstrate how to solve an example with a certain procedure. The first column states the formal step, the second column is in words as the teacher would explain the process, and then the third column is the actual math. A How To procedure box follows each of these How To examples and summarizes the series of steps from the example. These procedure boxes provide an easy reference for students.

**Media**

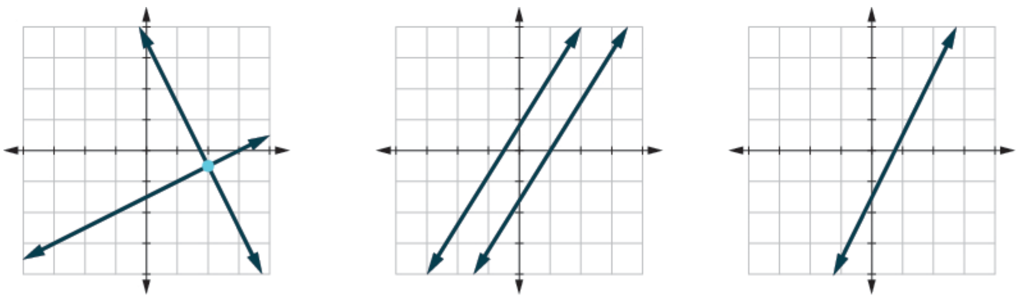
**Media** The “Media” icon appears at the conclusion of each section, just prior to the Self Check. This icon marks a list of links to online video tutorials that reinforce the concepts and skills introduced in the section.

Disclaimer: While we have selected tutorials that closely align to our learning objectives, we did not produce these tutorials, nor were they specifically produced or tailored to accompany *Intermediate Algebra*.

**Self Check** The Self Check includes the learning objectives for the section so that students can self-assess their mastery and make concrete plans to improve.

## Art Program

*Intermediate Algebra* contains many figures and illustrations. Art throughout the text adheres to a clear, understated style, drawing the eye to the most important information in each figure while minimizing visual distractions.



Intersecting Parallel Coincident

## Section Exercises and Chapter Review

**Section Exercises** Each section of every chapter concludes with a well-rounded set of exercises that can be assigned as homework or used selectively for guided practice. Exercise sets are named *Practice Makes Perfect* to encourage completion of homework assignments.

- Exercises correlate to the learning objectives. This facilitates assignment of personalized study plans based on individual student needs.
- Exercises are carefully sequenced to promote building of skills.
- Values for constants and coefficients were chosen to practice and reinforce arithmetic facts.
- Even and odd-numbered exercises are paired.
- Exercises parallel and extend the text examples and use the same instructions as the examples to help students easily recognize the connection.
- Applications are drawn from many everyday experiences, as well as those traditionally found in college math texts.
- **Everyday Math** highlights practical situations using the concepts from that particular section
- **Writing Exercises** are included in every exercise set to encourage conceptual understanding, critical thinking, and literacy.

**Chapter review** Each chapter concludes with a review of the most important takeaways, as well as additional practice problems that students can use to prepare for exams.

- **Key Terms** provide a formal definition for each bold-faced term in the chapter.
- **Key Concepts** summarize the most important ideas introduced in each section, linking back to the relevant Example(s) in case students need to review.
- **Chapter Review Exercises** include practice problems that recall the most important concepts from each section.
- **Practice Test** includes additional problems assessing the most important learning objectives from the chapter.
- **Answer Key** includes the answers to all Try It exercises and every other exercise from the Section Exercises, Chapter Review Exercises, and Practice Test.

# Additional Resources

## Student and Instructor Resources

We’ve compiled additional resources for both students and instructors, including Getting Started Guides, manipulative mathematics worksheets, an answer key to the Be Prepared Exercises, and an answer guide to the section review exercises. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

## Partner Resources

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## About the Authors

## Senior Contributing Author

**Lynn Marecek, Santa Ana College**

Lynn Marecek has been teaching mathematics at Santa Ana College for many years has focused her career on meeting the needs of developmental math students. At Santa Ana College, she has been awarded the Distinguished Faculty Award, Innovation Award, and the Curriculum Development Award four times. She is a Coordinator of the Freshman Experience Program, the Department Facilitator for Redesign, and a member of the Student Success and Equity Committee, and the Basic Skills Initiative Task Force.

She is the coauthor with MaryAnne Anthony-Smith of *Strategies for Success: Study Skills for the College Math Student*, *Prealgebra* published by OpenStax and *Elementary Algebra* published by OpenStax.

## Reviewers

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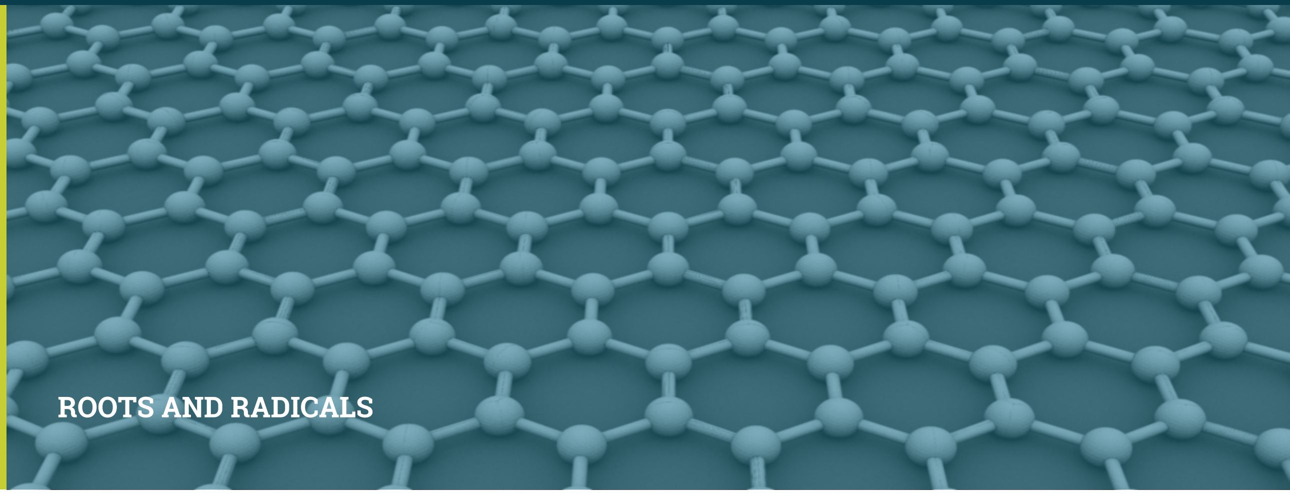


# 8 ROOTS AND RADICALS

**Figure 8.1** Graphene is an incredibly strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: “AlexanderAIUS” / Wikimedia Commons)

## Chapter Outline



## Introduction

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

## 8.1 Simplify Expressions with Roots

### Learning Objectives

**By the end of this section, you will be able to:**

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

**Be Prepared!**

Before you get started, take this readiness quiz.

1. Simplify: ⓐ $(-9)^2$ ⓑ $-9^2$ ⓒ $(-9)^3$.
If you missed this problem, review **Example 2.21**.

2. Round $3.846$ to the nearest hundredth.
   If you missed this problem, review **Example 1.34**.
3. Simplify: ⓐ $x^3 \cdot x^3$ ⓑ $y^2 \cdot y^2 \cdot y^2$ ⓒ $z^3 \cdot z^3 \cdot z^3 \cdot z^3$.
   If you missed this problem, review **Example 5.12**.

## Simplify Expressions with Roots

In **Foundations**, we briefly looked at square roots. Remember that when a real number *n* is multiplied by itself, we write $n^2$ and read it '*n* squared'. This number is called the **square** of *n*, and *n* is called the **square root**. For example,

$$13^2 \text{ is read "13 squared"}$$
$$169 \text{ is called the } \textit{square} \text{ of 13, since } 13^2 = 169$$
$$13 \text{ is a } \textit{square root} \text{ of } 169$$

**Square and Square Root of a number**

**Square**

$$\text{If } n^2 = m, \text{ then } m \text{ is the } \textbf{square} \text{ of } n.$$

**Square Root**

$$\text{If } n^2 = m, \text{ then } n \text{ is a } \textbf{square root} \text{ of } m.$$

Notice $(-13)^2 = 169$ also, so −13 is also a square root of 169. Therefore, both 13 and −13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of *m*. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

**Square Root Notation**

$$\sqrt{m} \text{ is read "the square root of } m\text{".}$$
$$\text{If } n^2 = m, \text{ then } n = \sqrt{m}, \text{ for } n \geq 0.$$


We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169} = 13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169} = -13$.

**EXAMPLE 8.1**

Simplify: ⓐ $\sqrt{144}$ ⓑ $-\sqrt{289}$.

### Solution

ⓐ

| | |
|---|---|
| | $\sqrt{144}$ |
| Since $12^2 = 144$. | $12$ |

ⓑ

| | |
|---|---|
| | $-\sqrt{289}$ |
| Since $17^2 = 289$ and the negative is in front of the radical sign. | $-17$ |

**TRY IT : :** 8.1 Simplify: ⓐ $-\sqrt{64}$ ⓑ $\sqrt{225}$.

**TRY IT : :** 8.2 Simplify: ⓐ $\sqrt{100}$ ⓑ $-\sqrt{121}$.

Can we simplify $\sqrt{-49}$? Is there a number whose square is $-49$?

$$(\ )^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

**EXAMPLE 8.2**

Simplify: ⓐ $\sqrt{-196}$ ⓑ $-\sqrt{64}$.

### Solution

ⓐ

| | |
|---|---|
| | $\sqrt{-196}$ |
| There is no real number whose square is $-196$. | $\sqrt{-196}$ is not a real number. |

ⓑ

| | |
|---|---|
| | $-\sqrt{64}$ |
| The negative is in front of the radical. | $-8$ |

**TRY IT : :** 8.3 Simplify: ⓐ $\sqrt{-169}$ ⓑ $-\sqrt{81}$.

**TRY IT : :** 8.4 Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{-121}$.

So far we have only talked about squares and square roots. Let's now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

| We write: | We say: |
|---|---|
| $n^2$ | $n$ squared |
| $n^3$ | $n$ cubed |
| $n^4$ | $n$ to the fourth power |
| $n^5$ | $n$ to the fi th power |

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from −5 to 5. See **Figure 8.2**.

| Number | Square | Cube | Fourth power | Fifth power |
|---|---|---|---|---|
| $n$ | $n^2$ | $n^3$ | $n^4$ | $n^5$ |
| 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 8 | 16 | 32 |
| 3 | 9 | 27 | 81 | 243 |
| 4 | 16 | 64 | 256 | 1024 |
| 5 | 25 | 125 | 625 | 3125 |
| $x$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ |
| $x^2$ | $x^4$ | $x^6$ | $x^8$ | $x^{10}$ |

| Number | Square | Cube | Fourth power | Fifth power |
|---|---|---|---|---|
| $n$ | $n^2$ | $n^3$ | $n^4$ | $n^5$ |
| –1 | 1 | –1 | 1 | –1 |
| –2 | 4 | –8 | 16 | –32 |
| –3 | 9 | –27 | 81 | –243 |
| –4 | 16 | –64 | 256 | –1024 |
| –5 | 25 | –125 | 625 | –3125 |

**Figure 8.2**

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of −2 to help you see this.

| $n$ | $n^2$ | $n^3$ | $n^4$ | $n^5$ |
|---|---|---|---|---|
| –2 | 4 | –8 | 16 | –32 |

Even power
Positive result

Odd power
Negative result

We will now extend the square root definition to higher roots.

### $n^{\text{th}}$ Root of a Number

$$\text{If } b^n = a, \text{ then } b \text{ is an } n^{th} \text{ root of } a.$$
$$\text{The principal } n^{th} \text{ root of } a \text{ is written } \sqrt[n]{a}.$$
$$n \text{ is called the } \textbf{index} \text{ of the radical.}$$

Just like we use the word 'cubed' for $b^3$, we use the term 'cube root' for $\sqrt[3]{a}$.

We can refer to **Figure 8.2** to help find higher roots.

$$\begin{aligned} 4^3 &= 64 \\ 3^4 &= 81 \\ (-2)^5 &= -32 \end{aligned} \qquad \begin{aligned} \sqrt[3]{64} &= 4 \\ \sqrt[4]{81} &= 3 \\ \sqrt[5]{-32} &= -2 \end{aligned}$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

### Properties of $\sqrt[n]{a}$

When $n$ is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When $n$ is an odd number, $\sqrt[n]{a}$ is a real number for all values of $a$.

We will apply these properties in the next two examples.

### EXAMPLE 8.3

Simplify: ⓐ $\sqrt[3]{64}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{32}$.

#### Solution

ⓐ

| | $\sqrt[3]{64}$ |
|---|---|
| Since $4^3 = 64$. | $4$ |

ⓑ

| | $\sqrt[4]{81}$ |
|---|---|
| Since $(3)^4 = 81$. | $3$ |

ⓒ

| | $\sqrt[5]{32}$ |
|---|---|
| Since $(2)^5 = 32$. | $2$ |

**TRY IT : :** 8.5 Simplify: ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{256}$ ⓒ $\sqrt[5]{243}$.

**TRY IT : :** 8.6 Simplify: ⓐ $\sqrt[3]{1000}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$.

In this example be alert for the negative signs as well as even and odd powers.

**EXAMPLE 8.4**

Simplify: ⓐ $\sqrt[3]{-125}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{-243}$.

### Solution

ⓐ

| | |
|---|---|
| | $\sqrt[3]{-125}$ |
| Since $(-5)^3 = -125$. | $-5$ |

ⓑ

| | |
|---|---|
| | $\sqrt[4]{-16}$ |
| Think, $(\,?\,)^4 = -16$. No real number raised to the fourth power is negative. | Not a real number. |

ⓒ

| | |
|---|---|
| | $\sqrt[5]{-243}$ |
| Since $(-3)^5 = -243$. | $-3$ |

**TRY IT : :** 8.7 Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-256}$ ⓒ $\sqrt[5]{-32}$.

**TRY IT : :** 8.8 Simplify: ⓐ $\sqrt[3]{-216}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1024}$.

## Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4} = 2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for perfect square numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

| Number | Square Root |
|---|---|
| 4 | 2 |
| 9 | 3 |
| 16 | 4 |
| 25 | 5 |

11 → between 9 and 16; $\sqrt{11}$ → between 3 and 4

$9 < 11 < 16$

$3 < \sqrt{11} < 4$

| Number | Cube Root |
|---|---|
| 8 | 2 |
| 27 | 3 |
| 64 | 4 |
| 125 | 5 |

91 → between 64 and 125; $\sqrt[3]{91}$ → between 4 and 5

$64 < 91 < 125$

$4 < \sqrt[3]{91} < 5$

Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

**EXAMPLE 8.5**

Estimate each root between two consecutive whole numbers: ⓐ $\sqrt{105}$ ⓑ $\sqrt[3]{43}$.

### Solution

ⓐ Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

$\sqrt{105}$

| Number | Square Root |
|---|---|
| 81 | 9 |
| 100 | 10 |
| 121 | 11 |
| 144 | 12 |

105    $\sqrt{105}$

| | |
|---|---|
| Locate 105 between two consecutive perfect squares. | $100 < 105 < 121$ |
| $\sqrt{105}$ is between their square roots. | $10 < \sqrt{105} < 11$ |

ⓑ Similarly we locate 43 between two perfect cube numbers.

$\sqrt[3]{43}$

| Number | Cube Root |
|---|---|
| 8 | 2 |
| 27 | 3 |
| 64 | 4 |
| 125 | 5 |

43    $\sqrt[3]{43}$

| | |
|---|---|
| Locate 43 between two consecutive perfect cubes. | $27 < 43 < 64$ |
| $\sqrt[3]{43}$ is between their cube roots. | $3 < \sqrt[3]{43} < 4$ |

**TRY IT : :** 8.9 Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{38}$ ⓑ $\sqrt[3]{93}$

**TRY IT : :** 8.10 Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{84}$ ⓑ $\sqrt[3]{152}$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{ rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{ rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$\begin{aligned}(2.236067978)^2 &= 5.000000002 \qquad & (3.105422799)^4 &= 92.999999991\\ (2.24)^2 &= 5.0176 & (3.11)^4 &= 93.54951841\end{aligned}$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

**EXAMPLE 8.6**

Round to two decimal places: ⓐ $\sqrt{17}$ ⓑ $\sqrt[3]{49}$ ⓒ $\sqrt[4]{51}$.

**Solution**

ⓐ

| | $\sqrt{17}$ |
|---|---|
| Use the calculator square root key. | $4.123105626\ldots$ |
| Round to two decimal places. | $4.12$ |
| | $\sqrt{17} \approx 4.12$ |

ⓑ

| | $\sqrt[3]{49}$ |
|---|---|
| Use the calculator $\sqrt[y]{x}$ key. | $3.659305710\ldots$ |
| Round to two decimal places. | $3.66$ |
| | $\sqrt[3]{49} \approx 3.66$ |

ⓒ

| | $\sqrt[4]{51}$ |
|---|---|
| Use the calculator $\sqrt[y]{x}$ key. | $2.6723451177\ldots$ |
| Round to two decimal places. | $2.67$ |
| | $\sqrt[4]{51} \approx 2.67$ |

**TRY IT : :** 8.11 Round to two decimal places:

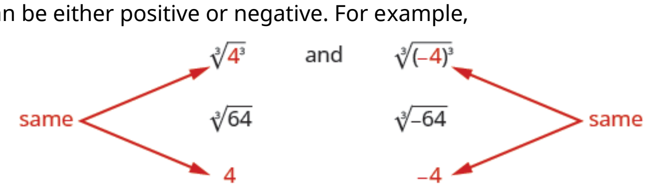
ⓐ $\sqrt{11}$ ⓑ $\sqrt[3]{71}$ ⓒ $\sqrt[4]{127}$.

**TRY IT : :** 8.12 Round to two decimal places:

ⓐ $\sqrt{13}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{98}$.

## Simplify Variable Expressions with Roots

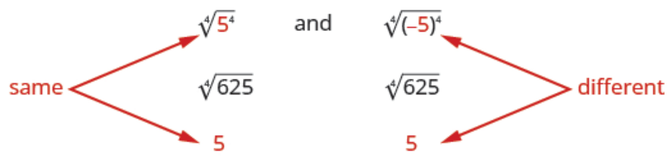
The odd root of a number can be either positive or negative. For example,




In either case, when $n$ is odd, $\sqrt[n]{a^n} = a$.

But what about an even root? We want the principal root, so $\sqrt[4]{625} = 5$.

But notice,




Here we see that sometimes when $n$ is even, $\sqrt[n]{a^n} \neq a$.

How can we make sure the fourth root of −5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when $n$ is even $\sqrt[n]{a^n} = |a|$. This guarantees the principal root is positive.

**Simplifying Odd and Even Roots**

For any integer $n \geq 2$,

| | |
|---|---|
| when the index $n$ is odd | $\sqrt[n]{a^n} = a$ |
| when the index $n$ is even | $\sqrt[n]{a^n} = \lvert a \rvert$ |

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

**EXAMPLE 8.7**

Simplify: ⓐ $\sqrt{x^2}$ ⓑ $\sqrt[3]{n^3}$ ⓒ $\sqrt[4]{p^4}$ ⓓ $\sqrt[5]{y^5}$.

**Solution**

ⓐ We use the absolute value to be sure to get the positive root.

| | |
|---|---|
| | $\sqrt{x^2}$ |
| Since the index $n$ is even, $\sqrt[n]{a^n} = \lvert a \rvert$. | $\lvert x \rvert$ |

ⓑ This is an odd indexed root so there is no need for an absolute value sign.

| | |
|---|---|
| | $\sqrt[3]{m^3}$ |
| Since the index $n$ is odd, $\sqrt[n]{a^n} = a$. | $m$ |

ⓒ

| | |
|---|---|
| | $\sqrt[4]{p^4}$ |
| Since the index $n$ is even $\sqrt[n]{a^n} = \lvert a \rvert$. | $\lvert p \rvert$ |

ⓓ

| | |
|---|---|
| | $\sqrt[5]{y^5}$ |
| Since the index $n$ is odd, $\sqrt[n]{a^n} = a$. | $y$ |

**TRY IT : :** 8.13 Simplify: ⓐ $\sqrt{b^2}$ ⓑ $\sqrt[3]{w^3}$ ⓒ $\sqrt[4]{m^4}$ ⓓ $\sqrt[5]{q^5}$.

**TRY IT : :** 8.14 Simplify: ⓐ $\sqrt{y^2}$ ⓑ $\sqrt[3]{p^3}$ ⓒ $\sqrt[4]{z^4}$ ⓓ $\sqrt[5]{q^5}$.

What about square roots of higher powers of variables? The Power Property of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square $a^m$, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root,

| | |
|---|---|
| | $\sqrt{a^{2m}}$ |
| Since $(a^m)^2 = a^{2m}$. | $\sqrt{(a^m)^2}$ |
| Since $n$ is even $\sqrt[n]{a^n} = \lvert a \rvert$. | $\lvert a^m \rvert$ |

$$\text{So } \sqrt{a^{2m}} = \lvert a^m \rvert.$$

We apply this concept in the next example.

**EXAMPLE 8.8**

Simplify: ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt{x^6}$ |
| Since $\left(x^3\right)^2 = x^6$. | $\sqrt{\left(x^3\right)^2}$ |
| Since the index $n$ is even $\sqrt{a^n} = \lvert a \rvert$. | $\lvert x^3 \rvert$ |

ⓑ

| | |
|---|---|
| | $\sqrt{y^{16}}$ |
| Since $\left(y^8\right)^2 = y^{16}$. | $\sqrt{\left(y^8\right)^2}$ |
| Since the index $n$ is even $\sqrt[n]{a^n} = \lvert a \rvert$. | $y^8$ |
| In this case the absolute value sign is not needed as $y^8$ is positive. | |

**TRY IT : :** 8.15 Simplify: ⓐ $\sqrt{y^{18}}$ ⓑ $\sqrt{z^{12}}$.

**TRY IT : :** 8.16 Simplify: ⓐ $\sqrt{m^4}$ ⓑ $\sqrt{b^{10}}$.

The next example uses the same idea for highter roots.

**EXAMPLE 8.9**

Simplify: ⓐ $\sqrt[3]{y^{18}}$ ⓑ $\sqrt[4]{z^8}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt[3]{y^{18}}$ |
| Since $\left(y^6\right)^3 = y^{18}$. | $\sqrt[3]{\left(y^6\right)^3}$ |
| Since $n$ is odd, $\sqrt[n]{a^n} = a$. | $y^6$ |

ⓑ

| | |
|---|---|
| | $\sqrt[4]{z^8}$ |
| Since $\left(z^2\right)^4 = z^8$. | $\sqrt[4]{\left(z^2\right)^4}$ |
| Since $z^2$ is positive, we do not need an absolute value sign. | $z^2$ |

> **TRY IT : : 8.17** Simplify: ⓐ $\sqrt[4]{u^{12}}$ ⓑ $\sqrt[3]{v^{15}}$.

> **TRY IT : : 8.18** Simplify: ⓐ $\sqrt[5]{c^{20}}$ ⓑ $\sqrt[6]{d^{24}}$

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = \left|a^m\right|$ works in much the same way.

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as $u^4$ is always positive.

## EXAMPLE 8.10

Simplify: ⓐ $\sqrt{16n^2}$ ⓑ $-\sqrt{81c^2}$.

### Solution

ⓐ

| | |
|---|---|
| | $\sqrt{16n^2}$ |
| Since $(4n)^2 = 16n^2$. | $\sqrt{(4n)^2}$ |
| Since the index $n$ is even $\sqrt[n]{a^n} = \lvert a\rvert$. | $4\lvert n\rvert$ |

ⓑ

| | |
|---|---|
| | $-\sqrt{81c^2}$ |
| Since $(9c)^2 = 81c^2$. | $-\sqrt{(9c)^2}$ |
| Since the index $n$ is even $\sqrt[n]{a^n} = \lvert a\rvert$. | $-9\lvert c\rvert$ |

> **TRY IT : : 8.19** Simplify: ⓐ $\sqrt{64x^2}$ ⓑ $-\sqrt{100p^2}$.

> **TRY IT : : 8.20** Simplify: ⓐ $\sqrt{169y^2}$ ⓑ $-\sqrt{121y^2}$.

This example just takes the idea farther as it has roots of higher index.

## EXAMPLE 8.11

Simplify: ⓐ $\sqrt[3]{64p^6}$ ⓑ $\sqrt[4]{16q^{12}}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt[3]{64p^6}$ |
| Rewrite $64p^6$ as $\left(4p^2\right)^3$. | $\sqrt[3]{\left(4p^2\right)^3}$ |
| Take the cube root. | $4p^2$ |

ⓑ

| | |
|---|---|
| | $\sqrt[4]{16q^{12}}$ |
| Rewrite the radicand as a fourth power. | $\sqrt[4]{\left(2q^3\right)^4}$ |
| Take the fourth root. | $2\left\lvert q^3\right\rvert$ |

**TRY IT : :** 8.21 Simplify: ⓐ $\sqrt[3]{27x^{27}}$ ⓑ $\sqrt[4]{81q^{28}}$.

**TRY IT : :** 8.22 Simplify: ⓐ $\sqrt[3]{125q^9}$ ⓑ $\sqrt[5]{243q^{25}}$.

The next examples have two variables.

**EXAMPLE 8.12**

Simplify: ⓐ $\sqrt{36x^2y^2}$ ⓑ $\sqrt{121a^6b^8}$ ⓒ $\sqrt[3]{64p^{63}q^9}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt{36x^2y^2}$ |
| Since $(6xy)^2 = 36x^2y^2$ | $\sqrt{(6xy)^2}$ |
| Take the square root. | $6\lvert xy\rvert$ |

ⓑ

| | |
|---|---|
| | $\sqrt{121a^6b^8}$ |
| Since $\left(11a^3b^4\right)^2 = 121a^6b^8$ | $\sqrt{\left(11a^3b^4\right)^2}$ |
| Take the square root. | $11\left\lvert a^3\right\rvert b^4$ |

ⓒ

| | |
|---|---|
| | $\sqrt[3]{64p^{63}q^9}$ |
| Since $\left(4p^{21}q^3\right)^3 = 64p^{63}q^9$ | $\sqrt[3]{\left(4p^{21}q^3\right)^3}$ |
| Take the cube root. | $4p^{21}q^3$ |

**TRY IT : :** 8.23

Simplify: ⓐ $\sqrt{100a^2b^2}$ ⓑ $\sqrt{144p^{12}q^{20}}$ ⓒ $\sqrt[3]{8x^{30}y^{12}}$

**TRY IT : :** 8.24

Simplify: ⓐ $\sqrt{225m^2n^2}$ ⓑ $\sqrt{169x^{10}y^{14}}$ ⓒ $\sqrt[3]{27w^{36}z^{15}}$

**MEDIA : :**

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- **Simplifying Variables Exponents with Roots using Absolute Values (https://openstax.org/l/37SimVarAbVal)**

## 8.1 EXERCISES

### Practice Makes Perfect

#### Simplify Expressions with Roots

*In the following exercises, simplify.*

**1.** ⓐ $\sqrt{64}$ ⓑ $-\sqrt{81}$

**2.** ⓐ $\sqrt{169}$ ⓑ $-\sqrt{100}$

**3.** ⓐ $\sqrt{196}$ ⓑ $-\sqrt{1}$

**4.** ⓐ $\sqrt{144}$ ⓑ $-\sqrt{121}$

**5.** ⓐ $\sqrt{\frac{4}{9}}$ ⓑ $-\sqrt{0.01}$

**6.** ⓐ $\sqrt{\frac{64}{121}}$ ⓑ $-\sqrt{0.16}$

**7.** ⓐ $\sqrt{-121}$ ⓑ $-\sqrt{289}$

**8.** ⓐ $-\sqrt{400}$ ⓑ $\sqrt{-36}$

**9.** ⓐ $-\sqrt{225}$ ⓑ $\sqrt{-9}$

**10.** ⓐ $\sqrt{-49}$ ⓑ $-\sqrt{256}$

**11.** ⓐ $\sqrt[3]{216}$ ⓑ $\sqrt[4]{256}$

**12.** ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$

**13.** ⓐ $\sqrt[3]{512}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{1}$

**14.** ⓐ $\sqrt[3]{125}$ ⓑ $\sqrt[4]{1296}$ ⓒ $\sqrt[5]{1024}$

**15.** ⓐ $\sqrt[3]{-8}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-32}$

**16.**
ⓐ $\sqrt[3]{-64}$
ⓑ $\sqrt[4]{-16}$
ⓒ $\sqrt[5]{-243}$

**17.**
ⓐ $\sqrt[3]{-125}$
ⓑ $\sqrt[4]{-1296}$
ⓒ $\sqrt[5]{-1024}$

**18.**
ⓐ $\sqrt[3]{-512}$
ⓑ $\sqrt[4]{-81}$
ⓒ $\sqrt[5]{-1}$

#### Estimate and Approximate Roots

*In the following exercises, estimate each root between two consecutive whole numbers.*

**19.** ⓐ $\sqrt{70}$ ⓑ $\sqrt[3]{71}$

**20.** ⓐ $\sqrt{55}$ ⓑ $\sqrt[3]{119}$

**21.** ⓐ $\sqrt{200}$ ⓑ $\sqrt[3]{137}$

**22.** ⓐ $\sqrt{172}$ ⓑ $\sqrt[3]{200}$

*In the following exercises, approximate each root and round to two decimal places.*

**23.** ⓐ $\sqrt{19}$ ⓑ $\sqrt[3]{89}$ ⓒ $\sqrt[4]{97}$

**24.** ⓐ $\sqrt{21}$ ⓑ $\sqrt[3]{93}$ ⓒ $\sqrt[4]{101}$

**25.** ⓐ $\sqrt{53}$ ⓑ $\sqrt[3]{147}$ ⓒ $\sqrt[4]{452}$

**26.** ⓐ $\sqrt{47}$ ⓑ $\sqrt[3]{163}$ ⓒ $\sqrt[4]{527}$

#### Simplify Variable Expressions with Roots

*In the following exercises, simplify using absolute values as necessary.*

**27.** ⓐ $\sqrt[5]{u^5}$ ⓑ $\sqrt[8]{v^8}$

**28.** ⓐ $\sqrt[3]{a^3}$ ⓑ $\sqrt[9]{b^9}$

**29.** ⓐ $\sqrt[4]{y^4}$ ⓑ $\sqrt[7]{m^7}$

**30.** ⓐ $\sqrt[8]{k^8}$ ⓑ $\sqrt[6]{p^6}$

**31.** ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$

**32.** ⓐ $\sqrt{a^{14}}$ ⓑ $\sqrt{w^{24}}$

**33.** ⓐ $\sqrt{x^{24}}$ ⓑ $\sqrt{y^{22}}$

**34.** ⓐ $\sqrt{a^{12}}$ ⓑ $\sqrt{b^{26}}$

**35.** ⓐ $\sqrt[3]{x^9}$ ⓑ $\sqrt[4]{y^{12}}$

**36.** ⓐ $\sqrt[5]{a^{10}}$ ⓑ $\sqrt[3]{b^{27}}$

**37.** ⓐ $\sqrt[4]{m^8}$ ⓑ $\sqrt[5]{n^{20}}$

**38.** ⓐ $\sqrt[6]{r^{12}}$ ⓑ $\sqrt[3]{s^{30}}$

**39.** ⓐ $\sqrt{49x^2}$ ⓑ $-\sqrt{81x^{18}}$

**40.** ⓐ $\sqrt{100y^2}$ ⓑ $-\sqrt{100m^{32}}$

**41.** ⓐ $\sqrt{121m^{20}}$ ⓑ $-\sqrt{64a^2}$

**42.**
ⓐ $\sqrt{81x^{36}}$
ⓑ $-\sqrt{25x^2}$

**43.**
ⓐ $\sqrt[4]{16x^8}$
ⓑ $\sqrt[6]{64y^{12}}$

**44.**
ⓐ $\sqrt[3]{-8c^9}$
ⓑ $\sqrt[3]{125d^{15}}$

**45.**
ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

**46.**
ⓐ $\sqrt[7]{128r^{14}}$
ⓑ $\sqrt[4]{81s^{24}}$

**47.**
ⓐ $\sqrt{144x^2y^2}$
ⓑ $\sqrt{169w^8y^{10}}$
ⓒ $\sqrt[3]{8a^{51}b^6}$

**48.**
ⓐ $\sqrt{196a^2b^2}$
ⓑ $\sqrt{81p^{24}q^6}$
ⓒ $\sqrt[3]{27p^{45}q^9}$

**49.**
ⓐ $\sqrt{121a^2b^2}$
ⓑ $\sqrt{9c^8d^{12}}$
ⓒ $\sqrt[3]{64x^{15}y^{66}}$

**50.**
ⓐ $\sqrt{225x^2y^2z^2}$
ⓑ $\sqrt{36r^6s^{20}}$
ⓒ $\sqrt[3]{125y^{18}z^{27}}$

## Writing Exercises

**51.** Why is there no real number equal to $\sqrt{-64}$?

**52.** What is the difference between $9^2$ and $\sqrt{9}$?

**53.** Explain what is meant by the $n^{th}$ root of a number.

**54.** Explain the difference of finding the $n^{th}$ root of a number when the index is even compared to when the index is odd.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| simplify expressions with roots. | | | |
| estimate and approximate roots. | | | |
| simplify variable expressions with roots. | | | |

ⓑ *If most of your checks were:*

**...confidently.** *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

**...with some help.** *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

**...no - I don't get it!** *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

# 8.2 Simplify Radical Expressions

## Learning Objectives

**By the end of this section, you will be able to:**

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

### Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{x^9}{x^4}$.

   If you missed this problem, review Example 5.13.

2. Simplify: $\frac{y^3}{y^{11}}$.

   If you missed this problem, review Example 5.13.

3. Simplify: $\left(n^2\right)^6$.

   If you missed this problem, review Example 5.17.

## Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A radical expression, $\sqrt[n]{a}$, is considered simplified if it has no factors of $m^n$. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

### Simplified Radical Expression

For real numbers *a* and *m*, and $n \geq 2$,

$$\sqrt[n]{a} \text{ is considered simplified i } \; a \text{ has no factors of } m^n$$

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

Similarly, $\sqrt[3]{4}$ is simplified because there are no perfect cube factors in 4. But $\sqrt[3]{24}$ is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that $(ab)^n = a^n b^n$. The corresponding of **Product Property of Roots** says that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

### Product Property of $n^{\text{th}}$ Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $n \geq 2$ is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

We use the Product Property of Roots to remove all perfect square factors from a square root.

**EXAMPLE 8.13** SIMPLIFY SQUARE ROOTS USING THE PRODUCT PROPERTY OF ROOTS

Simplify: $\sqrt{98}$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor. | We see that 49 is the largest factor of 98 that has a power of 2. In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first. | $\sqrt{98}$ $\sqrt{49 \cdot 2}$ |
| **Step 2.** Use the product rule to rewrite the radical as the product of two radicals. | | $\sqrt{49} \cdot \sqrt{2}$ |
| **Step 3.** Simplify the root of the perfect power. | | $7\sqrt{2}$ |

**TRY IT : :** 8.25 Simplify: $\sqrt{48}$.

**TRY IT : :** 8.26 Simplify: $\sqrt{45}$.

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

**HOW TO : :** SIMPLIFY A RADICAL EXPRESSION USING THE PRODUCT PROPERTY.

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.

Step 2. Use the product rule to rewrite the radical as the product of two radicals.

Step 3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

**EXAMPLE 8.14**

Simplify: ⓐ $\sqrt{500}$ ⓑ $\sqrt[3]{16}$ ⓒ $\sqrt[4]{243}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt{500}$ |
| Rewrite the radicand as a product using the largest perfect square factor. | $\sqrt{100 \cdot 5}$ |
| Rewrite the radical as the product of two radicals | $\sqrt{100} \cdot \sqrt{5}$ |
| Simplify. | $10\sqrt{5}$ |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{16}$ |
| Rewrite the radicand as a product using the greatest perfect cube factor. $2^3 = 8$ | $\sqrt[3]{8 \cdot 2}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt[3]{8} \cdot \sqrt[3]{2}$ |
| Simplify. | $2\sqrt[3]{2}$ |

ⓒ

| | |
|---|---|
| | $\sqrt[4]{243}$ |
| Rewrite the radicand as a product using the greatest perfect fourth power factor. $3^4 = 81$ | $\sqrt[4]{81 \cdot 3}$ |
| Rewrite the radical as the product of two radicals | $\sqrt[4]{81} \cdot \sqrt[4]{3}$ |
| Simplify. | $3\sqrt[4]{3}$ |

**TRY IT : :** 8.27 Simplify: ⓐ $\sqrt{288}$ ⓑ $\sqrt[3]{81}$ ⓒ $\sqrt[4]{64}$.

**TRY IT : :** 8.28 Simplify: ⓐ $\sqrt{432}$ ⓑ $\sqrt[3]{625}$ ⓒ $\sqrt[4]{729}$.

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

**EXAMPLE 8.15**

Simplify: ⓐ $\sqrt{x^3}$ ⓑ $\sqrt[3]{x^4}$ ⓒ $\sqrt[4]{x^7}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt{x^3}$ |
| Rewrite the radicand as a product using the largest perfect square factor. | $\sqrt{x^2 \cdot x}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt{x^2} \cdot \sqrt{x}$ |
| Simplify. | $\lvert x \rvert \sqrt{x}$ |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{x^4}$ |
| Rewrite the radicand as a product using the largest perfect cube factor. | $\sqrt[3]{x^3 \cdot x}.$ |
| Rewrite the radical as the product of two radicals. | $\sqrt[3]{x^3} \cdot \sqrt[3]{x}$ |
| Simplify. | $x\sqrt[3]{x}$ |

ⓒ

| | |
|---|---|
| | $\sqrt[4]{x^7}$ |
| Rewrite the radicand as a product using the greatest perfect fourth power factor. | $\sqrt[4]{x^4 \cdot x^3}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$ |
| Simplify. | $\lvert x \rvert \sqrt[4]{x^3}$ |

**TRY IT : :** 8.29 Simplify: ⓐ $\sqrt{b^5}$ ⓑ $\sqrt[4]{y^6}$ ⓒ $\sqrt[3]{z^5}$

**TRY IT : :** 8.30 Simplify: ⓐ $\sqrt{p^9}$ ⓑ $\sqrt[5]{y^8}$ ⓒ $\sqrt[6]{q^{13}}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

**EXAMPLE 8.16**

Simplify: ⓐ $\sqrt{72n^7}$ ⓑ $\sqrt[3]{24x^7}$ ⓒ $\sqrt[4]{80y^{14}}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt{72n^7}$ |
| Rewrite the radicand as a product using the largest perfect square factor. | $\sqrt{36n^6 \cdot 2n}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt{36n^6} \cdot \sqrt{2n}$ |
| Simplify. | $6\lvert n^3 \rvert \sqrt{2n}$ |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{24x^7}$ |
| Rewrite the radicand as a product using perfect cube factors. | $\sqrt[3]{8x^6 \cdot 3x}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$ |
| Rewrite the fir t radicand as $(2x^2)^3$. | $\sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}$ |
| Simplify. | $2x^2 \sqrt[3]{3x}$ |

ⓒ

| | |
|---|---|
| | $\sqrt[4]{80y^{14}}$ |
| Rewrite the radicand as a product using perfect fourth power factors. | $\sqrt[4]{16y^{12} \cdot 5\,y^2}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt[4]{16y^{12}} \cdot \sqrt[4]{5y^2}$ |
| Rewrite the fir t radicand as $\left(2y^3\right)^4$. | $\sqrt[4]{\left(2\,y^3\right)^4} \cdot \sqrt[4]{5y^2}$ |
| Simplify. | $2\left\|y^3\right\| \sqrt[4]{5\,y^2}$ |

**TRY IT : :** 8.31 Simplify: ⓐ $\sqrt{32y^5}$ ⓑ $\sqrt[3]{54p^{10}}$ ⓒ $\sqrt[4]{64q^{10}}$.

**TRY IT : :** 8.32 Simplify: ⓐ $\sqrt{75a^9}$ ⓑ $\sqrt[3]{128m^{11}}$ ⓒ $\sqrt[4]{162n^7}$.

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

**EXAMPLE 8.17**

Simplify: ⓐ $\sqrt{63u^3v^5}$ ⓑ $\sqrt[3]{40x^4y^5}$ ⓒ $\sqrt[4]{48x^4y^7}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt{63u^3v^5}$ |
| Rewrite the radicand as a product using the largest perfect square factor. | $\sqrt{9u^2v^4 \cdot 7uv}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt{9u^2v^4} \cdot \sqrt{7uv}$ |
| Rewrite the fir t radicand as $\left(3uv^2\right)^2$. | $\sqrt{\left(3uv^2\right)^2} \cdot \sqrt{7uv}$ |
| Simplify. | $3\|u\|v^2\sqrt{7uv}$ |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{40x^4y^5}$ |
| Rewrite the radicand as a product using the largest perfect cube factor. | $\sqrt[3]{8x^3y^3 \cdot 5xy^2}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{5xy^2}$ |
| Rewrite the fir t radicand as $(2xy)^3$. | $\sqrt[3]{(2xy)^3} \cdot \sqrt[3]{5xy^2}$ |
| Simplify. | $2xy\sqrt[3]{5xy^2}$ |

ⓒ

| | |
|---|---|
| | $\sqrt[4]{48x^4 y^7}$ |
| Rewrite the radicand as a product using the largest perfect fourth power factor. | $\sqrt[4]{16x^4 y^4 \cdot 3y^3}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt[4]{16x^4 y^4} \cdot \sqrt[4]{3y^3}$ |
| Rewrite the fir t radicand as $(2xy)^4$. | $\sqrt[4]{(2xy)^4} \cdot \sqrt[4]{3y^3}$ |
| Simplify. | $2\lvert xy\rvert \sqrt[4]{3y^3}$ |

**TRY IT : :** 8.33 Simplify: ⓐ $\sqrt{98a^7 b^5}$ ⓑ $\sqrt[3]{56x^5 y^4}$ ⓒ $\sqrt[4]{32x^5 y^8}$.

**TRY IT : :** 8.34 Simplify: ⓐ $\sqrt{180m^9 n^{11}}$ ⓑ $\sqrt[3]{72x^6 y^5}$ ⓒ $\sqrt[4]{80x^7 y^4}$.

**EXAMPLE 8.18**

Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-16}$.

**Solution**

ⓐ

| | |
|---|---|
| | $\sqrt[3]{-27}$ |
| Rewrite the radicand as a product using perfect cube factors. | $\sqrt[3]{(-3)^3}$ |
| Take the cube root. | $-3$ |

ⓑ

| | |
|---|---|
| | $\sqrt[4]{-16}$ |
| There is no real number $n$ where $n^4 = -16$. | Not a real number. |

**TRY IT : :** 8.35 Simplify: ⓐ $\sqrt[3]{-64}$ ⓑ $\sqrt[4]{-81}$.

**TRY IT : :** 8.36 Simplify: ⓐ $\sqrt[3]{-625}$ ⓑ $\sqrt[4]{-324}$.

We have seen how to use the order of operations to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

**EXAMPLE 8.19**

Simplify: ⓐ $3 + \sqrt{32}$ ⓑ $\frac{4 - \sqrt{48}}{2}$.

**Solution**

ⓐ

| | |
|---|---|
| | $3+\sqrt{32}$ |
| Rewrite the radicand as a product using the largest perfect square factor. | $3+\sqrt{16\cdot 2}$ |
| Rewrite the radical as the product of two radicals. | $3+\sqrt{16}\cdot\sqrt{2}$ |
| Simplify. | $3+4\sqrt{2}$ |

The terms cannot be added as one has a radical and the other does not. Trying to add an integer and a radical is like trying to add an integer and a variable. They are not like terms!

ⓑ

| | |
|---|---|
| | $\dfrac{4-\sqrt{48}}{2}$ |
| Rewrite the radicand as a product using the largest perfect square factor. | $\dfrac{4-\sqrt{16\cdot 3}}{2}$ |
| Rewrite the radical as the product of two radicals. | $\dfrac{4-\sqrt{16}\cdot\sqrt{3}}{2}$ |
| Simplify. | $\dfrac{4-4\sqrt{3}}{2}$ |
| Factor the common factor from the numerator. | $\dfrac{4(1-\sqrt{3})}{2}$ |
| Remove the common factor, 2, from the numerator and denominator. | $\dfrac{\cancel{2}\cdot 2(1-\sqrt{3})}{\cancel{2}}$ |
| Simplify. | $2(1-\sqrt{3})$ |

**TRY IT : :** 8.37 Simplify: ⓐ $5+\sqrt{75}$ ⓑ $\dfrac{10-\sqrt{75}}{5}$

**TRY IT : :** 8.38 Simplify: ⓐ $2+\sqrt{98}$ ⓑ $\dfrac{6-\sqrt{45}}{3}$

## Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

**EXAMPLE 8.20**

Simplify: ⓐ $\sqrt{\dfrac{45}{80}}$ ⓑ $\sqrt[3]{\dfrac{16}{54}}$ ⓒ $\sqrt[4]{\dfrac{5}{80}}$.

### Solution

ⓐ

| | |
|---|---|
| | $\sqrt{\dfrac{45}{80}}$ |
| Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator. | $\sqrt{\dfrac{5\cdot 9}{5\cdot 16}}$ |
| Simplify the fraction by removing common factors. | $\sqrt{\dfrac{9}{16}}$ |
| Simplify. Note $\left(\dfrac{3}{4}\right)^2=\dfrac{9}{16}$. | $\dfrac{3}{4}$ |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{\frac{16}{54}}$ |
| Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator. | $\sqrt[3]{\frac{2\cdot 8}{2\cdot 27}}$ |
| Simplify the fraction by removing common factors. | $\sqrt[3]{\frac{8}{27}}$ |
| Simplify. Note $\left(\frac{2}{3}\right)^3=\frac{8}{27}$. | $\frac{2}{3}$ |

ⓒ

| | |
|---|---|
| | $\sqrt[4]{\frac{5}{80}}$ |
| Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator. | $\sqrt[4]{\frac{5\cdot 1}{5\cdot 16}}$ |
| Simplify the fraction by removing common factors. | $\sqrt[4]{\frac{1}{16}}$ |
| Simplify. Note $\left(\frac{1}{2}\right)^4=\frac{1}{16}$. | $\frac{1}{2}$ |

**TRY IT : :** 8.39 Simplify: ⓐ $\sqrt{\frac{75}{48}}$ ⓑ $\sqrt[3]{\frac{54}{250}}$ ⓒ $\sqrt[4]{\frac{32}{162}}$.

**TRY IT : :** 8.40 Simplify: ⓐ $\sqrt{\frac{98}{162}}$ ⓑ $\sqrt[3]{\frac{24}{375}}$ ⓒ $\sqrt[4]{\frac{4}{324}}$.

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n}=a^{m-n}, \quad a\neq 0$$

**EXAMPLE 8.21**

Simplify: ⓐ $\sqrt{\frac{m^6}{m^4}}$ ⓑ $\sqrt[3]{\frac{a^8}{a^5}}$ ⓒ $\sqrt[4]{\frac{a^{10}}{a^2}}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt{\frac{m^6}{m^4}}$ |
| Simplify the fraction inside the radical fir t. Divide the like bases by subtracting the exponents. | $\sqrt{m^2}$ |
| Simplify. | $\lvert m\rvert$ |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{\frac{a^8}{a^5}}$ |
| Use the Quotient Property of exponents to simplify the fraction under the radical fir t. | $\sqrt[3]{a^3}$ |
| Simplify. | $a$ |

ⓒ

| | |
|---|---|
| | $\sqrt[4]{\frac{a^{10}}{a^2}}$ |
| Use the Quotient Property of exponents to simplify the fraction under the radical fir t. | $\sqrt[4]{a^8}$ |
| Rewrite the radicand using perfect fourth power factors. | $\sqrt[4]{\left(a^2\right)^4}$ |
| Simplify. | $a^2$ |

**TRY IT : :** 8.41 Simplify: ⓐ $\sqrt{\frac{a^8}{a^6}}$ ⓑ $\sqrt[4]{\frac{x^7}{x^3}}$ ⓒ $\sqrt[4]{\frac{y^{17}}{y^5}}$.

**TRY IT : :** 8.42 Simplify: ⓐ $\sqrt{\frac{x^{14}}{x^{10}}}$ ⓑ $\sqrt[3]{\frac{m^{13}}{m^7}}$ ⓒ $\sqrt[5]{\frac{n^{12}}{n^2}}$.

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$$

We can use a similar property to simplify a root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect power of the index, we simplify the numerator and denominator separately.

### Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

**EXAMPLE 8.22** HOW TO SIMPLIFY THE QUOTIENT OF RADICAL EXPRESSIONS

Simplify: $\sqrt{\frac{27m^3}{196}}$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Simplify the fraction in the radicand, if possible. | $\frac{27m^3}{196}$ cannot be simplified. | $\sqrt{\frac{27m^3}{196}}$ |

| | | |
|---|---|---|
| **Step 2.** Use the Quotient Property to rewrite the radical as the quotient of two radicals. | We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$. | $\frac{\sqrt{27m^3}}{\sqrt{196}}$ |
| **Step 3.** Simplify the radicals in the numerator and the denominator. | $9m^2$ and 196 are perfect squares. | $\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$ <br> $\frac{3m\sqrt{3m}}{14}$ |

**TRY IT : :** 8.43 Simplify: $\sqrt{\frac{24p^3}{49}}$.

**TRY IT : :** 8.44 Simplify: $\sqrt{\frac{48x^5}{100}}$.

**HOW TO : : SIMPLIFY A SQUARE ROOT USING THE QUOTIENT PROPERTY.**

Step 1. Simplify the fraction in the radicand, if possible.
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
Step 3. Simplify the radicals in the numerator and the denominator.

**EXAMPLE 8.23**

Simplify: ⓐ $\sqrt{\frac{45x^5}{y^4}}$ ⓑ $\sqrt[3]{\frac{24x^7}{y^3}}$ ⓒ $\sqrt[4]{\frac{48x^{10}}{y^8}}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt{\frac{45x^5}{y^4}}$ |
| We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property. | $\frac{\sqrt{45x^5}}{\sqrt{y^4}}$ |
| Simplify the radicals in the numerator and the denominator. | $\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$ |
| Simplify. | $\frac{3x^2\sqrt{5x}}{y^2}$ |

ⓑ

$$\sqrt[3]{\frac{24x^7}{y^3}}$$

The fraction in the radicand cannot be simplified. Use  he Quotient Property to write as two radicals.

$$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$$

Rewrite each radicand as a product using perfect cube factors.

$$\frac{\sqrt[3]{8x^6 \cdot 3x}}{\sqrt[3]{y^3}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}}{\sqrt[3]{y^3}}$$

Simplify.

$$\frac{2x^2\sqrt[3]{3x}}{y}$$

ⓒ

$$\sqrt[4]{\frac{48x^{10}}{y^8}}$$

The fraction in the radicand cannot be simplified

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{(2x^2)^4} \cdot \sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$$

Simplify.

$$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$$

**TRY IT : :** 8.45

Simplify: ⓐ $\sqrt{\frac{80m^3}{n^6}}$ ⓑ $\sqrt[3]{\frac{108c^{10}}{d^6}}$ ⓒ $\sqrt[4]{\frac{80x^{10}}{y^4}}$.

**TRY IT : :** 8.46

Simplify: ⓐ $\sqrt{\frac{54u^7}{v^8}}$ ⓑ $\sqrt[3]{\frac{40r^3}{s^6}}$ ⓒ $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$.

Be sure to simplify the fraction in the radicand first, if possible.

**EXAMPLE 8.24**

Simplify: ⓐ $\sqrt{\frac{18p^5q^7}{32pq^2}}$ ⓑ $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$.

## Solution

ⓐ

$$\sqrt{\frac{18p^5q^7}{32pq^2}}$$

| | |
|---|---|
| Simplify the fraction in the radicand, if possible. | $\sqrt{\frac{9p^4q^5}{16}}$ |
| Rewrite using the Quotient Property. | $\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$ |
| Simplify the radicals in the numerator and the denominator. | $\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$ |
| Simplify. | $\frac{3p^2q^2\sqrt{q}}{4}$ |

ⓑ

$$\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$$

| | |
|---|---|
| Simplify the fraction in the radicand, if possible. | $\sqrt[3]{\frac{8x^3y^5}{27}}$ |
| Rewrite using the Quotient Property. | $\frac{\sqrt[3]{8x^3y^5}}{\sqrt[3]{27}}$ |
| Simplify the radicals in the numerator and the denominator. | $\frac{\sqrt[3]{8x^3y^3}\cdot\sqrt[3]{y^2}}{\sqrt[3]{27}}$ |
| Simplify. | $\frac{2xy\sqrt[3]{y^2}}{3}$ |

ⓒ

| | |
|---|---|
| | $\sqrt[4]{\dfrac{5a^8b^6}{80a^3b^2}}$ |
| Simplify the fraction in the radicand, if possible. | $\sqrt[4]{\dfrac{a^5b^4}{16}}$ |
| Rewrite using the Quotient Property. | $\dfrac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$ |
| Simplify the radicals in the numerator and the denominator. | $\dfrac{\sqrt[4]{a^4b^4}\cdot\sqrt[4]{a}}{\sqrt[4]{16}}$ |
| Simplify. | $\dfrac{\lvert ab\rvert\sqrt[4]{a}}{2}$ |

**TRY IT : :** 8.47

Simplify: ⓐ $\sqrt{\dfrac{50x^5y^3}{72x^4y}}$ ⓑ $\sqrt[3]{\dfrac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\dfrac{5a^8b^6}{80a^3b^2}}$.

**TRY IT : :** 8.48

Simplify: ⓐ $\sqrt{\dfrac{48m^7n^2}{100m^5n^8}}$ ⓑ $\sqrt[3]{\dfrac{54x^7y^5}{250x^2y^2}}$ ⓒ $\sqrt[4]{\dfrac{32a^9b^7}{162a^3b^3}}$.

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the Quotient Property again, to combine them into one radical. We will then look to see if we can simplify the expression.

**EXAMPLE 8.25**

Simplify: ⓐ $\dfrac{\sqrt{48a^7}}{\sqrt{3a}}$ ⓑ $\dfrac{\sqrt[3]{-108}}{\sqrt[3]{2}}$ ⓒ $\dfrac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$.

## Solution

ⓐ

| | |
|---|---|
| | $\dfrac{\sqrt{48a^7}}{\sqrt{3a}}$ |
| The denominator cannot be simplified, s use the Quotient Property to write as one radical. | $\sqrt{\dfrac{48a^7}{3a}}$ |
| Simplify the fraction under the radical. | $\sqrt{16a^6}$ |
| Simplify. | $4\lvert a^3\rvert$ |

ⓑ

| | |
|---|---|
| | $\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$ |
| The denominator cannot be simplified, s use the Quotient Property to write as one radical. | $\sqrt[3]{\frac{-108}{2}}$ |
| Simplify the fraction under the radical. | $\sqrt[3]{-54}$ |
| Rewrite the radicand as a product using perfect cube factors. | $\sqrt[3]{(-3)^3 \cdot 2}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt[3]{(-3)^3} \cdot \sqrt[3]{2}$ |
| Simplify. | $-3\sqrt[3]{2}$ |

ⓒ

| | |
|---|---|
| | $\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$ |
| The denominator cannot be simplified, s use the Quotient Property to write as one radical. | $\sqrt[4]{\frac{96x^7}{3x^2}}$ |
| Simplify the fraction under the radical. | $\sqrt[4]{32x^5}$ |
| Rewrite the radicand as a product using perfect fourth power factors. | $\sqrt[4]{16x^4} \cdot \sqrt[4]{2x}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$ |
| Simplify. | $2\|x\|\sqrt[4]{2x}$ |

**TRY IT : :** 8.49

Simplify: ⓐ $\frac{\sqrt{98z^5}}{\sqrt{2z}}$ ⓑ $\frac{\sqrt[3]{-500}}{\sqrt[3]{2}}$ ⓒ $\frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$.

**TRY IT : :** 8.50

Simplify: ⓐ $\frac{\sqrt{128m^9}}{\sqrt{2m}}$ ⓑ $\frac{\sqrt[3]{-192}}{\sqrt[3]{3}}$ ⓒ $\frac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$.

**MEDIA : :**

Access these online resources for additional instruction and practice with simplifying radical expressions.

- **Simplifying Square Root and Cube Root with Variables (https://openstax.org/l/37SimRtwithVar1)**
- **Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents (https://openstax.org/l/37SimRtwithVar2)**
- **Simplifying Cube Roots (https://openstax.org/l/37SimRtwithVar3)**

## 8.2 EXERCISES

### Practice Makes Perfect

#### Use the Product Property to Simplify Radical Expressions

*In the following exercises, use the Product Property to simplify radical expressions.*

**55.** $\sqrt{27}$

**56.** $\sqrt{80}$

**57.** $\sqrt{125}$

**58.** $\sqrt{96}$

**59.** $\sqrt{147}$

**60.** $\sqrt{450}$

**61.** $\sqrt{800}$

**62.** $\sqrt{675}$

**63.** ⓐ $\sqrt[4]{32}$ ⓑ $\sqrt[5]{64}$

**64.** ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

**65.** ⓐ $\sqrt[5]{64}$ ⓑ $\sqrt[3]{256}$

**66.** ⓐ $\sqrt[4]{3125}$ ⓑ $\sqrt[3]{81}$

*In the following exercises, simplify using absolute value signs as needed.*

**67.**
ⓐ $\sqrt{y^{11}}$
ⓑ $\sqrt[3]{r^5}$
ⓒ $\sqrt[4]{s^{10}}$

**68.**
ⓐ $\sqrt{m^{13}}$
ⓑ $\sqrt[5]{u^7}$
ⓒ $\sqrt[6]{v^{11}}$

**69.**
ⓐ $\sqrt{n^{21}}$
ⓑ $\sqrt[3]{q^8}$
ⓒ $\sqrt[8]{n^{10}}$

**70.**
ⓐ $\sqrt{r^{25}}$
ⓑ $\sqrt[5]{p^8}$
ⓒ $\sqrt[4]{m^5}$

**71.**
ⓐ $\sqrt{125r^{13}}$
ⓑ $\sqrt[3]{108x^5}$
ⓒ $\sqrt[4]{48y^6}$

**72.**
ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
ⓒ $\sqrt[6]{128b^7}$

**73.**
ⓐ $\sqrt{242m^{23}}$
ⓑ $\sqrt[4]{405m10}$
ⓒ $\sqrt[5]{160n^8}$

**74.**
ⓐ $\sqrt{175n^{13}}$
ⓑ $\sqrt[5]{512p^5}$
ⓒ $\sqrt[4]{324q^7}$

**75.**
ⓐ $\sqrt{147m^7n^{11}}$
ⓑ $\sqrt[3]{48x^6y^7}$
ⓒ $\sqrt[4]{32x^5y^4}$

**76.**
ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
ⓒ $\sqrt[4]{80x^8y^9}$

**77.**
ⓐ $\sqrt{192q^3r^7}$
ⓑ $\sqrt[3]{54m^9n^{10}}$
ⓒ $\sqrt[4]{81a^9b^8}$

**78.**
ⓐ $\sqrt{150m^9n^3}$
ⓑ $\sqrt[3]{81p^7q^8}$
ⓒ $\sqrt[4]{162c^{11}d^{12}}$

**79.**
ⓐ $\sqrt[3]{-864}$
ⓑ $\sqrt[4]{-256}$

**80.**
ⓐ $\sqrt[5]{-486}$
ⓑ $\sqrt[6]{-64}$

**81.**
ⓐ $\sqrt[5]{-32}$
ⓑ $\sqrt[8]{-1}$

**82.**
ⓐ $\sqrt[3]{-8}$
ⓑ $\sqrt[4]{-16}$

**83.**
ⓐ $5+\sqrt{12}$
ⓑ $\dfrac{10-\sqrt{24}}{2}$

**84.**
ⓐ $8+\sqrt{96}$
ⓑ $\dfrac{8-\sqrt{80}}{4}$

**85.**
ⓐ $1+\sqrt{45}$
ⓑ $\frac{3+\sqrt{90}}{3}$

**86.**
ⓐ $3+\sqrt{125}$
ⓑ $\frac{15+\sqrt{75}}{5}$

## Use the Quotient Property to Simplify Radical Expressions

*In the following exercises, use the Quotient Property to simplify square roots.*

**87.** ⓐ $\sqrt{\frac{45}{80}}$ ⓑ $\sqrt[3]{\frac{8}{27}}$ ⓒ $\sqrt[4]{\frac{1}{81}}$

**88.** ⓐ $\sqrt{\frac{72}{98}}$ ⓑ $\sqrt[3]{\frac{24}{81}}$ ⓒ $\sqrt[4]{\frac{6}{96}}$

**89.** ⓐ $\sqrt{\frac{100}{36}}$ ⓑ $\sqrt[3]{\frac{81}{375}}$ ⓒ $\sqrt[4]{\frac{1}{256}}$

**90.** ⓐ $\sqrt{\frac{121}{16}}$ ⓑ $\sqrt[3]{\frac{16}{250}}$ ⓒ $\sqrt[4]{\frac{32}{162}}$

**91.** ⓐ $\sqrt{\frac{x^{10}}{x^6}}$ ⓑ $\sqrt[3]{\frac{p^{11}}{p^2}}$ ⓒ $\sqrt[4]{\frac{q^{17}}{q^{13}}}$

**92.** ⓐ $\sqrt{\frac{p^{20}}{p^{10}}}$ ⓑ $\sqrt[5]{\frac{d^{12}}{d^7}}$ ⓒ $\sqrt[8]{\frac{m^{12}}{m^4}}$

**93.** ⓐ $\sqrt{\frac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

**94.** ⓐ $\sqrt{\frac{q^8}{q^{14}}}$ ⓑ $\sqrt[3]{\frac{r^{14}}{r^5}}$ ⓒ $\sqrt[4]{\frac{c^{21}}{c^9}}$

**95.** $\sqrt{\frac{96x^7}{121}}$

**96.** $\sqrt{\frac{108y^4}{49}}$

**97.** $\sqrt{\frac{300m^5}{64}}$

**98.** $\sqrt{\frac{125n^7}{169}}$

**99.** $\sqrt{\frac{98r^5}{100}}$

**100.** $\sqrt{\frac{180s^{10}}{144}}$

**101.** $\sqrt{\frac{28q^6}{225}}$

**102.** $\sqrt{\frac{150r^3}{256}}$

**103.**
ⓐ $\sqrt{\frac{75r^9}{s^8}}$
ⓑ $\sqrt[3]{\frac{54a^8}{b^3}}$
ⓒ $\sqrt[4]{\frac{64c^5}{d^4}}$

**104.**
ⓐ $\sqrt{\frac{72x^5}{y^6}}$
ⓑ $\sqrt[5]{\frac{96r^{11}}{s^5}}$
ⓒ $\sqrt[6]{\frac{128u^7}{v^{12}}}$

**105.**
ⓐ $\sqrt{\frac{28p^7}{q^2}}$
ⓑ $\sqrt[3]{\frac{81s^8}{t^3}}$
ⓒ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

**106.**
ⓐ $\sqrt{\frac{45r^3}{s^{10}}}$
ⓑ $\sqrt[3]{\frac{625u^{10}}{v^3}}$
ⓒ $\sqrt[4]{\frac{729c^{21}}{d^8}}$

**107.**
ⓐ $\sqrt{\frac{32x^5y^3}{18x^3y}}$
ⓑ $\sqrt[3]{\frac{5x^6y^9}{40x^5y^3}}$
ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

**108.**
ⓐ $\sqrt{\frac{75r^6s^8}{48rs^4}}$
ⓑ $\sqrt[3]{\frac{24x^8y^4}{81x^2y}}$
ⓒ $\sqrt[4]{\frac{32m^9n^2}{162mn^2}}$

**109.**
ⓐ $\sqrt{\frac{27p^2q}{108p^4q^3}}$
ⓑ $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$
ⓒ $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

**110.**
ⓐ $\sqrt{\frac{50r^5s^2}{128r^2s^6}}$
ⓑ $\sqrt[3]{\frac{24m^9n^7}{375m^4n}}$
ⓒ $\sqrt[4]{\frac{81m^2n^8}{256m^1n^2}}$

**111.**

ⓐ $\frac{\sqrt{45p^9}}{\sqrt{5q^2}}$

ⓑ $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$

ⓒ $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

**112.**

ⓐ $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

ⓑ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

ⓒ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

**113.**

ⓐ $\frac{\sqrt{50m^7}}{\sqrt{2m}}$

ⓑ $\sqrt[3]{\frac{1250}{2}}$

ⓒ $\sqrt[4]{\frac{486y^9}{2y^3}}$

**114.**

ⓐ $\frac{\sqrt{72n^{11}}}{\sqrt{2n}}$

ⓑ $\sqrt[3]{\frac{162}{6}}$

ⓒ $\sqrt[4]{\frac{160r^{10}}{5r^3}}$

## Writing Exercises

**115.** Explain why $\sqrt{x^4} = x^2$. Then explain why $\sqrt{x^{16}} = x^8$.

**116.** Explain why $7 + \sqrt{9}$ is not equal to $\sqrt{7 + 9}$.

**117.** Explain how you know that $\sqrt[5]{x^{10}} = x^2$.

**118.** Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can… | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| use the Product Property to simplify radical expressions. | | | |
| use the Quotient Property to simplify radical expressions. | | | |

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

# 8.3 Simplify Rational Exponents

## Learning Objectives

**By the end of this section, you will be able to:**

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

### Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15} + \frac{5}{12}$.
   If you missed this problem, review **Example 1.28**.
2. Simplify: $\left(4x^2 y^5\right)^3$.
   If you missed this problem, review **Example 5.18**.
3. Simplify: $5^{-3}$.
   If you missed this problem, review **Example 5.14**.

## Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n = a^{m \cdot n}$ when *m* and *n* are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $(8^p)^3 = 8$. We will use the Power Property of Exponents to find the value of *p*.

| | |
|---|---|
| | $(8^p)^3 = 8$ |
| Multiply the exponents on the left. | $8^{3p} = 8$ |
| Write the exponent 1 on the right. | $8^{3p} = 8^1$ |
| Since the bases are the same, the exponents must be equal. | $3p = 1$ |
| Solve for *p*. | $p = \frac{1}{3}$ |

So $\left(8^{\frac{1}{3}}\right)^3 = 8$. But we know also $\left(\sqrt[3]{8}\right)^3 = 8$. Then it must be that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This same logic can be used for any positive integer exponent *n* to show that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

### Rational Exponent $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

**EXAMPLE 8.26**

Write as a radical expression: ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$.

## Solution

We want to write each expression in the form $\sqrt[n]{a}$.

ⓐ

| | |
|---|---|
| | $x^{\frac{1}{2}}$ |
| The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2. | $\sqrt{x}$ |

ⓑ

| | |
|---|---|
| | $y^{\frac{1}{3}}$ |
| The denominator of the exponent is 3, so the index is 3. | $\sqrt[3]{y}$ |

ⓒ

| | |
|---|---|
| | $z^{\frac{1}{4}}$ |
| The denominator of the exponent is 4, so the index is 4. | $\sqrt[4]{z}$ |

**TRY IT : :** 8.51 Write as a radical expression: ⓐ $t^{\frac{1}{2}}$ ⓑ $m^{\frac{1}{3}}$ ⓒ $r^{\frac{1}{4}}$.

**TRY IT : :** 8.52 Write as a radial expression: ⓐ $b^{\frac{1}{6}}$ ⓑ $z^{\frac{1}{5}}$ ⓒ $p^{\frac{1}{4}}$.

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

**EXAMPLE 8.27**

Write with a rational exponent: ⓐ $\sqrt{5y}$ ⓑ $\sqrt[3]{4x}$ ⓒ $3\sqrt[4]{5z}$.

## Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

| | |
|---|---|
| | $\sqrt{5y}$ |
| No index is shown, so it is 2. The denominator of the exponent will be 2. Put parentheses around the entire expression $5y$. | $(5y)^{\frac{1}{2}}$ |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{4x}$ |
| The index is 3, so the denominator of the exponent is 3. Include parentheses $(4x)$. | $(4x)^{\frac{1}{3}}$ |

ⓒ

| | |
|---|---|
| | $3\sqrt[4]{5z}$ |
| The index is 4, so the denominator of the exponent is 4. Put parentheses only around the $5z$ since 3 is not under the radical sign. | $3(5z)^{\frac{1}{4}}$ |

**TRY IT : :** 8.53 Write with a rational exponent: ⓐ $\sqrt{10m}$ ⓑ $\sqrt[5]{3n}$ ⓒ $3\sqrt[4]{6y}$.

**TRY IT : :** 8.54 Write with a rational exponent: ⓐ $\sqrt[7]{3k}$ ⓑ $\sqrt[4]{5j}$ ⓒ $8\sqrt[3]{2a}$.

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

## EXAMPLE 8.28

Simplify: ⓐ $25^{\frac{1}{2}}$ ⓑ $64^{\frac{1}{3}}$ ⓒ $256^{\frac{1}{4}}$.

### Solution

ⓐ

| | |
|---|---|
| | $25^{\frac{1}{2}}$ |
| Rewrite as a square root. | $\sqrt{25}$ |
| Simplify. | $5$ |

ⓑ

| | |
|---|---|
| | $64^{\frac{1}{3}}$ |
| Rewrite as a cube root. | $\sqrt[3]{64}$ |
| Recognize 64 is a perfect cube. | $\sqrt[3]{4^3}$ |
| Simplify. | $4$ |

ⓒ

| | |
|---|---|
| | $256^{\frac{1}{4}}$ |
| Rewrite as a fourth root. | $\sqrt[4]{256}$ |
| Recognize 256 is a perfect fourth power. | $\sqrt[4]{4^4}$ |
| Simplify. | $4$ |

**TRY IT : :** 8.55 Simplify: ⓐ $36^{\frac{1}{2}}$ ⓑ $8^{\frac{1}{3}}$ ⓒ $16^{\frac{1}{4}}$.

**TRY IT : :** 8.56 Simplify: ⓐ $100^{\frac{1}{2}}$ ⓑ $27^{\frac{1}{3}}$ ⓒ $81^{\frac{1}{4}}$.

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

**EXAMPLE 8.29**

Simplify: ⓐ $(-16)^{\frac{1}{4}}$ ⓑ $-16^{\frac{1}{4}}$ ⓒ $(16)^{-\frac{1}{4}}$.

### Solution

ⓐ

| | |
|---|---|
| | $(-16)^{\frac{1}{4}}$ |
| Rewrite as a fourth root. | $\sqrt[4]{-16}$ |
| | $\sqrt[4]{(-2)^4}$ |
| Simplify. | No real solution. |

ⓑ

| | |
|---|---|
| | $-16^{\frac{1}{4}}$ |
| The exponent only applies to the 16. Rewrite as a fouth root. | $-\sqrt[4]{16}$ |
| Rewrite 16 as $2^4$. | $-\sqrt[4]{2^4}$ |
| Simplify. | $-2$ |

ⓒ

| | |
|---|---|
| | $(16)^{-\frac{1}{4}}$ |
| Rewrite using the property $a^{-n} = \frac{1}{a^n}$. | $\frac{1}{(16)^{\frac{1}{4}}}$ |
| Rewrite as a fourth root. | $\frac{1}{\sqrt[4]{16}}$ |
| Rewrite 16 as $2^4$. | $\frac{1}{\sqrt[4]{2^4}}$ |
| Simplify. | $\frac{1}{2}$ |

**TRY IT : :** 8.57 Simplify: ⓐ $(-64)^{-\frac{1}{2}}$ ⓑ $-64^{\frac{1}{2}}$ ⓒ $(64)^{-\frac{1}{2}}$.

**TRY IT : :** 8.58 Simplify: ⓐ $(-256)^{\frac{1}{4}}$ ⓑ $-256^{\frac{1}{4}}$ ⓒ $(256)^{-\frac{1}{4}}$.

## Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and

$(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the following definition.

**Rational Exponent $a^{\frac{m}{n}}$**

For any positive integers *m* and *n*,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

**EXAMPLE 8.30**

Write with a rational exponent: ⓐ $\sqrt{y^3}$ ⓑ $\left(\sqrt[3]{2x}\right)^4$ ⓒ $\sqrt{\left(\frac{3a}{4b}\right)^3}$.

## Solution

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$.

ⓐ

| | |
|---|---|
| | $\sqrt{y^3}$ |
| The numerator of the exponent is the exponent, 3. | |
| The denominator of the exponent is the index of the radical, 2. | $y^{\frac{3}{2}}$ |

ⓑ

| | |
|---|---|
| | $\left(\sqrt[3]{2x}\right)^4$ |
| The numerator of the exponent is the exponent, 4. | |
| The denominator of the exponent is the index of the radical, 3. | $(2x)^{\frac{4}{3}}$ |

ⓒ

| | |
|---|---|
| | $\sqrt{\left(\frac{3a}{4b}\right)^3}$ |
| The numerator of the exponent is the exponent, 3. | |
| The denominator of the exponent is the index of the radical, 2. | $\left(\frac{3a}{4b}\right)^{\frac{3}{2}}$ |

**TRY IT : :** 8.59

Write with a rational exponent: ⓐ $\sqrt{x^5}$ ⓑ $\left(\sqrt[4]{3y}\right)^3$ ⓒ $\sqrt{\left(\frac{2m}{3n}\right)^5}$.

**TRY IT : :** 8.60

Write with a rational exponent: ⓐ $\sqrt[5]{a^2}$ ⓑ $\left(\sqrt[3]{5ab}\right)^5$ ⓒ $\sqrt{\left(\frac{7xy}{z}\right)^3}$.

Remember that $a^{-n} = \frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

**EXAMPLE 8.31**

Simplify: ⓐ $125^{\frac{2}{3}}$ ⓑ $16^{-\frac{3}{2}}$ ⓒ $32^{-\frac{2}{5}}$.

## Solution

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

| | |
|---|---|
| | $125^{\frac{2}{3}}$ |
| The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3. | $\left(\sqrt[3]{125}\right)^2$ |
| Simplify. | $(5)^2$ |
| | $25$ |

ⓑ We will rewrite each expression first using $a^{-n} = \frac{1}{a^n}$ and then change to radical form.

| | |
|---|---|
| | $16^{-\frac{3}{2}}$ |
| Rewrite using $a^{-n} = \frac{1}{a^n}$ | $\frac{1}{16^{\frac{3}{2}}}$ |
| Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2. | $\frac{1}{(\sqrt{16})^3}$ |
| Simplify. | $\frac{1}{4^3}$ |
| | $\frac{1}{64}$ |

ⓒ

| | |
|---|---|
| | $32^{-\frac{2}{5}}$ |
| Rewrite using $a^{-n} = \frac{1}{a^n}$. | $\frac{1}{32^{\frac{2}{5}}}$ |
| Change to radical form. | $\frac{1}{\left(\sqrt[5]{32}\right)^2}$ |
| Rewrite the radicand as a power. | $\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$ |
| Simplify. | $\frac{1}{2^2}$ |
| | $\frac{1}{4}$ |

**TRY IT : :** 8.61

Simplify: ⓐ $27^{\frac{2}{3}}$ ⓑ $81^{-\frac{3}{2}}$ ⓒ $16^{-\frac{3}{4}}$.

**TRY IT : :** 8.62

Simplify: ⓐ $4^{\frac{3}{2}}$ ⓑ $27^{-\frac{2}{3}}$ ⓒ $625^{-\frac{3}{4}}$.

**EXAMPLE 8.32**

Simplify: ⓐ $-25^{\frac{3}{2}}$ ⓑ $-25^{-\frac{3}{2}}$ ⓒ $(-25)^{\frac{3}{2}}$.

### Solution

ⓐ

| | |
|---|---|
| | $-25^{\frac{3}{2}}$ |
| Rewrite in radical form. | $-\left(\sqrt{25}\right)^3$ |
| Simplify the radical. | $-(5)^3$ |
| Simplify. | $-125$ |

ⓑ

| | |
|---|---|
| | $-25^{-\frac{3}{2}}$ |
| Rewrite using $a^{-n} = \frac{1}{a^n}$. | $-\left(\frac{1}{25^{\frac{3}{2}}}\right)$ |
| Rewrite in radical form. | $-\left(\frac{1}{(\sqrt{25})^3}\right)$ |
| Simplify the radical. | $-\left(\frac{1}{(5)^3}\right)$ |
| Simplify. | $-\frac{1}{125}$ |

ⓒ

| | |
|---|---|
| | $(-25)^{\frac{3}{2}}$ |
| Rewrite in radical form. | $(\sqrt{-25})^3$ |
| There is no real number whose square root is $-25$. | Not a real number. |

> **TRY IT : : 8.63** Simplify: ⓐ $-16^{\frac{3}{2}}$ ⓑ $-16^{-\frac{3}{2}}$ ⓒ $(-16)^{-\frac{3}{2}}$.

> **TRY IT : : 8.64** Simplify: ⓐ $-81^{\frac{3}{2}}$ ⓑ $-81^{-\frac{3}{2}}$ ⓒ $(-81)^{-\frac{3}{2}}$.

## Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponenets here to have them for reference as we simplify expressions.

**Properties of Exponents**

If *a* and *b* are real numbers and *m* and *n* are rational numbers, then

| | | | |
|---|---|---|---|
| **Product Property** | $a^m \cdot a^n$ | $=$ | $a^{m+n}$ |
| **Power Property** | $(a^m)^n$ | $=$ | $a^{m \cdot n}$ |
| **Product to a Power** | $(ab)^m$ | $=$ | $a^m b^m$ |
| **Quotient Property** | $\frac{a^m}{a^n}$ | $=$ | $a^{m-n},\ a \neq 0$ |
| **Zero Exponent Definitio** | $a^0$ | $=$ | $1,\ a \neq 0$ |
| **Quotient to a Power Property** | $\left(\frac{a}{b}\right)^m$ | $=$ | $\frac{a^m}{b^m},\ b \neq 0$ |
| **Negative Exponent Property** | $a^{-n}$ | $=$ | $\frac{1}{a^n},\ a \neq 0$ |

We will apply these properties in the next example.

**EXAMPLE 8.33**

Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$ ⓑ $\left(z^9\right)^{\frac{2}{3}}$ ⓒ $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$.

## Solution

ⓐ The Product Property tells us that when we multiply the same base, we add the exponents.

| | |
|---|---|
| | $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$ |
| The bases are the same, so we add the exponents. | $x^{\frac{1}{2}+\frac{5}{6}}$ |
| Add the fractions. | $x^{\frac{8}{6}}$ |
| Simplify the exponent. | $x^{\frac{4}{3}}$ |

ⓑ The Power Property tells us that when we raise a power to a power, we multiply the exponents.

| | |
|---|---|
| | $\left(z^9\right)^{\frac{2}{3}}$ |
| To raise a power to a power, we multiply the exponents. | $z^{9 \cdot \frac{2}{3}}$ |
| Simplify. | $z^6$ |

ⓒ The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

| | |
|---|---|
| | $\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$ |
| | $\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$ |
| To divide with the same base, we subtract the exponents. | $\dfrac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$ |
| Simplify. | $\dfrac{1}{x^{\frac{4}{3}}}$ |

**TRY IT : :** 8.65

Simplify: ⓐ $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$ ⓑ $\left(x^6\right)^{\frac{4}{3}}$ ⓒ $\dfrac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$.

**TRY IT : :** 8.66

Simplify: ⓐ $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$ ⓑ $\left(m^9\right)^{\frac{2}{9}}$ ⓒ $\dfrac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$.

Sometimes we need to use more than one property. In the next example, we will use both the Product to a Power Property and then the Power Property.

**EXAMPLE 8.34**

Simplify: ⓐ $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ ⓑ $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$.

## Solution

ⓐ

| | |
|---|---|
| | $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ |
| First we use the Product to a Power Property. | $(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ |
| Rewrite 27 as a power of 3. | $\left(3^3\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ |
| To raise a power to a power, we multiply the exponents. | $\left(3^2\right)\left(u^{\frac{1}{3}}\right)$ |
| Simplify. | $9u^{\frac{1}{3}}$ |

ⓑ

| | |
|---|---|
| | $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$ |
| First we use the Product to a Power Property. | $\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$ |
| To raise a power to a power, we multiply the exponents. | $mn^{\frac{3}{4}}$ |

**TRY IT : :** 8.67

Simplify: ⓐ $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ ⓑ $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$.

**TRY IT : :** 8.68

Simplify: ⓐ $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$ ⓑ $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$.

We will use both the Product Property and the Quotient Property in the next example.

**EXAMPLE 8.35**

Simplify: ⓐ $\dfrac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ ⓑ $\left(\dfrac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$.

## Solution

ⓐ

$$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents. $\dfrac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$

Use the Quotient Property, subtract the exponents. $x^{\frac{8}{4}}$

Simplify. $x^2$

ⓑ Follow the order of operations to simplify inside the parenthese first.

$$\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$$

Use the Quotient Property, subtract the exponents. $\left(\dfrac{16x^{\frac{6}{3}}}{y^{\frac{6}{6}}}\right)^{\frac{1}{2}}$

Simplify. $\left(\dfrac{16x^2}{y}\right)^{\frac{1}{2}}$

Use the Product to a Power Property, multiply the exponents. $\dfrac{4x}{y^{\frac{1}{2}}}$

**TRY IT : :** 8.69

Simplify: ⓐ $\dfrac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$ ⓑ $\left(\dfrac{25m^{\frac{1}{6}}n^{\frac{11}{6}}}{m^{\frac{2}{3}}n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$.

**TRY IT : :** 8.70

Simplify: ⓐ $\dfrac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$ ⓑ $\left(\dfrac{27x^{\frac{4}{5}}y^{\frac{1}{6}}}{x^{\frac{1}{5}}y^{-\frac{5}{6}}}\right)^{\frac{1}{3}}$.

**MEDIA : :**

Access these online resources for additional instruction and practice with simplifying rational exponents.

- **Review-Rational Exponents (https://openstax.org/l/37RatExpont1)**
- **Using Laws of Exponents on Radicals: Properties of Rational Exponents (https://openstax.org/l/37RatExpont2)**

## 8.3 EXERCISES

### Practice Makes Perfect

#### Simplify expressions with $a^{\frac{1}{n}}$

*In the following exercises, write as a radical expression.*

**119.** ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$

**120.** ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

**121.** ⓐ $u^{\frac{1}{5}}$ ⓑ $v^{\frac{1}{9}}$ ⓒ $w^{\frac{1}{20}}$

**122.** ⓐ $g^{\frac{1}{7}}$ ⓑ $h^{\frac{1}{5}}$ ⓒ $j^{\frac{1}{25}}$

*In the following exercises, write with a rational exponent.*

**123.** ⓐ $\sqrt[7]{x}$ ⓑ $\sqrt[9]{y}$ ⓒ $\sqrt[5]{f}$

**124.** ⓐ $\sqrt[8]{r}$ ⓑ $\sqrt[10]{s}$ ⓒ $\sqrt[4]{t}$

**125.** ⓐ $\sqrt[3]{7c}$ ⓑ $\sqrt[7]{12d}$ ⓒ $2\sqrt[4]{6b}$

**126.** ⓐ $\sqrt[4]{5x}$ ⓑ $\sqrt[8]{9y}$ ⓒ $7\sqrt[5]{3z}$

**127.** ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $4\sqrt[6]{36r}$

**128.** ⓐ $\sqrt[3]{25a}$ ⓑ $\sqrt{3b}$ ⓒ $\sqrt[8]{40c}$

*In the following exercises, simplify.*

**129.**
ⓐ $81^{\frac{1}{2}}$
ⓑ $125^{\frac{1}{3}}$
ⓒ $64^{\frac{1}{2}}$

**130.**
ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
ⓒ $32^{\frac{1}{5}}$

**131.**
ⓐ $16^{\frac{1}{4}}$
ⓑ $16^{\frac{1}{2}}$
ⓒ $625^{\frac{1}{4}}$

**132.**
ⓐ $64^{\frac{1}{3}}$
ⓑ $32^{\frac{1}{5}}$
ⓒ $81^{\frac{1}{4}}$

**133.**
ⓐ $(-216)^{\frac{1}{3}}$
ⓑ $-216^{\frac{1}{3}}$
ⓒ $(216)^{-\frac{1}{3}}$

**134.**
ⓐ $(-1000)^{\frac{1}{3}}$
ⓑ $-1000^{\frac{1}{3}}$
ⓒ $(1000)^{-\frac{1}{3}}$

**135.**
ⓐ $(-81)^{\frac{1}{4}}$
ⓑ $-81^{\frac{1}{4}}$
ⓒ $(81)^{-\frac{1}{4}}$

**136.**
ⓐ $(-49)^{\frac{1}{2}}$
ⓑ $-49^{\frac{1}{2}}$
ⓒ $(49)^{-\frac{1}{2}}$

**137.**
ⓐ $(-36)^{\frac{1}{2}}$
ⓑ $-36^{\frac{1}{2}}$
ⓒ $(36)^{-\frac{1}{2}}$

**138.**
ⓐ $(-16)^{\frac{1}{4}}$
ⓑ $-16^{\frac{1}{4}}$
ⓒ $16^{-\frac{1}{4}}$

**139.**
ⓐ $(-100)^{\frac{1}{2}}$
ⓑ $-100^{\frac{1}{2}}$
ⓒ $(100)^{-\frac{1}{2}}$

**140.**
ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

## Simplify Expressions with $a^{\frac{m}{n}}$

*In the following exercises, write with a rational exponent.*

**141.**
ⓐ $\sqrt{m^5}$
ⓑ $\left(\sqrt[3]{3y}\right)^7$
ⓒ $\sqrt[5]{\left(\frac{4x}{5y}\right)^3}$

**142.**
ⓐ $\sqrt[4]{r^7}$
ⓑ $\left(\sqrt[5]{2pq}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

**143.**
ⓐ $\sqrt[5]{u^2}$
ⓑ $\left(\sqrt[3]{6x}\right)^5$
ⓒ $\sqrt[4]{\left(\frac{18a}{5b}\right)^7}$

**144.**
ⓐ $\sqrt[3]{a}$
ⓑ $\left(\sqrt[4]{21v}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{2xy}{5z}\right)^2}$

*In the following exercises, simplify.*

**145.**
ⓐ $64^{\frac{5}{2}}$
ⓑ $81^{\frac{-3}{2}}$
ⓒ $(-27)^{\frac{2}{3}}$

**146.**
ⓐ $25^{\frac{3}{2}}$
ⓑ $9^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{2}{3}}$

**147.**
ⓐ $32^{\frac{2}{5}}$
ⓑ $27^{-\frac{2}{3}}$
ⓒ $(-25)^{\frac{1}{2}}$

**148.**
ⓐ $100^{\frac{3}{2}}$
ⓑ $49^{-\frac{5}{2}}$
ⓒ $(-100)^{\frac{3}{2}}$

**149.**
ⓐ $-9^{\frac{3}{2}}$
ⓑ $-9^{-\frac{3}{2}}$
ⓒ $(-9)^{\frac{3}{2}}$

**150.**
ⓐ $-64^{\frac{3}{2}}$
ⓑ $-64^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{3}{2}}$

## Use the Laws of Exponents to Simplify Expressions with Rational Exponents

*In the following exercises, simplify.*

**151.**
ⓐ $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$
ⓑ $\left(p^{12}\right)^{\frac{3}{4}}$
ⓒ $\frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

**152.**
ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$
ⓒ $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

**153.**
ⓐ $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$
ⓑ $\left(x^{12}\right)^{\frac{2}{3}}$
ⓒ $\frac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$

**154.**

ⓐ $q^{\frac{2}{3}} \cdot q^{\frac{5}{6}}$

ⓑ $\left(h^6\right)^{\frac{4}{3}}$

ⓒ $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

**155.**

ⓐ $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$

ⓑ $\left(a^{\frac{1}{3}} b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

**156.**

ⓐ $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$

ⓑ $\left(m^{\frac{4}{3}} n^{\frac{1}{2}}\right)^{\frac{3}{4}}$

**157.**

ⓐ $\left(16\,u^{\frac{1}{3}}\right)^{\frac{3}{4}}$

ⓑ $\left(4\,p^{\frac{1}{3}} q^{\frac{1}{2}}\right)^{\frac{3}{2}}$

**158.**

ⓐ $\left(625\,n^{\frac{8}{3}}\right)^{\frac{3}{4}}$

ⓑ $\left(9\,x^{\frac{2}{5}} y^{\frac{3}{5}}\right)^{\frac{5}{2}}$

**159.**

ⓐ $\dfrac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$

ⓑ $\left(\dfrac{36\,s^{\frac{1}{5}} t^{-\frac{3}{2}}}{s^{-\frac{9}{5}} t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$

**160.**

ⓐ $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

ⓑ $\left(\dfrac{27\,b^{\frac{2}{3}} c^{-\frac{5}{2}}}{b^{-\frac{7}{3}} c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

**161.**

ⓐ $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$

ⓑ $\left(\dfrac{8\,x^{\frac{5}{3}} y^{-\frac{1}{2}}}{27\,x^{-\frac{4}{3}} y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$

**162.**

ⓐ $\dfrac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

ⓑ $\left(\dfrac{16\,m^{\frac{1}{5}} n^{\frac{3}{2}}}{81\,m^{\frac{9}{5}} n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

## Writing Exercises

**163.** Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

**164.** Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| simplify expressions with $a^{\frac{1}{n}}$. | | | |
| simplify expressions with $a^{\frac{m}{n}}$. | | | |
| use the Laws of Exponents to simply expressions with rational exponents. | | | |

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

# 8.4 Add, Subtract, and Multiply Radical Expressions

## Learning Objectives

**By the end of this section, you will be able to:**

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

### Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $3x^2 + 9x - 5 - \left(x^2 - 2x + 3\right)$.
   If you missed this problem, review Example 5.5.
2. Simplify: $(2 + a)(4 - a)$.
   If you missed this problem, review Example 5.28.
3. Simplify: $(9 - 5y)^2$.
   If you missed this problem, review Example 5.31.

## Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

### Like Radicals

**Like radicals** are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x + 8x$ is $11x$. Similarly we add $3\sqrt{x} + 8\sqrt{x}$ and the result is $11\sqrt{x}$.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

### EXAMPLE 8.36

Simplify: ⓐ $2\sqrt{2} - 7\sqrt{2}$ ⓑ $5\sqrt[3]{y} + 4\sqrt[3]{y}$ ⓒ $7\sqrt[4]{x} - 2\sqrt[4]{y}$.

#### Solution

ⓐ

| | |
|---|---|
| | $2\sqrt{2} - 7\sqrt{2}$ |
| Since the radicals are like, we subtract the coefficient | $-5\sqrt{2}$ |

ⓑ

| | |
|---|---|
| | $5\sqrt[3]{y} + 4\sqrt[3]{y}$ |
| Since the radicals are like, we add the coefficient | $9\sqrt[3]{y}$ |

ⓒ

$$7\sqrt[4]{x} - 2\sqrt[4]{y}$$

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

**TRY IT : :** 8.71 Simplify: ⓐ $8\sqrt{2} - 9\sqrt{2}$ ⓑ $4\sqrt[3]{x} + 7\sqrt[3]{x}$ ⓒ $3\sqrt[4]{x} - 5\sqrt[4]{y}$.

**TRY IT : :** 8.72 Simplify: ⓐ $5\sqrt{3}-9\sqrt{3}$ ⓑ $5\sqrt[3]{y}+3\sqrt[3]{y}$ ⓒ $5\sqrt[4]{m}-2\sqrt[3]{m}$.

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

**EXAMPLE 8.37**

Simplify: ⓐ $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$ ⓑ $\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$.

## Solution

ⓐ

| | |
|---|---|
| | $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$ |
| Since the radicals are like, we combine them. | $0\sqrt{5n}$ |
| Simplify. | $0$ |

ⓑ

| | |
|---|---|
| | $\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$ |
| Since the radicals are like, we combine them. | $2\sqrt[4]{3xy}$ |

**TRY IT : :** 8.73 Simplify: ⓐ $\sqrt{7x}-7\sqrt{7x}+4\sqrt{7x}$ ⓑ $4\sqrt[4]{5xy}+2\sqrt[4]{5xy}-7\sqrt[4]{5xy}$.

**TRY IT : :** 8.74 Simplify: ⓐ $4\sqrt{3y}-7\sqrt{3y}+2\sqrt{3y}$ ⓑ $6\sqrt[3]{7mn}+\sqrt[3]{7mn}-4\sqrt[3]{7mn}$.

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

**EXAMPLE 8.38**

Simplify: ⓐ $\sqrt{20}+3\sqrt{5}$ ⓑ $\sqrt[3]{24}-\sqrt[3]{375}$ ⓒ $\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$.

## Solution

ⓐ

| | |
|---|---|
| | $\sqrt{20}+3\sqrt{5}$ |
| Simplify the radicals, when possible. | $\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$ |
| | $2\sqrt{5}+3\sqrt{5}$ |
| Combine the like radicals. | $5\sqrt{5}$ |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{24}-\sqrt[3]{375}$ |
| Simplify the radicals. | $\sqrt[3]{8}\cdot\sqrt[3]{3}-\sqrt[3]{125}\cdot\sqrt[3]{3}$ |
| | $2\sqrt[3]{3}-5\sqrt[3]{3}$ |
| Combine the like radicals. | $-3\sqrt[3]{3}$ |

ⓒ

| | |
|---|---|
| | $\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$ |
| Simplify the radicals. | $\frac{1}{2}\sqrt[4]{16}\cdot\sqrt[4]{3}-\frac{2}{3}\sqrt[4]{81}\cdot\sqrt[4]{3}$ |
| | $\frac{1}{2}\cdot 2\cdot\sqrt[4]{3}-\frac{2}{3}\cdot 3\cdot\sqrt[4]{3}$ |
| | $\sqrt[4]{3}-2\sqrt[4]{3}$ |
| Combine the like radicals. | $-\sqrt[4]{3}$ |

**TRY IT : :** 8.75 Simplify: ⓐ $\sqrt{18}+6\sqrt{2}$ ⓑ $6\sqrt[3]{16}-2\sqrt[3]{250}$ ⓒ $\frac{2}{3}\sqrt[3]{81}-\frac{1}{2}\sqrt[3]{24}$.

**TRY IT : :** 8.76 Simplify: ⓐ $\sqrt{27}+4\sqrt{3}$ ⓑ $4\sqrt[3]{5}-7\sqrt[3]{40}$ ⓒ $\frac{1}{2}\sqrt[3]{128}-\frac{5}{3}\sqrt[3]{54}$.

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

**EXAMPLE 8.39**

Simplify: ⓐ $9\sqrt{50m^2}-6\sqrt{48m^2}$ ⓑ $\sqrt[3]{54n^5}-\sqrt[3]{16n^5}$.

**Solution**

ⓐ

| | |
|---|---|
| | $9\sqrt{50m^2}-6\sqrt{48m^2}$ |
| Simplify the radicals. | $9\sqrt{25m^2}\cdot\sqrt{2}-6\sqrt{16m^2}\cdot\sqrt{3}$ |
| | $9\cdot 5m\cdot\sqrt{2}-6\cdot 4m\cdot\sqrt{3}$ |
| | $45m\sqrt{2}-24m\sqrt{3}$ |
| The radicals are not like and so cannot be combined. | |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{54n^5}-\sqrt[3]{16n^5}$ |
| Simplify the radicals. | $\sqrt[3]{27n^3}\cdot\sqrt[3]{2n^2}-\sqrt[3]{8n^3}\cdot\sqrt[3]{2n^2}$ |
| | $3n\sqrt[3]{2n^2}-2n\sqrt[3]{2n^2}$ |
| Combine the like radicals. | $n\sqrt[3]{2n^2}$ |

**TRY IT : :** 8.77 Simplify: ⓐ $\sqrt{32m^7}-\sqrt{50m^7}$ ⓑ $\sqrt[3]{135x^7}-\sqrt[3]{40x^7}$.

**TRY IT : :** 8.78 Simplify: ⓐ $\sqrt{27p^3}-\sqrt{48p^3}$ ⓑ $\sqrt[3]{256y^5}-\sqrt[3]{32n^5}$.

## Multiply Radical Expressions

We have used the Product Property of Roots to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots ‘in reverse’ to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

**Product Property of Roots**

For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is 12*xy*. Keep this in mind as you do these examples.

**EXAMPLE 8.40**

Simplify: ⓐ $(6\sqrt{2})(3\sqrt{10})$ ⓑ $\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$.

**Solution**

ⓐ

| | |
|---|---|
| | $(6\sqrt{2})(3\sqrt{10})$ |
| Multiply using the Product Property. | $18\sqrt{20}$ |
| Simplify the radical. | $18\sqrt{4} \cdot \sqrt{5}$ |
| Simplify. | $18 \cdot 2 \cdot \sqrt{5}$ |
| | $36\sqrt{5}$ |

ⓑ

| | |
|---|---|
| | $\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$ |
| Multiply using the Product Property. | $20\sqrt[3]{24}$ |
| Simplify the radical. | $20\sqrt[3]{8} \cdot \sqrt[3]{3}$ |
| Simplify. | $20 \cdot 2 \cdot \sqrt[3]{3}$ |
| | $40\sqrt[3]{3}$ |

**TRY IT : :** 8.79 Simplify: ⓐ $(3\sqrt{2})(2\sqrt{30})$ ⓑ $\left(2\sqrt[3]{18}\right)\left(-3\sqrt[3]{6}\right)$.

**TRY IT : :** 8.80 Simplify: ⓐ $(3\sqrt{3})(3\sqrt{6})$ ⓑ $\left(-4\sqrt[3]{9}\right)\left(3\sqrt[3]{6}\right)$.

We follow the same procedures when there are variables in the radicands.

**EXAMPLE 8.41**

Simplify: ⓐ $\left(10\sqrt{6p^3}\right)(4\sqrt{3p})$ ⓑ $\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$.

**Solution**

ⓐ

| | |
|---|---|
| | $\left(10\sqrt{6p^3}\right)(4\sqrt{3p})$ |
| Multiply. | $40\sqrt{18p^4}$ |
| Simplify the radical. | $40\sqrt{9p^4}\cdot\sqrt{2}$ |
| Simplify. | $40\cdot 3p^2\cdot\sqrt{3}$ |
| | $120p^2\sqrt{3}$ |

ⓑ When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

| | |
|---|---|
| | $\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$ |
| Multiply. | $6\sqrt[4]{4\cdot 5\cdot 4\cdot 7y^5}$ |
| Simplify the radical. | $6\sqrt[4]{16y^4}\cdot\sqrt[4]{35y}$ |
| Simplify. | $6\cdot 2y\sqrt[4]{35y}$ |
| Multiply. | $12y\sqrt[4]{35y}$ |

**TRY IT : :** 8.81 Simplify: ⓐ $\left(6\sqrt{6x^2}\right)\left(8\sqrt{30x^4}\right)$ ⓑ $\left(-4\sqrt[4]{12y^3}\right)\left(-\sqrt[4]{8y^3}\right)$.

**TRY IT : :** 8.82 Simplify: ⓐ $\left(2\sqrt{6y^4}\right)(12\sqrt{30y})$ ⓑ $\left(-4\sqrt[4]{9a^3}\right)\left(3\sqrt[4]{27a^2}\right)$.

## Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the Distributive Property to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

**EXAMPLE 8.42**

Simplify: ⓐ $\sqrt{6}(\sqrt{2}+\sqrt{18})$ ⓑ $\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$.

### Solution

ⓐ

| | |
|---|---|
| | $\sqrt{6}(\sqrt{2}+\sqrt{18})$ |
| Multiply. | $\sqrt{12}+\sqrt{108}$ |
| Simplify. | $\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$ |
| Simplify. | $2\sqrt{3}+6\sqrt{3}$ |
| Combine like radicals. | $8\sqrt{3}$ |

ⓑ

| | |
|---|---|
| | $\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$ |
| Distribute. | $5\sqrt[3]{9}-\sqrt[3]{162}$ |
| Simplify. | $5\sqrt[3]{9}-\sqrt[3]{27}\cdot\sqrt[3]{6}$ |
| Simplify. | $5\sqrt[3]{9}-3\sqrt[3]{6}$ |

> **TRY IT : :** 8.83 Simplify: ⓐ $\sqrt{6}(1+3\sqrt{6})$ ⓑ $\sqrt[3]{4}\left(-2-\sqrt[3]{6}\right)$.

> **TRY IT : :** 8.84 Simplify: ⓐ $\sqrt{8}(2-5\sqrt{8})$ ⓑ $\sqrt[3]{3}\left(-\sqrt[3]{9}-\sqrt[3]{6}\right)$.

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

**EXAMPLE 8.43**

Simplify: ⓐ $(3-2\sqrt{7})(4-2\sqrt{7})$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$.

**Solution**

ⓐ

| | |
|---|---|
| | $(3-2\sqrt{7})(4-2\sqrt{7})$ |
| Multiply | $12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$ |
| Simplify. | $12-6\sqrt{7}-8\sqrt{7}+28$ |
| Combine like terms. | $40-14\sqrt{7}$ |

ⓑ

| | |
|---|---|
| | $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$ |
| Multiply. | $\sqrt[3]{x^2}+4\sqrt[3]{x}-2\sqrt[3]{x}-8$ |
| Combine like terms. | $\sqrt[3]{x^2}+2\sqrt[3]{x}-8$ |

> **TRY IT : :** 8.85 Simplify: ⓐ $(6-3\sqrt{7})(3+4\sqrt{7})$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}-3\right)$.

> **TRY IT : :** 8.86 Simplify: ⓐ $(2-3\sqrt{11})(4-\sqrt{11})$ ⓑ $\left(\sqrt[3]{x}+1\right)\left(\sqrt[3]{x}+3\right)$.

**EXAMPLE 8.44**

Simplify: $\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$.

**Solution**

| | |
|---|---|
| | $\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$ |
| Multiply. | $3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$ |
| Simplify. | $6+12\sqrt{10}-\sqrt{10}-20$ |
| Combine like terms. | $-14+11\sqrt{10}$ |

> **TRY IT : :** 8.87 Simplify: $\left(5\sqrt{3}-\sqrt{7}\right)\left(\sqrt{3}+2\sqrt{7}\right)$

> **TRY IT : :** 8.88 Simplify: $\left(\sqrt{6}-3\sqrt{8}\right)\left(2\sqrt{6}+\sqrt{8}\right)$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

**Special Products**

| Binomial Squares | Product of Conjugates |
|---|---|
| $(a+b)^2 = a^2 + 2ab + b^2$ | $(a+b)(a-b) = a^2 - b^2$ |
| $(a-b)^2 = a^2 - 2ab + b^2$ | |

We will use the special product formulas in the next few examples. We will start with the Product of Binomial Squares Pattern.

**EXAMPLE 8.45**

Simplify: ⓐ $(2+\sqrt{3})^2$ ⓑ $(4-2\sqrt{5})^2$.

**Solution**

Be sure to include the $2ab$ term when squaring a binomial.

ⓐ

| | |
|---|---|
| | $(a+b)^2$ <br> $(2+\sqrt{3})^2$ |
| Multiply, using the Product of Binomial Squares Pattern. | $a^2 + 2\,a\,b + b^2$ <br> $2^2 + 2\cdot 2\cdot\sqrt{3} + (\sqrt{3})^2$ |
| Simplify. | $4 + 4\sqrt{3} + 3$ |
| Combine like terms. | $7 + 4\sqrt{3}$ |

ⓑ

| | |
|---|---|
| | $(a-b)^2$ <br> $(4-2\sqrt{5})^2$ |
| Multiply, using the Product of Binomial Squares Pattern. | $a^2 - 2\,a\,b + b^2$ <br> $4^2 - 2\cdot 4\cdot 2\sqrt{5} + (2\sqrt{5})^2$ |
| Simplify. | $16 - 16\sqrt{5} + 4\cdot 5$ |
| | $16 - 16\sqrt{5} + 20$ |
| Combine like terms. | $36 - 16\sqrt{5}$ |

**TRY IT : :** 8.89 Simplify: ⓐ $(10+\sqrt{2})^2$ ⓑ $(1+3\sqrt{6})^2$.

**TRY IT : :** 8.90 Simplify: ⓐ $(6-\sqrt{5})^2$ ⓑ $(9-2\sqrt{10})^2$.

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

**EXAMPLE 8.46**

Simplify: $(5-2\sqrt{3})(5+2\sqrt{3})$.

## Solution

| | |
|---|---|
| | $(a-b)(a+b)$<br>$(5-2\sqrt{3})(5+2\sqrt{3})$ |
| Multiply, using the Product of Conjugates Pattern. | $a^2-b^2$<br>$5^2-(2\sqrt{3})^2$ |
| Simplify. | $25-4\cdot 3$ |
| | $13$ |

**TRY IT : :** 8.91 Simplify: $(3-2\sqrt{5})(3+2\sqrt{5})$

**TRY IT : :** 8.92 Simplify: $(4+5\sqrt{7})(4-5\sqrt{7})$.

**MEDIA : :**

Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- **Multiplying Adding Subtracting Radicals (https://openstax.org/l/37Radicals1)**
- **Multiplying Special Products: Square Binomials Containing Square Roots (https://openstax.org/l/37Radicals2)**
- **Multiplying Conjugates (https://openstax.org/l/37Radicals3)**

# 8.4 EXERCISES

## Practice Makes Perfect

### Add and Subtract Radical Expressions

*In the following exercises, simplify.*

**165.**
ⓐ $8\sqrt{2} - 5\sqrt{2}$
ⓑ $5\sqrt[3]{m} + 2\sqrt[3]{m}$
ⓒ $8\sqrt[4]{m} - 2\sqrt[4]{n}$

**166.**
ⓐ $7\sqrt{2} - 3\sqrt{2}$
ⓑ $7\sqrt[3]{p} + 2\sqrt[3]{p}$
ⓒ $5\sqrt[3]{x} - 3\sqrt[3]{x}$

**167.**
ⓐ $3\sqrt{5} + 6\sqrt{5}$
ⓑ $9\sqrt[3]{a} + 3\sqrt[3]{a}$
ⓒ $5\sqrt[4]{2z} + \sqrt[4]{2z}$

**168.**
ⓐ $4\sqrt{5} + 8\sqrt{5}$
ⓑ $\sqrt[3]{m} - 4\sqrt[3]{m}$
ⓒ $\sqrt{n} + 3\sqrt{n}$

**169.**
ⓐ $3\sqrt{2a} - 4\sqrt{2a} + 5\sqrt{2a}$
ⓑ $5\sqrt[4]{3ab} - 3\sqrt[4]{3ab} - 2\sqrt[4]{3ab}$

**170.**
ⓐ $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$
ⓑ $8\sqrt[4]{11cd} + 5\sqrt[4]{11cd} - 9\sqrt[4]{11cd}$

**171.**
ⓐ $8\sqrt{3c} + 2\sqrt{3c} - 9\sqrt{3c}$
ⓑ $2\sqrt[3]{4pq} - 5\sqrt[3]{4pq} + 4\sqrt[3]{4pq}$

**172.**
ⓐ $3\sqrt{5d} + 8\sqrt{5d} - 11\sqrt{5d}$
ⓑ $11\sqrt[3]{2rs} - 9\sqrt[3]{2rs} + 3\sqrt[3]{2rs}$

**173.**
ⓐ $\sqrt{27} - \sqrt{75}$
ⓑ $\sqrt[3]{40} - \sqrt[3]{320}$
ⓒ $\frac{1}{2}\sqrt[4]{32} + \frac{2}{3}\sqrt[4]{162}$

**174.**
ⓐ $\sqrt{72} - \sqrt{98}$
ⓑ $\sqrt[3]{24} + \sqrt[3]{81}$
ⓒ $\frac{1}{2}\sqrt[4]{80} - \frac{2}{3}\sqrt[4]{405}$

**175.**
ⓐ $\sqrt{48} + \sqrt{27}$
ⓑ $\sqrt[3]{54} + \sqrt[3]{128}$
ⓒ $6\sqrt[4]{5} - \frac{3}{2}\sqrt[4]{320}$

**176.**
ⓐ $\sqrt{45} + \sqrt{80}$
ⓑ $\sqrt[3]{81} - \sqrt[3]{192}$
ⓒ $\frac{5}{2}\sqrt[4]{80} + \frac{7}{3}\sqrt[4]{405}$

**177.**
ⓐ $\sqrt{72a^5} - \sqrt{50a^5}$
ⓑ $9\sqrt[4]{80p^4} - 6\sqrt[4]{405p^4}$

**178.**
ⓐ $\sqrt{48b^5} - \sqrt{75b^5}$
ⓑ $8\sqrt[3]{64q^6} - 3\sqrt[3]{125q^6}$

**179.**
ⓐ $\sqrt{80c^7} - \sqrt{20c^7}$
ⓑ $2\sqrt[4]{162r^{10}} + 4\sqrt[4]{32r^{10}}$

**180.**
ⓐ $\sqrt{96d^9} - \sqrt{24d^9}$
ⓑ $5\sqrt[4]{243s^6} + 2\sqrt[4]{3s^6}$

**181.**
$3\sqrt{128y^2} + 4y\sqrt{162} - 8\sqrt{98y^2}$

**182.** $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

### Multiply Radical Expressions

*In the following exercises, simplify.*

**183.**
ⓐ $(-2\sqrt{3})(3\sqrt{18})$
ⓑ $\left(8\sqrt[3]{4}\right)\left(-4\sqrt[3]{18}\right)$

**184.**
ⓐ $(-4\sqrt{5})(5\sqrt{10})$
ⓑ $\left(-2\sqrt[3]{9}\right)\left(7\sqrt[3]{9}\right)$

**185.**
ⓐ $(5\sqrt{6})(-\sqrt{12})$
ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

**186.**
ⓐ $(-2\sqrt{7})(-2\sqrt{14})$
ⓑ $\left(-3\sqrt[4]{8}\right)\left(-5\sqrt[4]{6}\right)$

**187.**
ⓐ $\left(4\sqrt{12z^3}\right)\left(3\sqrt{9z}\right)$
ⓑ $\left(5\sqrt[3]{3x^3}\right)\left(3\sqrt[3]{18x^3}\right)$

**188.**
ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$
ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

**189.**
ⓐ $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$
ⓑ $\left(2\sqrt[4]{8y^2}\right)\left(-2\sqrt[4]{12y^3}\right)$

**190.**
ⓐ $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$
ⓑ $\left(-\sqrt[4]{6b^3}\right)\left(3\sqrt[4]{8b^3}\right)$

## Use Polynomial Multiplication to Multiply Radical Expressions

*In the following exercises, multiply.*

**191.**
ⓐ $\sqrt{7}\left(5+2\sqrt{7}\right)$
ⓑ $\sqrt[3]{6}\left(4+\sqrt[3]{18}\right)$

**192.**
ⓐ $\sqrt{11}\left(8+4\sqrt{11}\right)$
ⓑ $\sqrt[3]{3}\left(\sqrt[3]{9}+\sqrt[3]{18}\right)$

**193.**
ⓐ $\sqrt{11}\left(-3+4\sqrt{11}\right)$
ⓑ $\sqrt[4]{3}\left(\sqrt[4]{54}+\sqrt[4]{18}\right)$

**194.**
ⓐ $\sqrt{2}\left(-5+9\sqrt{2}\right)$
ⓑ $\sqrt[4]{2}\left(\sqrt[4]{12}+\sqrt[4]{24}\right)$

**195.** $\left(7+\sqrt{3}\right)\left(9-\sqrt{3}\right)$

**196.** $\left(8-\sqrt{2}\right)\left(3+\sqrt{2}\right)$

**197.**
ⓐ $\left(9-3\sqrt{2}\right)\left(6+4\sqrt{2}\right)$
ⓑ $\left(\sqrt[3]{x}-3\right)\left(\sqrt[3]{x}+1\right)$

**198.**
ⓐ $(3-2\sqrt{7})(5-4\sqrt{7})$
ⓑ $\left(\sqrt[3]{x}-5\right)\left(\sqrt[3]{x}-3\right)$

**199.**
ⓐ $\left(1+3\sqrt{10}\right)\left(5-2\sqrt{10}\right)$
ⓑ $\left(2\sqrt[3]{x}+6\right)\left(\sqrt[3]{x}+1\right)$

**200.**
ⓐ $\left(7-2\sqrt{5}\right)\left(4+9\sqrt{5}\right)$
ⓑ $\left(3\sqrt[3]{x}+2\right)\left(\sqrt[3]{x}-2\right)$

**201.** $\left(\sqrt{3}+\sqrt{10}\right)\left(\sqrt{3}+2\sqrt{10}\right)$

**202.** $\left(\sqrt{11}+\sqrt{5}\right)\left(\sqrt{11}+6\sqrt{5}\right)$

**203.** $\left(2\sqrt{7}-5\sqrt{11}\right)\left(4\sqrt{7}+9\sqrt{11}\right)$

**204.** $\left(4\sqrt{6}+7\sqrt{13}\right)\left(8\sqrt{6}-3\sqrt{13}\right)$

**205.** ⓐ $\left(3+\sqrt{5}\right)^2$ ⓑ $\left(2-5\sqrt{3}\right)^2$

**206.** ⓐ $\left(4+\sqrt{11}\right)^2$ ⓑ $\left(3-2\sqrt{5}\right)^2$

**207.** ⓐ $\left(9-\sqrt{6}\right)^2$ ⓑ $\left(10+3\sqrt{7}\right)^2$

**208.** ⓐ $\left(5-\sqrt{10}\right)^2$ ⓑ $\left(8+3\sqrt{2}\right)^2$

**209.** $\left(4+\sqrt{2}\right)\left(4-\sqrt{2}\right)$

**210.** $\left(7+\sqrt{10}\right)\left(7-\sqrt{10}\right)$

**211.** $\left(4+9\sqrt{3}\right)\left(4-9\sqrt{3}\right)$

**212.** $\left(1+8\sqrt{2}\right)\left(1-8\sqrt{2}\right)$

**213.** $\left(12-5\sqrt{5}\right)\left(12+5\sqrt{5}\right)$

**214.** $\left(9-4\sqrt{3}\right)\left(9+4\sqrt{3}\right)$

**215.** $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

**216.** $\left(\sqrt[3]{4x}+3\right)\left(\sqrt[3]{4x}-3\right)$

## Mixed Practice

**217.** $\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$

**218.** $\sqrt{175k^4}-\sqrt{63k^4}$

**219.** $\frac{5}{6}\sqrt{162}+\frac{3}{16}\sqrt{128}$

**220.** $\sqrt[3]{24}+\sqrt[3]{/81}$

**221.** $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$

**222.** $8\sqrt[4]{13}-4\sqrt[4]{13}-3\sqrt[4]{13}$

**223.** $5\sqrt{12c^4}-3\sqrt{27c^6}$

**224.** $\sqrt{80a^5}-\sqrt{45a^5}$

**225.** $\frac{3}{5}\sqrt{75}-\frac{1}{4}\sqrt{48}$

**226.** $21\sqrt[3]{9}-2\sqrt[3]{9}$

**227.** $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$

**228.** $11\sqrt{11}-10\sqrt{11}$

**229.** $\sqrt{3}\cdot\sqrt{21}$

**230.** $(4\sqrt{6})(-\sqrt{18})$

**231.** $\left(7\sqrt[3]{4}\right)\left(-3\sqrt[3]{18}\right)$

**232.** $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$

**233.** $(\sqrt{29})^2$

**234.** $(-4\sqrt{17})(-3\sqrt{17})$

**235.** $(-4+\sqrt{17})(-3+\sqrt{17})$

**236.** $\left(3\sqrt[4]{8a^2}\right)\left(\sqrt[4]{12a^3}\right)$

**237.** $(6-3\sqrt{2})^2$

**238.** $\sqrt{3}(4-3\sqrt{3})$

**239.** $\sqrt[3]{3}\left(2\sqrt[3]{9}+\sqrt[3]{18}\right)$

**240.** $(\sqrt{6}+\sqrt{3})(\sqrt{6}+6\sqrt{3})$

## Writing Exercises

**241.** Explain the when a radical expression is in simplest form.

**242.** Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

**243.**

ⓐ Explain why $(-\sqrt{n})^2$ is always non-negative, for $n \geq 0$.

ⓑ Explain why $-(\sqrt{n})^2$ is always non-positive, for $n \geq 0$.

**244.** Use the binomial square pattern to simplify $(3+\sqrt{2})^2$. Explain all your steps.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| add and subtract radical expressions. | | | |
| multiply radical expressions. | | | |
| use polynomial multiplication to multiply radical expressions. | | | |

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

# 8.5 Divide Radical Expressions

## Learning Objectives

**By the end of this section, you will be able to:**

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

### Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{30}{48}$.
   If you missed this problem, review **Example 1.24**.
2. Simplify: $x^2 \cdot x^4$.
   If you missed this problem, review **Example 5.12**.
3. Multiply: $(7+3x)(7-3x)$.
   If you missed this problem, review **Example 5.32**.

## Divide Radical Expressions

We have used the Quotient Property of Radical Expressions to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals.

We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

### Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

### EXAMPLE 8.47

Simplify: ⓐ $\frac{\sqrt{72x^3}}{\sqrt{162x}}$ ⓑ $\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$.

### Solution

ⓐ

| | |
|---|---|
| | $\dfrac{\sqrt{72x^3}}{\sqrt{162x}}$ |
| Rewrite using the quotient property, $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$. | $\sqrt{\dfrac{72x^3}{162x}}$ |
| Remove common factors. | $\sqrt{\dfrac{\cancel{18} \cdot 4 \cdot x^2 \cdot \cancel{x}}{\cancel{18} \cdot 9 \cdot \cancel{x}}}$ |
| Simplify. | $\sqrt{\dfrac{4x^2}{9}}$ |
| Simplify the radical. | $\dfrac{2x}{3}$ |

ⓑ

| | |
|---|---|
| | $\dfrac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$ |
| Rewrite using the quotient property, $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$. | $\sqrt[3]{\dfrac{32x^2}{4x^5}}$ |
| Simplify the fraction under the radical. | $\sqrt[3]{\dfrac{8}{x^3}}$ |
| Simplify the radical. | $\dfrac{2}{x}$ |

**TRY IT : :** 8.93

Simplify: ⓐ $\dfrac{\sqrt{50s^3}}{\sqrt{128s}}$ ⓑ $\dfrac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$.

**TRY IT : :** 8.94

Simplify: ⓐ $\dfrac{\sqrt{75q^5}}{\sqrt{108q}}$ ⓑ $\dfrac{\sqrt[3]{72b^2}}{\sqrt[3]{9b^5}}$.

**EXAMPLE 8.48**

Simplify: ⓐ $\dfrac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$ ⓑ $\dfrac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$.

## Solution

ⓐ

| | |
|---|---|
| | $\dfrac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$ |
| Rewrite using the quotient property. | $\sqrt{\dfrac{147ab^8}{3a^3b^4}}$ |
| Remove common factors in the fraction. | $\sqrt{\dfrac{49b^4}{a^2}}$ |
| Simplify the radical. | $\dfrac{7b^2}{a}$ |

ⓑ

| | |
|---|---|
| | $\dfrac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$ |
| Rewrite using the quotient property. | $\sqrt[3]{\dfrac{-250mn^{-2}}{2m^{-2}n^4}}$ |
| Simplify the fraction under the radical. | $\sqrt[3]{\dfrac{-125m^3}{n^6}}$ |
| Simplify the radical. | $-\dfrac{5m}{n^2}$ |

**TRY IT : :** 8.95

Simplify: ⓐ $\dfrac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$ ⓑ $\dfrac{\sqrt[3]{-128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$.

**TRY IT : :** 8.96

Simplify: ⓐ $\dfrac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$ ⓑ $\dfrac{\sqrt[3]{-81pq^{-1}}}{\sqrt[3]{3p^{-2}q^5}}$.

**EXAMPLE 8.49**

Simplify: $\dfrac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$.

**Solution**

| | |
|---|---|
| | $\dfrac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$ |
| Rewrite using the quotient property. | $\sqrt{\dfrac{54x^5y^3}{3x^2y}}$ |
| Remove common factors in the fraction. | $\sqrt{18x^3y^2}$ |
| Rewrite the radicand as a product using the largest perfect square factor. | $\sqrt{9x^2y^2\cdot 2x}$ |
| Rewrite the radical as the product of two radicals. | $\sqrt{9x^2y^2}\cdot\sqrt{2x}$ |
| Simplify. | $3xy\sqrt{2x}$ |

**TRY IT : :** 8.97

Simplify: $\dfrac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$.

**TRY IT : :** 8.98

Simplify: $\dfrac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$.

## Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator.

This process is still used today, and is useful in other areas of mathematics, too.

### Rationalizing the Denominator

**Rationalizing the denominator** is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a radical expression is not considered simplified if the radicand contains a fraction.

### Simplified Radical Expressions

A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

**EXAMPLE 8.50**

Simplify: ⓐ $\frac{4}{\sqrt{3}}$ ⓑ $\sqrt{\frac{3}{20}}$ ⓒ $\frac{3}{\sqrt{6x}}$.

## Solution

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

| | |
|---|---|
| | $\frac{4}{\sqrt{3}}$ |
| Multiply both the numerator and denominator by $\sqrt{3}$. | $\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$ |
| Simplify. | $\frac{4\sqrt{3}}{3}$ |

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

| | |
|---|---|
| | $\sqrt{\frac{3}{20}}$ |
| The fraction is not a perfect square, so rewrite using the Quotient Property. | $\frac{\sqrt{3}}{\sqrt{20}}$ |
| Simplify the denominator. | $\frac{\sqrt{3}}{2\sqrt{5}}$ |
| Multiply the numerator and denominator by $\sqrt{5}$. | $\frac{\sqrt{3} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}}$ |

| | |
|---|---|
| Simplify. | $\frac{\sqrt{15}}{2 \cdot 5}$ |
| Simplify. | $\frac{\sqrt{15}}{10}$ |

ⓒ

| | |
|---|---|
| | $\frac{3}{\sqrt{6x}}$ |
| Multiply the numerator and denominator by $\sqrt{6x}$. | $\frac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$ |
| Simplify. | $\frac{3\sqrt{6x}}{6x}$ |
| Simplify. | $\frac{\sqrt{6x}}{2x}$ |

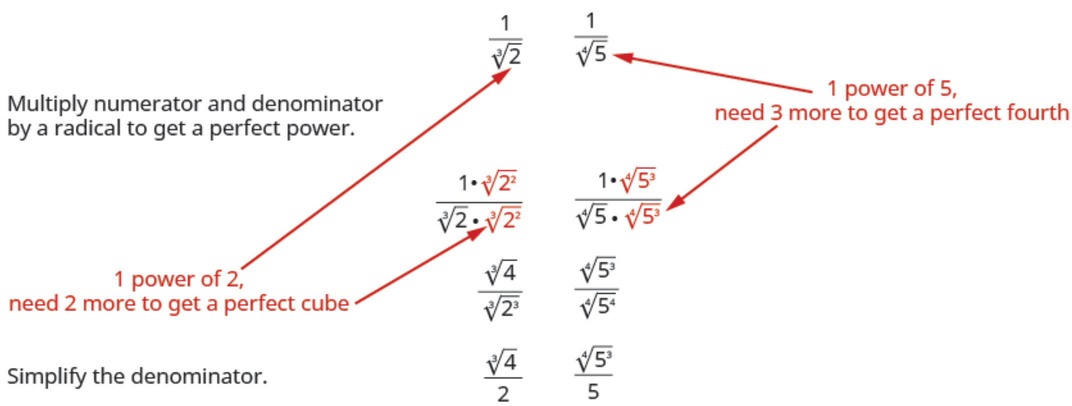
**TRY IT : :** 8.99 Simplify: ⓐ $\frac{5}{\sqrt{3}}$ ⓑ $\sqrt{\frac{3}{32}}$ ⓒ $\frac{2}{\sqrt{2x}}$.

**TRY IT : :** 8.100 Simplify: ⓐ $\frac{6}{\sqrt{5}}$ ⓑ $\sqrt{\frac{7}{18}}$ ⓒ $\frac{5}{\sqrt{5x}}$.

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,




We will use this technique in the next examples.

**EXAMPLE 8.51**

Simplify ⓐ $\frac{1}{\sqrt[3]{6}}$ ⓑ $\sqrt[3]{\frac{7}{24}}$ ⓒ $\frac{3}{\sqrt[3]{4x}}$.

## Solution

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

| | |
|---|---|
| | $\frac{1}{\sqrt[3]{6}}$ |
| The radical in the denominator has one factor of 6. Multiply both the numerator and denominator by $\sqrt[3]{6^2}$, which gives us 2 more factors of 6. | $\frac{1 \cdot \sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \sqrt[3]{6^2}}$ |
| Multiply. Notice the radicand in the denominator has 3 powers of 6. | $\frac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$ |
| Simplify the cube root in the denominator. | $\frac{\sqrt[3]{36}}{6}$ |

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

| | |
|---|---|
| | $\sqrt[3]{\frac{7}{24}}$ |
| The fraction is not a perfect cube, so rewrite using the Quotient Property. | $\frac{\sqrt[3]{7}}{\sqrt[3]{24}}$ |
| Simplify the denominator. | $\frac{\sqrt[3]{7}}{2\sqrt[3]{3}}$ |
| Multiply the numerator and denominator by $\sqrt[3]{3^2}$. This will give us 3 factors of 3. | $\frac{\sqrt[3]{7} \cdot \sqrt[3]{3^2}}{2\sqrt[3]{3} \cdot \sqrt[3]{3^2}}$ |
| Simplify. | $\frac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$ |
| Remember, $\sqrt[3]{3^3} = 3$. | $\frac{\sqrt[3]{63}}{2 \cdot 3}$ |
| Simplify. | $\frac{\sqrt[3]{63}}{6}$ |

ⓒ

| | |
|---|---|
| | $\frac{3}{\sqrt[3]{4x}}$ |
| Rewrite the radicand to show the factors. | $\frac{3}{\sqrt[3]{2^2 \cdot x}}$ |
| Multiply the numerator and denominator by $\sqrt[3]{2 \cdot x^2}$. This will get us 3 factors of 2 and 3 factors of *x*. | $\frac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$ |
| Simplify. | $\frac{3\sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$ |
| Simplify the radical in the denominator. | $\frac{3\sqrt[3]{2x^2}}{2x}$ |

**TRY IT : : 8.101** Simplify: ⓐ $\frac{1}{\sqrt[3]{7}}$ ⓑ $\sqrt[3]{\frac{5}{12}}$ ⓒ $\frac{5}{\sqrt[3]{9y}}$.

**TRY IT : : 8.102** Simplify: ⓐ $\frac{1}{\sqrt[3]{2}}$ ⓑ $\sqrt[3]{\frac{3}{20}}$ ⓒ $\frac{2}{\sqrt[3]{25n}}$.

**EXAMPLE 8.52**

Simplify: ⓐ $\frac{1}{\sqrt[4]{2}}$ ⓑ $\sqrt[4]{\frac{5}{64}}$ ⓒ $\frac{2}{\sqrt[4]{8x}}$.

## Solution

To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

| | |
|---|---|
| | $\frac{1}{\sqrt[4]{2}}$ |
| The radical in the denominator has one factor of 2. Multiply both the numerator and denominator by $\sqrt[4]{2^3}$, which gives us 3 more factors of 2. | $\frac{1 \cdot \sqrt[4]{2^3}}{\sqrt[4]{2} \cdot \sqrt[4]{2^3}}$ |
| Multiply. Notice the radicand in the denominator has 4 powers of 2. | $\frac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$ |
| Simplify the fourth root in the denominator. | $\frac{\sqrt[4]{8}}{2}$ |

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

| | |
|---|---|
| | $\sqrt[4]{\frac{5}{64}}$ |

| | |
|---|---|
| The fraction is not a perfect fourth power, so rewrite using the Quotient Property. | $\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$ |
| Rewrite the radicand in the denominator to show the factors. | $\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$ |
| Simplify the denominator. | $\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$ |
| Multiply the numerator and denominator by $\sqrt[4]{2^2}$. This will give us 4 factors of 2. | $\frac{\sqrt[4]{5} \cdot \sqrt[4]{2^2}}{2\sqrt[4]{2^2} \cdot \sqrt[4]{2^2}}$ |
| Simplify. | $\frac{\sqrt[4]{5} \cdot \sqrt[4]{4}}{2\sqrt[4]{2^4}}$ |
| Remember, $\sqrt[4]{2^4} = 2$. | $\frac{\sqrt[4]{20}}{2 \cdot 2}$ |
| Simplify. | $\frac{\sqrt[4]{20}}{4}$ |

ⓒ

| | |
|---|---|
| | $\frac{2}{\sqrt[4]{8x}}$ |
| Rewrite the radicand to show the factors. | $\frac{2}{\sqrt[4]{2^3 \cdot x}}$ |
| Multiply the numerator and denominator by $\sqrt[4]{2 \cdot x^3}$. This will get us 4 factors of 2 and 4 factors of *x*. | $\frac{2 \cdot \sqrt[4]{2 \cdot x^3}}{\sqrt[4]{2^3x} \cdot \sqrt[4]{2 \cdot x^3}}$ |
| Simplify. | $\frac{2\sqrt[4]{2x^3}}{\sqrt[4]{2^4x^4}}$ |
| Simplify the radical in the denominator. | $\frac{2\sqrt[4]{2x^3}}{2^4x^4}$ |
| Simplify the fraction. | $\frac{\sqrt[4]{2x^3}}{x}$ |

**TRY IT : :** 8.103 Simplify: ⓐ $\frac{1}{\sqrt[4]{3}}$ ⓑ $\sqrt[4]{\frac{3}{64}}$ ⓒ $\frac{3}{\sqrt[4]{125x}}$.

**TRY IT : :** 8.104 Simplify: ⓐ $\frac{1}{\sqrt[4]{5}}$ ⓑ $\sqrt[4]{\frac{7}{128}}$ ⓒ $\frac{4}{\sqrt[4]{4x}}$

## Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.

$$\begin{array}{cc} (a-b)(a+b) & (2-\sqrt{5})(2+\sqrt{5}) \\ a^2-b^2 & 2^2-(\sqrt{5})^2 \\ & 4-5 \\ & -1 \end{array}$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

**EXAMPLE 8.53**

Simplify: $\dfrac{5}{2-\sqrt{3}}$.

**Solution**

| | |
|---|---|
| | $\dfrac{5}{2-\sqrt{3}}$ |
| Multiply the numerator and denominator by the conjugate of the denominator. | $\dfrac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$ |
| Multiply the conjugates in the denominator. | $\dfrac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$ |
| Simplify the denominator. | $\dfrac{5(2+\sqrt{3})}{4-3}$ |
| Simplify the denominator. | $\dfrac{5(2+\sqrt{3})}{1}$ |
| Simplify. | $5(2+\sqrt{3})$ |

**TRY IT : :** 8.105 Simplify: $\dfrac{3}{1-\sqrt{5}}$.

**TRY IT : :** 8.106 Simplify: $\dfrac{2}{4-\sqrt{6}}$.

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

**EXAMPLE 8.54**

Simplify: $\dfrac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

**Solution**

| | |
|---|---|
| | $\dfrac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$ |
| Multiply the numerator and denominator by the conjugate of the denominator. | $\dfrac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$ |
| Multiply the conjugates in the denominator. | $\dfrac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$ |

| | |
|---|---|
| Simplify the denominator. | $\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$ |

**TRY IT : :** 8.107 Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

**TRY IT : :** 8.108 Simplify: $\frac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

## EXAMPLE 8.55

Simplify: $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

### Solution

| | |
|---|---|
| | $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$ |
| Multiply the numerator and denominator by the conjugate of the denominator. | $\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$ |
| Multiply the conjugates in the denominator. | $\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$ |
| Simplify the denominator. | $\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$ |

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

**TRY IT : :** 8.109 Simplify: $\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

**TRY IT : :** 8.110 Simplify: $\frac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$

**MEDIA : :**

Access these online resources for additional instruction and practice with dividing radical expressions.

- **Rationalize the Denominator (https://openstax.org/l/37RatDenom1)**
- **Dividing Radical Expressions and Rationalizing the Denominator (https://openstax.org/l/37RatDenom2)**
- **Simplifying a Radical Expression with a Conjugate (https://openstax.org/l/37RatDenom3)**
- **Rationalize the Denominator of a Radical Expression (https://openstax.org/l/37RatDenom4)**

# 8.5 EXERCISES

## Practice Makes Perfect

### Divide Square Roots

*In the following exercises, simplify.*

**245.** ⓐ $\frac{\sqrt{128}}{\sqrt{72}}$ ⓑ $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$

**246.** ⓐ $\frac{\sqrt{48}}{\sqrt{75}}$ ⓑ $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$

**247.** ⓐ $\frac{\sqrt{200m^5}}{\sqrt{98m}}$ ⓑ $\frac{\sqrt[3]{54y^2}}{\sqrt[3]{2y^5}}$

**248.** ⓐ $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$ ⓑ $\frac{\sqrt[3]{54y}}{\sqrt[3]{16y^4}}$

**249.** ⓐ $\frac{\sqrt{75r^3}}{\sqrt{108r^7}}$ ⓑ $\frac{\sqrt[3]{24x^7}}{\sqrt[3]{81x^4}}$

**250.** ⓐ $\frac{\sqrt{196q}}{\sqrt{484q^5}}$ ⓑ $\frac{\sqrt[3]{16m^4}}{\sqrt[3]{54m}}$

**251.** ⓐ $\frac{\sqrt{108p^5q^2}}{\sqrt{3p^3q^6}}$ ⓑ $\frac{\sqrt[3]{-16a^4b^{-2}}}{\sqrt[3]{2a^{-2}b}}$

**252.** ⓐ $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$ ⓑ $\frac{\sqrt[3]{-375y^4z^{-2}}}{\sqrt[3]{3y^{-2}z^4}}$

**253.** ⓐ $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$ ⓑ $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

**254.** ⓐ $\frac{\sqrt{810c^{-3}d^7}}{\sqrt{1000cd^{-1}}}$ ⓑ $\frac{\sqrt[3]{24a^7b^{-1}}}{\sqrt[3]{-81a^{-2}b^2}}$

**255.** $\frac{\sqrt{56x^5y^4}}{\sqrt{2xy^3}}$

**256.** $\frac{\sqrt{72a^3b^6}}{\sqrt{3ab^3}}$

**257.** $\frac{\sqrt[3]{48a^3b^6}}{\sqrt[3]{3a^{-1}b^3}}$

**258.** $\frac{\sqrt[3]{162x^{-3}y^6}}{\sqrt[3]{2x^3y^{-2}}}$

### Rationalize a One Term Denominator

*In the following exercises, rationalize the denominator.*

**259.** ⓐ $\frac{10}{\sqrt{6}}$ ⓑ $\sqrt{\frac{4}{27}}$ ⓒ $\frac{10}{\sqrt{5x}}$

**260.** ⓐ $\frac{8}{\sqrt{3}}$ ⓑ $\sqrt{\frac{7}{40}}$ ⓒ $\frac{8}{\sqrt{2y}}$

**261.** ⓐ $\frac{6}{\sqrt{7}}$ ⓑ $\sqrt{\frac{8}{45}}$ ⓒ $\frac{12}{\sqrt{3p}}$

**262.** ⓐ $\frac{4}{\sqrt{5}}$ ⓑ $\sqrt{\frac{27}{80}}$ ⓒ $\frac{18}{\sqrt{6q}}$

**263.** ⓐ $\frac{1}{\sqrt[3]{5}}$ ⓑ $\sqrt[3]{\frac{5}{24}}$ ⓒ $\frac{4}{\sqrt[3]{36a}}$

**264.** ⓐ $\frac{1}{\sqrt[3]{3}}$ ⓑ $\sqrt[3]{\frac{5}{32}}$ ⓒ $\frac{7}{\sqrt[3]{49b}}$

**265.** ⓐ $\frac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\frac{7}{54}}$ ⓒ $\frac{3}{\sqrt[3]{3x^2}}$

**266.** ⓐ $\frac{1}{\sqrt[3]{13}}$ ⓑ $\sqrt[3]{\frac{3}{128}}$ ⓒ $\frac{3}{\sqrt[3]{6y^2}}$

**267.** ⓐ $\frac{1}{\sqrt[4]{7}}$ ⓑ $\sqrt[4]{\frac{5}{32}}$ ⓒ $\frac{4}{\sqrt[4]{4x^2}}$

**268.** ⓐ $\frac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\frac{9}{32}}$ ⓒ $\frac{6}{\sqrt[4]{9x^3}}$

**269.** ⓐ $\frac{1}{\sqrt[4]{9}}$ ⓑ $\sqrt[4]{\frac{25}{128}}$ ⓒ $\frac{6}{\sqrt[4]{27a}}$

**270.** ⓐ $\frac{1}{\sqrt[4]{8}}$ ⓑ $\sqrt[4]{\frac{27}{128}}$ ⓒ $\frac{16}{\sqrt[4]{64b^2}}$

### Rationalize a Two Term Denominator

*In the following exercises, simplify.*

**271.** $\frac{8}{1-\sqrt{5}}$

**272.** $\frac{7}{2-\sqrt{6}}$

**273.** $\frac{6}{3-\sqrt{7}}$

**274.** $\frac{5}{4-\sqrt{11}}$

**275.** $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

**276.** $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

**277.** $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

**278.** $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

**279.** $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

**280.** $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

**281.** $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

**282.** $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

## Writing Exercises

**283.**

ⓐ Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

ⓑ Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

ⓒ Why are the two methods of simplifying square roots different?

**284.** Explain what is meant by the word rationalize in the phrase, “rationalize a denominator.”

**285.** Explain why multiplying $\sqrt{2x}-3$ by its conjugate results in an expression with no radicals.

**286.** Explain why multiplying $\frac{7}{\sqrt[3]{x}}$ by $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$ does not rationalize the denominator.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| divide radical expressions. | | | |
| rationalize a one-term denominator. | | | |
| rationalize a two-term denominator. | | | |

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

# 8.6 Solve Radical Equations

## Learning Objectives

**By the end of this section, you will be able to:**

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

### Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
   If you missed this problem, review Example 5.31.
2. Solve: $2x-5=0$.
   If you missed this problem, review Example 2.2.
3. Solve $n^2-6n+8=0$.
   If you missed this problem, review Example 6.45.

## Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

### Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index $n$ to the $n^{\text{th}}$ power. This will eliminate the radical.

$$\text{For } a \geq 0,\ \left(\sqrt[n]{a}\right)^n = a.$$

**EXAMPLE 8.56** HOW TO SOLVE A RADICAL EQUATION

Solve: $\sqrt{5n-4}-9=0$.

### Solution

| | | |
|---|---|---|
| **Step 1.** Isolate the radical on one side of the equation. | To isolate the radical, add 9 to both sides.<br>Simplify. | $\sqrt{5n-4}-9=0$<br>$\sqrt{5n-4}-9+9=0+9$<br>$\sqrt{5n-4}=9$ |
| **Step 2.** Raise both sides of the equation to the power of the index. | Since the index of a square root is 2, we square both sides. | $\left(\sqrt{5n-4}\right)^2=(9)^2$ |
| **Step 3.** Solve the new equation. | Remember, $\left(\sqrt{a}\right)^2=a$. | $5n-4=81$<br>$5n=85$<br>$n=17$ |

| | | |
|---|---|---|
| **Step 4.** Check the answer in the original equation. | | Check the answer.<br>$\sqrt{5n-4}-9=0$<br>$\sqrt{5(17)-4}-9\stackrel{?}{=}0$<br>$\sqrt{85-4}-9\stackrel{?}{=}0$<br>$\sqrt{81}-9\stackrel{?}{=}0$<br>$9-9\stackrel{?}{=}0$<br>$0=0\checkmark$<br>The solution is $n=17$. |

**TRY IT : :** 8.111 Solve: $\sqrt{3m+2}-5=0$.

**TRY IT : :** 8.112 Solve: $\sqrt{10z+1}-2=0$.

**HOW TO : : SOLVE A RADICAL EQUATION WITH ONE RADICAL.**

Step 1. Isolate the radical on one side of the equation.
Step 2. Raise both sides of the equation to the power of the index.
Step 3. Solve the new equation.
Step 4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

**EXAMPLE 8.57**

Solve: $\sqrt{9k-2}+1=0$.

**Solution**

| | |
|---|---|
| | $\sqrt{9k-2}+1=0$ |
| To isolate the radical, subtract 1 to both sides. | $\sqrt{9k-2}+1-1=0-1$ |
| Simplify. | $\sqrt{9k-2}=-1$ |

Because the square root is equal to a negative number, the equation has no solution.

**TRY IT : :** 8.113 Solve: $\sqrt{2r-3}+5=0$.

**TRY IT : :** 8.114 Solve: $\sqrt{7s-3}+2=0$.

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

**Binomial Squares**

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

Don't forget the middle term!

**EXAMPLE 8.58**

Solve: $\sqrt{p-1} + 1 = p$.

**Solution**

| | |
|---|---|
| | $\sqrt{p-1}+1=p$ |
| To isolate the radical, subtract 1 from both sides. | $\sqrt{p-1}+1-1=p-1$ |
| Simplify. | $\sqrt{p-1}=p-1$ |
| Square both sides of the equation. | $\left(\sqrt{p-1}\right)^2=(p-1)^2$ |
| Simplify, using the Product of Binomial Squares Pattern on the right. Then solve the new equation. | $p-1=p^2-2p+1$ |
| It is a quadratic equation, so get zero on one side. | $0=p^2-3p+2$ |
| Factor the right side. | $0=(p-1)(p-2)$ |
| Use the Zero Product Property. | $0=p-1 \quad 0=p-2$ |
| Solve each equation. | $p=1 \quad p=2$ |
| Check the answers. | |
| $p=1$: $\sqrt{p-1}+1=p$; $\sqrt{1-1}+1\overset{?}{=}1$; $\sqrt{0}+1\overset{?}{=}1$; $1=1\checkmark$    $p=2$: $\sqrt{p-1}+1=p$; $\sqrt{2-1}+1\overset{?}{=}2$; $\sqrt{1}+1\overset{?}{=}2$; $2=2\checkmark$ | |
| | The solutions are $p=1, \quad p=2$. |

**TRY IT : :** 8.115 Solve: $\sqrt{x-2}+2=x$.

**TRY IT : :** 8.116 Solve: $\sqrt{y-5}+5=y$.

When the index of the radical is 3, we cube both sides to remove the radical.

$$\left(\sqrt[3]{a}\right)^3 = a$$

**EXAMPLE 8.59**

Solve: $\sqrt[3]{5x+1}+8=4$.

## Solution

| | |
|---|---|
| | $\sqrt[3]{5x+1}+8=4$ |
| To isolate the radical, subtract 8 from both sides. | $\sqrt[3]{5x+1}=-4$ |
| Cube both sides of the equation. | $\left(\sqrt[3]{5x+1}\right)^3=(-4)^3$ |
| Simplify. | $5x+1=-64$ |
| Solve the equation. | $5x=-65$ |
| | $x=-13$ |
| Check the answer. | |
| $x=-13$ $\quad \sqrt[3]{5x+1}+8=4$ <br> $\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$ <br> $\sqrt[3]{-64}+8\stackrel{?}{=}4$ <br> $-4+8\stackrel{?}{=}4$ <br> $4=4$ ✓ | |
| | The solution is $x=-13$. |

**TRY IT : :** 8.117 Solve: $\sqrt[3]{4x-3}+8=5$

**TRY IT : :** 8.118 Solve: $\sqrt[3]{6x-10}+1=-3$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n=a^{m\cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2=x,\quad \left(x^{\frac{1}{3}}\right)^3=x$$

Remember, $x^{\frac{1}{2}}=\sqrt{x}$ and $x^{\frac{1}{3}}=\sqrt[3]{x}$.

### EXAMPLE 8.60

Solve: $(3x-2)^{\frac{1}{4}}+3=5$.

## Solution

$$(3x-2)^{\frac{1}{4}}+3=5$$

| | |
|---|---|
| To isolate the term with the rational exponent, subtract 3 from both sides. | $(3x-2)^{\frac{1}{4}}=2$ |
| Raise each side of the equation to the fourth power. | $\left((3x-2)^{\frac{1}{4}}\right)^4=(2)^4$ |
| Simplify. | $3x-2=16$ |
| Solve the equation. | $3x=18$ |
| | $x=6$ |
| Check the answer. | |
| $x=6 \quad (3x-2)^{\frac{1}{4}}+3=5$<br>$(3\cdot 6-2)^{\frac{1}{4}}+3\stackrel{?}{=}5$<br>$(16)^{\frac{1}{4}}+3\stackrel{?}{=}5$<br>$2+3\stackrel{?}{=}5$<br>$5=5\checkmark$ | |
| | The solution is $x=6$. |

**TRY IT : :** 8.119 Solve: $(9x+9)^{\frac{1}{4}}-2=1$.

**TRY IT : :** 8.120 Solve: $(4x-8)^{\frac{1}{4}}+5=7$.

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an extraneous solution!

**EXAMPLE 8.61**

Solve: $\sqrt{r+4}-r+2=0$.

## Solution

| | |
|---|---|
| | $\sqrt{r+4}-r+2=0$ |
| Isolate the radical. | $\sqrt{r+4}=r-2$ |
| Square both sides of the equation. | $\left(\sqrt{r+4}\right)^2=(r-2)^2$ |
| Simplify and then solve the equation | $r+4=r^2-4r+4$ |
| It is a quadratic equation, so get zero on one side. | $0=r^2-5r$ |
| Factor the right side. | $0=r(r-5)$ |
| Use the Zero Product Property. | $0=r \qquad 0=r-5$ |

| | |
|---|---|
| Solve the equation. | $r=0 \quad r=5$ |
| Check your answer. | |
| $r=0,\ \sqrt{r+4}-r+2=0$<br>$\sqrt{0+4}-0+2 \stackrel{?}{=} 0$<br>$\sqrt{4}+2 \stackrel{?}{=} 0$<br>$4 \neq 0$<br><br>$r=5,\ \sqrt{r+4}-r+2=0$<br>$\sqrt{5+4}-5+2 \stackrel{?}{=} 0$<br>$\sqrt{9}-3 \stackrel{?}{=} 0$<br>$0=0$ ✓ | The solution is $r=5$. |
| | $r=0$ is an extraneous solution. |

> **TRY IT : :** 8.121 Solve: $\sqrt{m+9}-m+3=0$.

> **TRY IT : :** 8.122 Solve: $\sqrt{n+1}-n+1=0$.

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

## EXAMPLE 8.62

Solve: $3\sqrt{3x-5}-8=4$.

### Solution

| | |
|---|---|
| | $3\sqrt{3x-5}-8=4$ |
| Isolate the radical term. | $3\sqrt{3x-5}=12$ |
| Isolate the radical by dividing both sides by 3. | $\sqrt{3x-5}=4$ |
| Square both sides of the equation. | $\left(\sqrt{3x-5}\right)^2=(4)^2$ |
| Simplify, then solve the new equation. | $3x-5=16$ |
| | $3x=21$ |
| Solve the equation. | $x=7$ |
| Check the answer. | |
| $x=7\quad 3\sqrt{3x-5}-8=4$<br>$3\sqrt{3(7)-5}-8 \stackrel{?}{=} 4$<br>$3\sqrt{21-5}-8 \stackrel{?}{=} 4$<br>$3\sqrt{16}-8 \stackrel{?}{=} 4$<br>$3(4)-8 \stackrel{?}{=} 4$<br>$4=4$ ✓ | |
| | The solution is $x=7$. |

> **TRY IT : :** 8.123 Solve: $2\sqrt{4a+4}-16=16$.

**TRY IT : :** 8.124 Solve: $3\sqrt{2b+3}-25=50$.

## Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

**EXAMPLE 8.63**

Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$.

### Solution

| | |
|---|---|
| The radical terms are isolated. | $\sqrt[3]{4x-3} = \sqrt[3]{3x+2}$ |
| Since the index is 3, cube both sides of the equation. | $\left(\sqrt[3]{4x-3}\right)^3 = \left(\sqrt[3]{3x+2}\right)^3$ |
| Simplify, then solve the new equation. | $4x-3 = 3x+2$ |
| | $x-3 = 2$ |
| | $x = 5$ |
| | The solution is $x = 5$. |
| Check the answer. We leave it to you to show that 5 checks! | |

**TRY IT : :** 8.125 Solve: $\sqrt[3]{5x-4}=\sqrt[3]{2x+5}$.

**TRY IT : :** 8.126 Solve: $\sqrt[3]{7x+1}=\sqrt[3]{2x-5}$.

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

**EXAMPLE 8.64** HOW TO SOLVE A RADICAL EQUATION

Solve: $\sqrt{m}+1=\sqrt{m+9}$.

### Solution

| | | |
|---|---|---|
| **Step 1.** Isolate one of the radical terms on one side of the equation. | The radical on the right is isolated. | $\sqrt{m}+1=\sqrt{m+9}$ |
| **Step 2.** Raise both sides of the equation to the power of the index. | We square both sides.<br>Simplify—be very careful as you multiply! | $\left(\sqrt{m}+1\right)^2=\left(\sqrt{m+9}\right)^2$ |
| **Step 3.** Are there any more radicals?<br>If yes, repeat Step 1 and Step 2 again.<br>If no, solve the new equation. | There is still a radical in the equation.<br>So we must repeat the previous steps. Isolate the radical term.<br>Here, we can easily isolate the radical by dividing both sides by 2.<br>Square both sides. | $m+2\sqrt{m}+1=m+9$<br>$2\sqrt{m}=8$<br>$\sqrt{m}=4$<br>$\left(\sqrt{m}\right)^2=(4)^2$<br>$m=16$ |

| | | |
|---|---|---|
| **Step 4.** Check the answer in the original equation. | | $\sqrt{m}+1=\sqrt{m+9}$<br>$\sqrt{16}+1\stackrel{?}{=}\sqrt{16+9}$<br>$4+1\stackrel{?}{=}5$<br>$5=5\checkmark$<br>The solution is $m=16$. |

**TRY IT : :** 8.127 Solve: $3-\sqrt{x}=\sqrt{x-3}$.

**TRY IT : :** 8.128 Solve: $\sqrt{x}+2=\sqrt{x+16}$.

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

**HOW TO : :** SOLVE A RADICAL EQUATION.

Step 1. Isolate one of the radical terms on one side of the equation.
Step 2. Raise both sides of the equation to the power of the index.
Step 3. Are there any more radicals?
If yes, repeat Step 1 and Step 2 again.
If no, solve the new equation.
Step 4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern is $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

**EXAMPLE 8.65**

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

## Solution

| | |
|---|---|
| | $\sqrt{q-2}+3=\sqrt{4q+1}$ |
| The radical on the right is isolated. Square both sides. | $\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$ |
| Simplify. | $q-2+6\sqrt{q-2}+9=4q+1$ |
| There is still a radical in the equation so we must repeat the previous steps. Isolate the radical. | $6\sqrt{q-2}=3q-6$ |
| Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$. | $\left(6\sqrt{q-2}\right)^2=\left(\overset{a-b}{3q-6}\right)^2$<br>$6^2\left(\sqrt{q-2}\right)^2=\overset{a^2}{(3q)^2}\overset{-}{-}\overset{2ab}{2\cdot 3q\cdot 6}\overset{+b^2}{+6^2}$ |
| Simplify, then solve the new equation. | $36(q-2)=9q^2-36q+36$ |

| | |
|---|---|
| Distribute. | $36q-72=9q^2-36q+36$ |
| It is a quadratic equation, so get zero on one side. | $0=9q^2-72q+108$ |
| Factor the right side. | $0=9(q^2-8q+12)$<br>$0=9(q-6)(q-2)$ |
| Use the Zero Product Property. | $q-6=0 \quad q-2=0$<br>$q=6 \quad q=2$ |
| The checks are left to you. | The solutions are $q=6$ and $q=2$. |

**TRY IT : :** 8.129 Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

**TRY IT : :** 8.130 Solve: $\sqrt{x}+2=\sqrt{3x+4}$

## Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

**HOW TO : :** USE A PROBLEM SOLVING STRATEGY FOR APPLICATIONS WITH FORMULAS.

Step 1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve the equation** using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

**Falling Objects**

On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula

$$t=\frac{\sqrt{h}}{4}.$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h=64$ into the formula.

| | |
|---|---|
| | $t = \frac{\sqrt{h}}{4}$ |
| | $t = \frac{\sqrt{64}}{4}$ |
| Take the square root of 64. | $t = \frac{8}{4}$ |
| Simplify the fraction. | $t = 2$ |

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

**EXAMPLE 8.66**

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

## Solution

| | |
|---|---|
| **Step 1. Read** the problem. | |
| **Step 2. Identify** what we are looking for. | the time it takes for the sunglasses to reach the river |
| **Step 3. Name** what we are looking. | Let $t =$ time. |
| **Step 4. Translate** into an equation by writing the appropriate formula. Substitute in the given information. | $t = \frac{\sqrt{h}}{4}$, and $h = 400$<br>$t = \frac{\sqrt{400}}{4}$ |
| **Step 5. Solve the equation.** | $t = \frac{20}{4}$ |
| | $t = 5$ |
| **Step 6. Check** the answer in the problem and make sure it makes sense. | $5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$<br>$5 \stackrel{?}{=} \frac{20}{4}$<br>$5 = 5 \checkmark$ |
| Does 5 seconds seem like a reasonable length of time? | Yes. |
| **Step 7. Answer** the question. | It will take 5 seconds for the sunglasses to reach the river. |

**TRY IT : :** 8.131

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

**TRY IT : :** 8.132

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

**Skid Marks and Speed of a Car**

If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

**EXAMPLE 8.67**

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

**Solution**

| | |
|---|---|
| **Step 1. Read** the problem | |
| **Step 2. Identify** what we are looking for. | the speed of a car |
| **Step 3. Name** what weare looking for, | Let $s =$ the speed. |
| **Step 4. Translate** into an equation by writing the appropriate formula. Substitute in the given information. | $s = \sqrt{24d}$, and $d = 190$<br>$s = \sqrt{24(190)}$ |
| **Step 5. Solve the equation.** | $s = \sqrt{4{,}560}$ |
| | $s = 67.52777...$ |
| Round to 1 decimal place. | $s \approx 67.5$ |
| | $67.5 \overset{?}{\approx} \sqrt{24(190)}$<br>$67.5 \overset{?}{\approx} \sqrt{4560}$<br>$67.5 \approx 67.5277... \checkmark$ |
| | The speed of the car before the brakes were applied was 67.5 miles per hour. |

**TRY IT : :** 8.133

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

**TRY IT : :** 8.134

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

**MEDIA : :**

Access these online resources for additional instruction and practice with solving radical equations.

- **Solving an Equation Involving a Single Radical (https://openstax.org/l/37RadEquat1)**
- **Solving Equations with Radicals and Rational Exponents (https://openstax.org/l/37RadEquat2)**
- **Solving Radical Equations (https://openstax.org/l/37RadEquat3)**
- **Solve Radical Equations (https://openstax.org/l/37RadEquat4)**
- **Radical Equation Application (https://openstax.org/l/37RadEquat5)**

## 8.6 EXERCISES

### Practice Makes Perfect

#### Solve Radical Equations

*In the following exercises, solve.*

**287.** $\sqrt{5x-6}=8$

**288.** $\sqrt{4x-3}=7$

**289.** $\sqrt{5x+1}=-3$

**290.** $\sqrt{3y-4}=-2$

**291.** $\sqrt[3]{2x}=-2$

**292.** $\sqrt[3]{4x-1}=3$

**293.** $\sqrt{2m-3}-5=0$

**294.** $\sqrt{2n-1}-3=0$

**295.** $\sqrt{6v-2}-10=0$

**296.** $\sqrt{12u+1}-11=0$

**297.** $\sqrt{4m+2}+2=6$

**298.** $\sqrt{6n+1}+4=8$

**299.** $\sqrt{2u-3}+2=0$

**300.** $\sqrt{5v-2}+5=0$

**301.** $\sqrt{u-3}-3=u$

**302.** $\sqrt{v-10}+10=v$

**303.** $\sqrt{r-1}=r-1$

**304.** $\sqrt{s-8}=s-8$

**305.** $\sqrt[3]{6x+4}=4$

**306.** $\sqrt[3]{11x+4}=5$

**307.** $\sqrt[3]{4x+5}-2=-5$

**308.** $\sqrt[3]{9x-1}-1=-5$

**309.** $(6x+1)^{\frac{1}{2}}-3=4$

**310.** $(3x-2)^{\frac{1}{2}}+1=6$

**311.** $(8x+5)^{\frac{1}{3}}+2=-1$

**312.** $(12x-5)^{\frac{1}{3}}+8=3$

**313.** $(12x-3)^{\frac{1}{4}}-5=-2$

**314.** $(5x-4)^{\frac{1}{4}}+7=9$

**315.** $\sqrt{x+1}-x+1=0$

**316.** $\sqrt{y+4}-y+2=0$

**317.** $\sqrt{z+100}-z=-10$

**318.** $\sqrt{w+25}-w=-5$

**319.** $3\sqrt{2x-3}-20=7$

**320.** $2\sqrt{5x+1}-8=0$

**321.** $2\sqrt{8r+1}-8=2$

**322.** $3\sqrt{7y+1}-10=8$

#### Solve Radical Equations with Two Radicals

*In the following exercises, solve.*

**323.** $\sqrt{3u+7}=\sqrt{5u+1}$

**324.** $\sqrt{4v+1}=\sqrt{3v+3}$

**325.** $\sqrt{8+2r}=\sqrt{3r+10}$

**326.** $\sqrt{10+2c}=\sqrt{4c+16}$

**327.** $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$

**328.** $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$

**329.** $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

**330.** $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$

**331.** $\sqrt{a}+2=\sqrt{a+4}$

**332.** $\sqrt{r}+6=\sqrt{r+8}$

**333.** $\sqrt{u}+1=\sqrt{u+4}$

**334.** $\sqrt{x}+1=\sqrt{x+2}$

**335.** $\sqrt{a+5}-\sqrt{a}=1$

**336.** $-2=\sqrt{d-20}-\sqrt{d}$

**337.** $\sqrt{2x+1}=1+\sqrt{x}$

**338.** $\sqrt{3x+1}=1+\sqrt{2x-1}$

**339.** $\sqrt{2x-1}-\sqrt{x-1}=1$

**340.** $\sqrt{x+1}-\sqrt{x-2}=1$

**341.** $\sqrt{x+7} - \sqrt{x-5} = 2$

**342.** $\sqrt{x+5} - \sqrt{x-3} = 2$

### Use Radicals in Applications

*In the following exercises, solve. Round approximations to one decimal place.*

**343. Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

**344. Landscaping** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

**345. Gravity** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.

**346. Gravity** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

**347. Accident investigation** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

**348. Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

## Writing Exercises

**349.** Explain why an equation of the form $\sqrt{x} + 1 = 0$ has no solution.

**350.**

ⓐ Solve the equation $\sqrt{r+4} - r + 2 = 0$.

ⓑ Explain why one of the “solutions” that was found was not actually a solution to the equation.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| solve radical equations. | | | |
| solve radical equations with two radicals. | | | |
| use radicals in applications. | | | |

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

# 8.7 Use Radicals in Functions

## Learning Objectives

**By the end of this section, you will be able to:**

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

### Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $1 - 2x \geq 0$.
   If you missed this problem, review **Example 2.50**.
2. For $f(x) = 3x - 4$, evaluate $f(2), f(-1), f(0)$.
   If you missed this problem, review **Example 3.48**.
3. Graph $f(x) = \sqrt{x}$. State the domain and range of the function in interval notation.
   If you missed this problem, review **Example 3.56**.

## Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

The square root function is $f(x) = \sqrt{x}$.

The cube root function is $f(x) = \sqrt[3]{x}$.

### Radical Function

A **radical function** is a function that is defined by a radical expression.

To evaluate a radical function, we find the value of *f*(*x*) for a given value of *x* just as we did in our previous work with functions.

**EXAMPLE 8.68**

For the function $f(x) = \sqrt{2x - 1}$, find ⓐ $f(5)$ ⓑ $f(-2)$.

### Solution

ⓐ

| | |
|---|---|
| | $f(x) = \sqrt{2x - 1}$ |
| To evaluate $f(5)$, substitute 5 for $x$. | $f(5) = \sqrt{2 \cdot 5 - 1}$ |
| Simplify. | $f(5) = \sqrt{9}$ |
| Take the square root. | $f(5) = 3$ |

ⓑ

| | |
|---|---|
| | $f(x) = \sqrt{2x - 1}$ |
| To evaluate $f(-2)$, substitute $-2$ for $x$. | $f(-2) = \sqrt{2(-2) - 1}$ |
| Simplify. | $f(-2) = \sqrt{-5}$ |

Since the square root of a negative number is not a real number, the function does not have a value at $x = -2$.

**TRY IT : :** 8.135 For the function $f(x) = \sqrt{3x - 2}$, find ⓐ $f(6)$ ⓑ $f(0)$.

**TRY IT : :** 8.136 For the function $g(x) = \sqrt{5x + 5}$, find ⓐ $g(4)$ ⓑ $g(-3)$.

We follow the same procedure to evaluate cube roots.

**EXAMPLE 8.69**

For the function $g(x) = \sqrt[3]{x - 6}$, find ⓐ $g(14)$ ⓑ $g(-2)$.

**Solution**

ⓐ

| | |
|---|---|
| | $g(x) = \sqrt[3]{x - 6}$ |
| To evaluate $g(14)$, substitute 14 for $x$. | $g(14) = \sqrt[3]{14 - 6}$ |
| Simplify. | $g(14) = \sqrt[3]{8}$ |
| Take the cube root. | $g(14) = 2$ |

ⓑ

| | |
|---|---|
| | $g(x) = \sqrt[3]{x - 6}$ |
| To evaluate $g(-2)$, substitute $-2$ for $x$. | $g(-2) = \sqrt[3]{-2 - 6}$ |
| Simplify. | $g(-2) = \sqrt[3]{-8}$ |
| Take the cube root. | $g(-2) = -2$ |

**TRY IT : :** 8.137 For the function $g(x) = \sqrt[3]{3x - 4}$, find ⓐ $g(4)$ ⓑ $g(1)$.

**TRY IT : :** 8.138 For the function $h(x) = \sqrt[3]{5x - 2}$, find ⓐ $h(2)$ ⓑ $h(-5)$.

The next example has fourth roots.

**EXAMPLE 8.70**

For the function $f(x) = \sqrt[4]{5x - 4}$, find ⓐ $f(4)$ ⓑ $f(-12)$

**Solution**

ⓐ

| | |
|---|---|
| | $f(x) = \sqrt[4]{5x - 4}$ |
| To evaluate $f(4)$, substitute 4 for $x$. | $f(4) = \sqrt[4]{5 \cdot 4 - 4}$ |
| Simplify. | $f(4) = \sqrt[4]{16}$ |
| Take the fourth root. | $f(4) = 2$ |

ⓑ

| | |
|---|---|
| | $f(x) = \sqrt[4]{5x - 4}$ |
| To evaluate $f(-12)$, substitute $-12$ for $x$. | $f(-12) = \sqrt[4]{5(-12) - 4}$ |
| Simplify. | $f(-12) = \sqrt[4]{-64}$ |

Since the fourth root of a negative number is not a real number, the function does not have a value at $x = -12$.

**TRY IT : :** 8.139 For the function $f(x) = \sqrt[4]{3x+4}$, find ⓐ $f(4)$ ⓑ $f(-1)$.

**TRY IT : :** 8.140 For the function $g(x) = \sqrt[4]{5x+1}$, find ⓐ $g(16)$ ⓑ $g(3)$.

## Find the Domain of a Radical Function

To find the domain and range of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

**Properties of $\sqrt[n]{a}$**

When *n* is an **even** number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When *n* is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of *a*.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

**Domain of a Radical Function**

When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
When the **index** of the radical is **odd**, the radicand can be any real number.

**EXAMPLE 8.71**

Find the domain of the function, $f(x) = \sqrt{3x-4}$. Write the domain in interval notation.

### Solution

Since the function, $f(x) = \sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

$$\begin{array}{lrcl} & 3x-4 & \geq & 0 \\ \text{Solve.} & 3x & \geq & 4 \\ & x & \geq & \frac{4}{3} \end{array}$$

The domain of $f(x) = \sqrt{3x-4}$ is all values $x \geq \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

**TRY IT : :** 8.141 Find the domain of the function, $f(x) = \sqrt{6x-5}$. Write the domain in interval notation.

**TRY IT : :** 8.142 Find the domain of the function, $f(x) = \sqrt{4-5x}$. Write the domain in interval notation.

**EXAMPLE 8.72**

Find the domain of the function, $g(x) = \sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

### Solution

Since the function, $g(x) = \sqrt{\frac{6}{x-1}}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

The radicand cannot be zero since the numerator is not zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive since the numerator is positive. We know a positive divided by a positive is positive.

We set $x - 1 > 0$ and solve.

$$\begin{array}{lrcl} & x-1 & > & 0 \\ \text{Solve.} & x & > & 1 \end{array}$$

Also, since the radicand is a fraction, we must realize that the denominator cannot be zero.

We solve $x - 1 = 0$ to find the value that must be eliminated from the domain.

$$\begin{array}{lrcl} & x-1 & = & 0 \\ \text{Solve.} & x & = & 1 \text{ so } x \neq 1 \text{ in the domain.} \end{array}$$

Putting this together we get the domain is $x > 1$ and we write it as $(1, \infty)$.

> **TRY IT : : 8.143** Find the domain of the function, $f(x) = \sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

> **TRY IT : : 8.144** Find the domain of the function, $h(x) = \sqrt{\frac{9}{x-5}}$. Write the domain in interval notation.

The next example involves a cube root and so will require different thinking.

**EXAMPLE 8.73**

Find the domain of the function, $f(x) = \sqrt[3]{2x^2+3}$. Write the domain in interval notation.

**Solution**

Since the function, $f(x) = \sqrt[3]{2x^2+3}$ has a radical with an index of 3, which is odd, we know the radicand can be any real number. This tells us the domain is any real number. In interval notation, we write $(-\infty, \infty)$.

The domain of $f(x) = \sqrt[3]{2x^2+3}$ is all real numbers and we write it in interval notation as $(-\infty, \infty)$.

> **TRY IT : : 8.145** Find the domain of the function, $f(x) = \sqrt[3]{3x^2-1}$. Write the domain in interval notation.

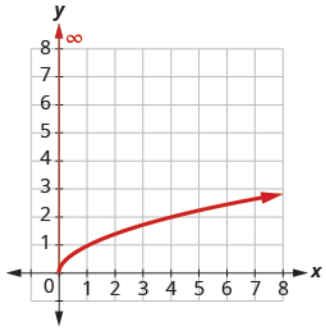
> **TRY IT : : 8.146** Find the domain of the function, $g(x) = \sqrt[3]{5x-4}$. Write the domain in interval notation.

## Graph Radical Functions

Before we graph any radical function, we first find the domain of the function. For the function, $f(x) = \sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x \geq 0$ and we write this in interval notation as $[0, \infty)$.

Previously we used point plotting to graph the function, $f(x) = \sqrt{x}$. We chose *x*-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.




| $x$ | $f(x)=\sqrt{x}$ | $(x, f(x))$ |
|---|---|---|
| 0 | 0 | (0, 0) |
| 1 | 1 | (1, 1) |
| 4 | 2 | (4, 2) |
| 9 | 3 | (9, 3) |

Once we see the graph, we can find the range of the function. The *y*-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

**EXAMPLE 8.74**

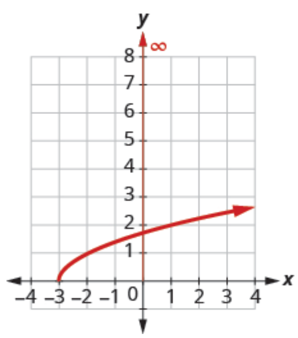
For the function $f(x) = \sqrt{x+3}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

## Solution

ⓐ Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x + 3 \geq 0$, then $x \geq -3$. This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty)$.

ⓑ To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.




| $x$ | $f(x)=\sqrt{x+3}$ | $(x, f(x))$ |
|---|---|---|
| –3 | 0 | (–3, 0) |
| –2 | 1 | (–2, 1) |
| 1 | 2 | (1, 2) |
| 6 | 3 | (6, 3) |

ⓒ Looking at the graph, we see the *y*-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

**TRY IT : :** 8.147

For the function $f(x) = \sqrt{x+2}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

**TRY IT : :** 8.148

For the function $f(x) = \sqrt{x-2}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

In our previous work graphing functions, we graphed $f(x) = x^3$ but we did not graph the function $f(x) = \sqrt[3]{x}$. We will do this now in the next example.

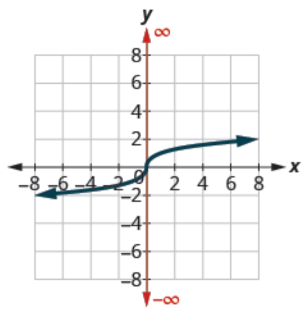
**EXAMPLE 8.75**

For the function $f(x) = \sqrt[3]{x}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

## Solution

ⓐ Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as $(-\infty, \infty)$

ⓑ To graph the function, we choose points in the interval $(-\infty, \infty)$ that will also give us a radicand which will be easy to take the cube root.




| $x$ | $f(x) = \sqrt[3]{x}$ | $(x, f(x))$ |
|---|---|---|
| −8 | −2 | (−8, −2) |
| −1 | −1 | (−1, −1) |
| 0 | 0 | (0, 0) |
| 1 | 1 | (1, 1) |
| 8 | 2 | (8, 2) |

ⓒ Looking at the graph, we see the *y*-values of the function are all real numbers. The range then is $(-\infty, \infty)$.

**TRY IT : :** 8.149

For the function $f(x) = -\sqrt[3]{x}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

**TRY IT : :** 8.150

For the function $f(x) = \sqrt[3]{x-2}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

**MEDIA : :**

Access these online resources for additional instruction and practice with radical functions.

- **Domain of a Radical Function (https://openstax.org/l/37RadFuncDom1)**
- **Domain of a Radical Function 2 (https://openstax.org/l/37RadFuncDom2)**
- **Finding Domain of a Radical Function (https://openstax.org/l/37RadFuncDom3)**

## 8.7 EXERCISES

### Practice Makes Perfect

#### Evaluate a Radical Function

*In the following exercises, evaluate each function.*

**351.** $f(x) = \sqrt{4x-4}$, find

ⓐ $f(5)$

ⓑ $f(0)$.

**352.** $f(x) = \sqrt{6x-5}$, find

ⓐ $f(5)$

ⓑ $f(-1)$.

**353.** $g(x) = \sqrt{6x+1}$, find

ⓐ $g(4)$

ⓑ $g(8)$.

**354.** $g(x) = \sqrt{3x+1}$, find

ⓐ $g(8)$

ⓑ $g(5)$.

**355.** $F(x) = \sqrt{3-2x}$, find

ⓐ $F(1)$

ⓑ $F(-11)$.

**356.** $F(x) = \sqrt{8-4x}$, find

ⓐ $F(1)$

ⓑ $F(-2)$.

**357.** $G(x) = \sqrt{5x-1}$, find

ⓐ $G(5)$

ⓑ $G(2)$.

**358.** $G(x) = \sqrt{4x+1}$, find

ⓐ $G(11)$

ⓑ $G(2)$.

**359.** $g(x) = \sqrt[3]{2x-4}$, find

ⓐ $g(6)$

ⓑ $g(-2)$.

**360.** $g(x) = \sqrt[3]{7x-1}$, find

ⓐ $g(4)$

ⓑ $g(-1)$.

**361.** $h(x) = \sqrt[3]{x^2-4}$, find

ⓐ $h(-2)$

ⓑ $h(6)$.

**362.** $h(x) = \sqrt[3]{x^2+4}$, find

ⓐ $h(-2)$

ⓑ $h(6)$.

**363.** For the function $f(x) = \sqrt[4]{2x^3}$, find

ⓐ $f(0)$

ⓑ $f(2)$.

**364.** For the function $f(x) = \sqrt[4]{3x^3}$, find

ⓐ $f(0)$

ⓑ $f(3)$.

**365.** For the function $g(x) = \sqrt[4]{4-4x}$, find

ⓐ $g(1)$

ⓑ $g(-3)$.

**366.** For the function $g(x) = \sqrt[4]{8-4x}$, find

ⓐ $g(-6)$

ⓑ $g(2)$.

#### Find the Domain of a Radical Function

*In the following exercises, find the domain of the function and write the domain in interval notation.*

**367.** $f(x) = \sqrt{3x-1}$

**368.** $f(x) = \sqrt{4x-2}$

**369.** $g(x) = \sqrt{2-3x}$

**370.** $g(x) = \sqrt{8-x}$

**371.** $h(x) = \sqrt{\dfrac{5}{x-2}}$

**372.** $h(x) = \sqrt{\dfrac{6}{x+3}}$

**373.** $f(x) = \sqrt{\dfrac{x+3}{x-2}}$

**374.** $f(x) = \sqrt{\dfrac{x-1}{x+4}}$

**375.** $g(x) = \sqrt[3]{8x-1}$

**376.** $g(x) = \sqrt[3]{6x+5}$

**377.** $f(x) = \sqrt[3]{4x^2-16}$

**378.** $f(x) = \sqrt[3]{6x^2-25}$

**379.** $F(x) = \sqrt[4]{8x + 3}$

**380.** $F(x) = \sqrt[4]{10 - 7x}$

**381.** $G(x) = \sqrt[5]{2x - 1}$

**382.** $G(x) = \sqrt[5]{6x - 3}$

### Graph Radical Functions

*In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.*

**383.** $f(x) = \sqrt{x + 1}$

**384.** $f(x) = \sqrt{x - 1}$

**385.** $g(x) = \sqrt{x + 4}$

**386.** $g(x) = \sqrt{x - 4}$

**387.** $f(x) = \sqrt{x} + 2$

**388.** $f(x) = \sqrt{x} - 2$

**389.** $g(x) = 2\sqrt{x}$

**390.** $g(x) = 3\sqrt{x}$

**391.** $f(x) = \sqrt{3 - x}$

**392.** $f(x) = \sqrt{4 - x}$

**393.** $g(x) = -\sqrt{x}$

**394.** $g(x) = -\sqrt{x} + 1$

**395.** $f(x) = \sqrt[3]{x + 1}$

**396.** $f(x) = \sqrt[3]{x - 1}$

**397.** $g(x) = \sqrt[3]{x + 2}$

**398.** $g(x) = \sqrt[3]{x - 2}$

**399.** $f(x) = \sqrt[3]{x} + 3$

**400.** $f(x) = \sqrt[3]{x} - 3$

**401.** $g(x) = \sqrt[3]{x}$

**402.** $g(x) = -\sqrt[3]{x}$

**403.** $f(x) = 2\sqrt[3]{x}$

**404.** $f(x) = -2\sqrt[3]{x}$

## Writing Exercises

**405.** Explain how to find the domain of a fourth root function.

**406.** Explain how to find the domain of a fifth root function.

**407.** Explain why $y = \sqrt[3]{x}$ is a function.

**408.** Explain why the process of finding the domain of a radical function with an even index is different from the process when the index is odd.

## Self Check

*ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| evaluate a radical function. | | | |
| find the domain of a radical function. | | | |
| graph a radical function. | | | |

*ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

# 8.8 Use the Complex Number System

## Learning Objectives

**By the end of this section, you will be able to:**

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of $i$

### Be Prepared!

Before you get started, take this readiness quiz.

1. Given the numbers $-4, \ -\sqrt{7}, 0.\overline{5}\,, \frac{7}{3}, 3, \sqrt{81},$ list the ⓐ rational numbers, ⓑ irrational numbers, ⓒ real numbers.
   If you missed this problem, review **Example 1.42**.
2. Multiply: $(x-3)(2x+5)$.
   If you missed this problem, review **Example 5.28**.
3. Rationalize the denominator: $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$.
   If you missed this problem, review **Example 5.32**.

## Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1},$ we are looking for a real number $x$ so that $x^2 = -1$. Since all real numbers squared are positive numbers, there is no real number that equals −1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** $i$ as the number whose square is −1.

### Imaginary Unit

The **imaginary unit** $i$ is the number whose square is −1.

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

We will use the imaginary unit to simplify the square roots of negative numbers.

### Square Root of a Negative Number

If $b$ is a positive real number, then

$$\sqrt{-b} = \sqrt{b}\, i$$

We will use this definition in the next example. Be careful that it is clear that the $i$ is not under the radical. Sometimes you will see this written as $\sqrt{-b} = i\sqrt{b}$ to emphasize the $i$ is not under the radical. But the $\sqrt{-b} = \sqrt{b}\, i$ is considered standard form.

**EXAMPLE 8.76**

Write each expression in terms of $i$ and simplify if possible:

ⓐ $\sqrt{-25}$ ⓑ $\sqrt{-7}$ ⓒ $\sqrt{-12}$.

## Solution

ⓐ

| | | |
|---|---|---|
| | $\sqrt{-25}$ | |
| Use the definition of he square root of negative numbers. | $\sqrt{25}\,i$ | |
| Simplify. | $5i$ | |

ⓑ

| | | |
|---|---|---|
| | $\sqrt{-7}$ | |
| Use the definition of he square root of negative numbers. | $\sqrt{7}i$ | |
| Simplify. | Be careful that it is clear that $i$ is not under the radical sign. | |

ⓒ

| | | |
|---|---|---|
| | $\sqrt{-12}$ | |
| Use the definition of he square root of negative numbers. | $\sqrt{12}\,i$ | |
| Simplify $\sqrt{12}$. | $2\sqrt{3}\,i$ | |

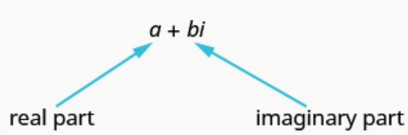
> **TRY IT : :** 8.151 Write each expression in terms of *i* and simplify if possible:
>
> ⓐ $\sqrt{-81}$ ⓑ $\sqrt{-5}$ ⓒ $\sqrt{-18}$.

> **TRY IT : :** 8.152 Write each expression in terms of *i* and simplify if possible:
>
> ⓐ $\sqrt{-36}$ ⓑ $\sqrt{-3}$ ⓒ $\sqrt{-27}$.

Now that we are familiar with the imaginary number *i*, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form *a* + *bi*, where *a, b* are real numbers. We call *a* the real part and *b* the imaginary part.

### Complex Number

A **complex number** is of the form *a* + *bi*, where *a* and *b* are real numbers.




A complex number is in standard form when written as $a + bi$, where *a* and *b* are real numbers.

If $b = 0$, then $a + bi$ becomes $a + 0 \cdot i = a$, and is a real number.

If $b \neq 0$, then $a + bi$ is an imaginary number.

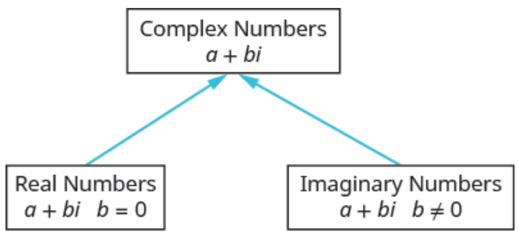
If $a = 0$, then $a + bi$ becomes $0 + bi = bi$, and is called a pure imaginary number.

We summarize this here.

| | $a+bi$ | |
|---|---|---|
| $b=0$ | $a+0\cdot i$<br>$a$ | Real number |
| $b\neq 0$ | $a+bi$ | Imaginary number |
| $a=0$ | $0+bi$<br>$bi$ | Pure imaginary number |

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $\sqrt{-b}=\sqrt{b}i$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.




## Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

**EXAMPLE 8.77**

Add: $\sqrt{-12}+\sqrt{-27}$.

### Solution

| | |
|---|---|
| | $\sqrt{-12}+\sqrt{-27}$ |
| Use the definition of he square root of negative numbers. | $\sqrt{12}\,i+\sqrt{27}\,i$ |
| Simplify the square roots. | $2\sqrt{3}\,i+3\sqrt{3}\,i$ |
| Add. | $5\sqrt{3}\,i$ |

**TRY IT : :** 8.153 Add: $\sqrt{-8}+\sqrt{-32}$.

**TRY IT : :** 8.154 Add: $\sqrt{-27}+\sqrt{-48}$.

Remember to add both the real parts and the imaginary parts in this next example.

**EXAMPLE 8.78**

Simplify: ⓐ $(4-3i)+(5+6i)$ ⓑ $(2-5i)-(5-2i)$.

### Solution

ⓐ

| | |
|---|---|
| | $(4-3i)+(5+6i)$ |
| Use the Associative Property to put the real parts and the imaginary parts together. | $(4+5)+(-3i+6i)$ |
| Simplify. | $9+3i$ |

ⓑ

| | |
|---|---|
| | $(2-5i)-(5-2i)$ |
| Distribute. | $2-5i-5+2i$ |
| Use the Associative Property to put the real parts and the imaginary parts together. | $2-5-5i+2i$ |
| Simplify. | $-3-3i$ |

**TRY IT : :** 8.155 Simplify: ⓐ $(2+7i)+(4-2i)$ ⓑ $(8-4i)-(2-i)$.

**TRY IT : :** 8.156 Simplify: ⓐ $(3-2i)+(-5-4i)$ ⓑ $(4+3i)-(2-6i)$.

## Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

**EXAMPLE 8.79**

Multiply: $2i(7-5i)$.

### Solution

| | |
|---|---|
| | $2i(7-5i)$ |
| Distribute. | $14i-10i^2$ |
| Simplify $i^2$. | $14i-10(-1)$ |
| Multiply. | $14i+10$ |
| Write in standard form. | $10+14i$ |

**TRY IT : :** 8.157 Multiply: $4i(5-3i)$.

**TRY IT : :** 8.158 Multiply: $-3i(2+4i)$.

In the next example, we multiply the binomials using the Distributive Property or FOIL.

**EXAMPLE 8.80**

Multiply: $(3+2i)(4-3i)$.

### Solution

| | |
|---|---|
| | $(3+2i)(4-3i)$ |
| Use FOIL. | $12-9i+8i-6i^2$ |
| Simplify $i^2$ and combine like terms. | $12-i-6(-1)$ |
| Multiply. | $12-i+6$ |
| Combine the real parts. | $18-i$ |

**TRY IT : :** 8.159 Multiply: $(5-3i)(-1-2i)$.

**TRY IT : :** 8.160 Multiply: $(-4-3i)(2+i)$.

In the next example, we could use FOIL or the Product of Binomial Squares Pattern.

**EXAMPLE 8.81**

Multiply: $(3+2i)^2$

**Solution**

| | |
|---|---|
| | $\begin{pmatrix} a+b \\ 3+2i \end{pmatrix}^2$ |
| Use the Product of Binomial Squares Pattern, $(a+b)^2 = a^2 + 2ab + b^2$. | $\overset{a^2}{3^2} + \overset{2}{2} \cdot \overset{a}{3} \cdot \overset{b}{2i} + \overset{b^2}{(2i)^2}$ |
| Simplify. | $9+12i+4i^2$ |
| Simplify $i^2$. | $9+12i+4(-1)$ |
| Simplify. | $5+12i$ |

**TRY IT : :** 8.161 Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

**TRY IT : :** 8.162 Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b} = \sqrt{b}i$. This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

**EXAMPLE 8.82**

Multiply: $\sqrt{-36} \cdot \sqrt{-4}$.

**Solution**

To multiply square roots of negative numbers, we first write them as complex numbers.

| | |
|---|---|
| | $\sqrt{-36} \cdot \sqrt{-4}$ |
| Write as complex numbers using $\sqrt{-b} = \sqrt{b}i$. | $\sqrt{36}\,i \cdot \sqrt{4}\,i$ |
| Simplify. | $6i \cdot 2i$ |
| Multiply. | $12i^2$ |
| Simplify $i^2$ and multiply. | $-12$ |

**TRY IT : :** 8.163 Multiply: $\sqrt{-49} \cdot \sqrt{-4}$.

**TRY IT : :** 8.164 Multiply: $\sqrt{-36} \cdot \sqrt{-81}$.

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

**EXAMPLE 8.83**

Multiply: $\left(3-\sqrt{-12}\right)\left(5+\sqrt{-27}\right)$.

## Solution

To multiply square roots of negative numbers, we first write them as complex numbers.

| | |
|---|---|
| | $\left(3-\sqrt{-12}\right)\left(5+\sqrt{-27}\right)$ |
| Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$. | $\left(3-2\sqrt{3}\,i\right)\left(5+3\sqrt{3}\,i\right)$ |
| Use FOIL. | $15+9\sqrt{3}\,i-10\sqrt{3}\,i-6\cdot 3i^2$ |
| Combine like terms and simplify $i^2$. | $15-\sqrt{3}\,i-6\cdot(-3)$ |
| Multiply and combine like terms. | $33-\sqrt{3}\,i$ |

**TRY IT : :** 8.165 Multiply: $\left(4-\sqrt{-12}\right)\left(3-\sqrt{-48}\right)$.

**TRY IT : :** 8.166 Multiply: $\left(-2+\sqrt{-8}\right)\left(3-\sqrt{-18}\right)$.

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),\ (a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

### Complex Conjugate Pair

A **complex conjugate pair** is of the form $a+bi,\quad a-bi$.

We will multiply a complex conjugate pair in the next example.

**EXAMPLE 8.84**

Multiply: $(3-2i)(3+2i)$.

## Solution

| | |
|---|---|
| | $(3-2i)(3+2i)$ |
| Use FOIL. | $9+6i-6i-4i^2$ |
| Combine like terms and simplify $i^2$. | $9-4(-1)$ |
| Multiply and combine like terms. | $13$ |

**TRY IT : :** 8.167 Multiply: $(4-3i)\cdot(4+3i)$.

**TRY IT : :** 8.168 Multiply: $(-2+5i)\cdot(-2-5i)$.

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$. The result is called a difference of squares. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the Product of Conjugates Pattern.

$$\overset{a\ -\ b}{(3-2i)}\overset{a\ +\ b}{(3+2i)}$$
$$\overset{a^2\ -\ b^2}{(3)^2-(2i)^2}$$
$$9-4i^2$$
$$9-4(-1)$$
$$13$$

Notice this is the same result we found in **Example 8.84**.

When we multiply complex conjugates, the product of the last terms will always have an $i^2$ which simplifies to $-1$.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)(a+bi)=a^2+b^2$

**Product of Complex Conjugates**

If *a* and *b* are real numbers, then

$$(a-bi)(a+bi)=a^2+b^2$$

**EXAMPLE 8.85**

Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

**Solution**

| | |
|---|---|
| | $\overset{(a-b)(a+b)}{(8-2i)(8+2i)}$ |
| Use the Product of Complex Conjugates Pattern, $(a-bi)(a+bi)=a^2+b^2$. | $\overset{a^2+b^2}{8^2+2^2}$ |
| Simplify the squares. | $64+4$ |
| Add. | $68$ |

> **TRY IT : : 8.169** Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

> **TRY IT : : 8.170** Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

## Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

**EXAMPLE 8.86** HOW TO DIVIDE COMPLEX NUMBERS

Divide: $\dfrac{4+3i}{3-4i}$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Write both the numerator and denominator in standard form. | They are both in standard form. | $\frac{4+3i}{3-4i}$ |
| **Step 2.** Multiply the numerator and denominator by the complex conjugate of the denominator. | The complex conjugate of $3-4i$ is $3+4i$. | $\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$ |
| **Step 3.** Simplify and write the result in standard form. | Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator. | $\frac{12+16i+9i+12i^2}{9+16}$ |
| | Combine like terms. | $\frac{12+25i-12}{25}$ |
| | Simplify. | $\frac{25i}{25}$ |
| | Write the result in standard form. | $i$ |

**TRY IT : :** 8.171 Divide: $\frac{2+5i}{5-2i}$.

**TRY IT : :** 8.172 Divide: $\frac{1+6i}{6-i}$.

We summarize the steps here.

**HOW TO : :** HOW TO DIVIDE COMPLEX NUMBERS.

Step 1. Write both the numerator and denominator in standard form.
Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.
Step 3. Simplify and write the result in standard form.

**EXAMPLE 8.87**

Divide, writing the answer in standard form: $\frac{-3}{5+2i}$.

## Solution

| | |
|---|---|
| | $\frac{-3}{5+2i}$ |
| Multiply the numerator and denominator by the complex conjugate of the denominator. | $\frac{-3(5-2i)}{(5+2i)(5-2i)}$ |
| Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator. | $\frac{-15+6i}{5^2+2^2}$ |
| Simplify. | $\frac{-15+6i}{29}$ |
| Write in standard form. | $-\frac{15}{29}+\frac{6}{29}i$ |

**TRY IT : :** 8.173 Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

**TRY IT : :** 8.174 Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Be careful as you find the conjugate of the denominator.

**EXAMPLE 8.88**

Divide: $\frac{5+3i}{4i}$.

**Solution**

| | |
|---|---|
| | $\frac{5+3i}{4i}$ |
| Write the denominator in standard form. | $\frac{5+3i}{0+4i}$ |
| Multiply the numerator and denominator by the complex conjugate of the denominator. | $\frac{(5+3i)(0-4i)}{(0+4i)(0-4i)}$ |
| Simplify. | $\frac{(5+3i)(-4i)}{(4i)(-4i)}$ |
| Multiply. | $\frac{-20i-12i^2}{-16i^2}$ |
| Simplify the $i^2$. | $\frac{-20i+12}{16}$ |
| Rewrite in standard form. | $\frac{12}{16}-\frac{20}{16}i$ |
| Simplify the fractions. | $\frac{3}{4}-\frac{5}{4}i$ |

**TRY IT : :** 8.175 Divide: $\frac{3+3i}{2i}$.

**TRY IT : :** 8.176 Divide: $\frac{2+4i}{5i}$.

## Simplify Powers of *i*

The powers of $i$ make an interesting pattern that will help us simplify higher powers of *i*. Let's evaluate the powers of $i$ to see the pattern.

| | | | |
|---|---|---|---|
| $i^1$ | $i^2$ | $i^3$ | $i^4$ |
| $i$ | $-1$ | $i^2 \cdot i$ | $i^2 \cdot i^2$ |
| | | $-1 \cdot i$ | $(-1)(-1)$ |
| | | $-i$ | $1$ |

| | | | |
|---|---|---|---|
| $i^5$ | $i^6$ | $i^7$ | $i^8$ |
| $i^4 \cdot i$ | $i^4 \cdot i^2$ | $i^4 \cdot i^3$ | $i^4 \cdot i^4$ |
| $1 \cdot i$ | $1 \cdot i^2$ | $1 \cdot i^3$ | $1 \cdot 1$ |
| $i$ | $i^2$ | $i^3$ | $1$ |
| | $-1$ | $-i$ | |

We summarize this now.

$$\begin{aligned} i^1 &= i & i^5 &= i \\ i^2 &= -1 & i^6 &= -1 \\ i^3 &= -i & i^7 &= -i \\ i^4 &= 1 & i^8 &= 1 \end{aligned}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of *i*. Since $i^4 = 1$, we rewrite each power, $i^n$, as a product using $i^4$ to a power and another power of *i*.

We rewrite it in the form $i^n = (i^4)^q \cdot i^r$, where the exponent, *q*, is the quotient of *n* divided by 4 and the exponent, *r*, is the remainder from this division. For example, to simplify $i^{57}$, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, $57 = 4 \cdot 14 + 1$. So we write $i^{57} = (1^4)^{14} \cdot i^1$ and then simplify from there.

$$\begin{array}{r} 14 \\ 4\overline{)57} \\ \underline{4} \\ 17 \\ \underline{16} \\ 1 \end{array} \qquad \begin{array}{c} i^{57} \\ (1^4)^{14} \cdot i^1 \\ 1 \cdot i \\ i \end{array}$$

**EXAMPLE 8.89**

Simplify: $i^{86}$.

**Solution**

| | |
|---|---|
| | $i^{86}$ |
| Divide 86 by 4 and rewrite $i^{86}$ in the $i^n = (i^4)^q \cdot i^r$ form. | $(1^4)^{21} \cdot i^2$ |
| | $\begin{array}{r} 21 \\ 4\overline{)86} \\ \underline{8} \\ 6 \\ \underline{4} \\ 2 \end{array}$ |
| Simplify. | $(1)^{21} \cdot (-1)$ |
| Simplify. | $-1$ |

**TRY IT : :** 8.177 Simplify: $i^{75}$.

**TRY IT : :** 8.178 Simplify: $i^{92}$.

**MEDIA : :**

Access these online resources for additional instruction and practice with the complex number system.

- **Expressing Square Roots of Negative Numbers with i (https://openstax.org/l/37CompNumb1)**
- **Subtract and Multiply Complex Numbers (https://openstax.org/l/37CompNumb2)**
- **Dividing Complex Numbers (https://openstax.org/l/37CompNumb3)**
- **Rewriting Powers of i (https://openstax.org/l/37CompNumb4)**

## 8.8 EXERCISES

### Practice Makes Perfect

#### Evaluate the Square Root of a Negative Number

*In the following exercises, write each expression in terms of i and simplify if possible.*

**409.**
ⓐ $\sqrt{-16}$
ⓑ $\sqrt{-11}$
ⓒ $\sqrt{-8}$

**410.**
ⓐ $\sqrt{-121}$
ⓑ $\sqrt{-1}$
ⓒ $\sqrt{-20}$

**411.**
ⓐ $\sqrt{-100}$
ⓑ $\sqrt{-13}$
ⓒ $\sqrt{-45}$

**412.**
ⓐ $\sqrt{-49}$
ⓑ $\sqrt{-15}$
ⓒ $\sqrt{-75}$

#### Add or Subtract Complex Numbers

*In the following exercises, add or subtract.*

**413.** $\sqrt{-75}+\sqrt{-48}$

**414.** $\sqrt{-12}+\sqrt{-75}$

**415.** $\sqrt{-50}+\sqrt{-18}$

**416.** $\sqrt{-72}+\sqrt{-8}$

**417.** $(1+3i)+(7+4i)$

**418.** $(6+2i)+(3-4i)$

**419.** $(8-i)+(6+3i)$

**420.** $(7-4i)+(-2-6i)$

**421.** $(1-4i)-(3-6i)$

**422.** $(8-4i)-(3+7i)$

**423.** $(6+i)-(-2-4i)$

**424.** $(-2+5i)-(-5+6i)$

**425.** $\left(5-\sqrt{-36}\right)+\left(2-\sqrt{-49}\right)$

**426.** $\left(-3+\sqrt{-64}\right)+\left(5-\sqrt{-16}\right)$

**427.** $\left(-7-\sqrt{-50}\right)-\left(-32-\sqrt{-18}\right)$

**428.** $\left(-5+\sqrt{-27}\right)-\left(-4-\sqrt{-48}\right)$

#### Multiply Complex Numbers

*In the following exercises, multiply.*

**429.** $4i(5-3i)$

**430.** $2i(-3+4i)$

**431.** $-6i(-3-2i)$

**432.** $-i(6+5i)$

**433.** $(4+3i)(-5+6i)$

**434.** $(-2-5i)(-4+3i)$

**435.** $(-3+3i)(-2-7i)$

**436.** $(-6-2i)(-3-5i)$

*In the following exercises, multiply using the Product of Binomial Squares Pattern.*

**437.** $(3+4i)^2$

**438.** $(-1+5i)^2$

**439.** $(-2-3i)^2$

**440.** $(-6-5i)^2$

*In the following exercises, multiply.*

**441.** $\sqrt{-25}\cdot\sqrt{-36}$

**442.** $\sqrt{-4}\cdot\sqrt{-16}$

**443.** $\sqrt{-9}\cdot\sqrt{-100}$

**444.** $\sqrt{-64} \cdot \sqrt{-9}$

**445.** $(-2-\sqrt{-27})(4-\sqrt{-48})$

**446.** $(5-\sqrt{-12})(-3+\sqrt{-75})$

**447.** $(2+\sqrt{-8})(-4+\sqrt{-18})$

**448.** $(5+\sqrt{-18})(-2-\sqrt{-50})$

**449.** $(2-i)(2+i)$

**450.** $(4-5i)(4+5i)$

**451.** $(7-2i)(7+2i)$

**452.** $(-3-8i)(-3+8i)$

*In the following exercises, multiply using the Product of Complex Conjugates Pattern.*

**453.** $(7-i)(7+i)$

**454.** $(6-5i)(6+5i)$

**455.** $(9-2i)(9+2i)$

**456.** $(-3-4i)(-3+4i)$

### Divide Complex Numbers

*In the following exercises, divide.*

**457.** $\frac{3+4i}{4-3i}$

**458.** $\frac{5-2i}{2+5i}$

**459.** $\frac{2+i}{3-4i}$

**460.** $\frac{3-2i}{6+i}$

**461.** $\frac{3}{2-3i}$

**462.** $\frac{2}{4-5i}$

**463.** $\frac{-4}{3-2i}$

**464.** $\frac{-1}{3+2i}$

**465.** $\frac{1+4i}{3i}$

**466.** $\frac{4+3i}{7i}$

**467.** $\frac{-2-3i}{4i}$

**468.** $\frac{-3-5i}{2i}$

### Simplify Powers of *i*

*In the following exercises, simplify.*

**469.** $i^{41}$

**470.** $i^{39}$

**471.** $i^{66}$

**472.** $i^{48}$

**473.** $i^{128}$

**474.** $i^{162}$

**475.** $i^{137}$

**476.** $i^{255}$

## Writing Exercises

**477.** Explain the relationship between real numbers and complex numbers.

**478.** Aniket multiplied as follows and he got the wrong answer. What is wrong with his reasoning?

$$\sqrt{-7} \cdot \sqrt{-7}$$
$$\sqrt{49}$$
$$7$$

**479.** Why is $\sqrt{-64} = 8i$ but $\sqrt[3]{-64} = -4$.

**480.** Explain how dividing complex numbers is similar to rationalizing a denominator.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| evaluate the square root of a negative number. | | | |
| add or subtract complex numbers. | | | |
| multiply complex numbers. | | | |
| divide complex numbers. | | | |
| simplify powers of *i*. | | | |

ⓑ *On a scale of* $1-10$, *how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

# CHAPTER 8 REVIEW

## KEY TERMS

**complex conjugate pair** A complex conjugate pair is of the form *a* + *bi*, *a* – *bi*.

**complex number** A complex number is of the form *a* + *bi*, where *a* and *b* are real numbers. We call *a* the real part and *b* the imaginary part.

**complex number system** The complex number system is made up of both the real numbers and the imaginary numbers.

**imaginary unit** The imaginary unit $i$ is the number whose square is –1. $i^2 = -1$ or $i = \sqrt{-1}$.

**like radicals** Like radicals are radical expressions with the same index and the same radicand.

**radical equation** An equation in which a variable is in the radicand of a radical expression is called a radical equation.

**radical function** A radical function is a function that is defined by a radical expression.

**rationalizing the denominator** Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

**square of a number** If $n^2 = m$, then *m* is the square of *n*.

**square root of a number** If $n^2 = m$, then *n* is a square root of *m*.

**standard form** A complex number is in standard form when written as $a + bi,$ where *a, b* are real numbers.

## KEY CONCEPTS

### 8.1 Simplify Expressions with Roots

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of *m*'
  - If $n^2 = m$, then $n = \sqrt{m},$ for $n \geq 0.$


  - The square root of *m*, $\sqrt{m},$ is a positive number whose square is *m*.
- ***n*<sup></sup>th Root of a Number**
  - If $b^n = a,$ then *b* is an $n^{th}$ root of *a*.
  - The principal $n^{th}$ root of *a* is written $\sqrt[n]{a}.$
  - *n* is called the *index* of the radical.
- **Properties of $\sqrt[n]{a}$**
  - When *n* is an even number and
    - $a \geq 0,$ then $\sqrt[n]{a}$ is a real number
    - $a < 0,$ then $\sqrt[n]{a}$ is not a real number
  - When *n* is an odd number, $\sqrt[n]{a}$ is a real number for all values of *a*.
- **Simplifying Odd and Even Roots**
  - For any integer $n \geq 2,$
    - when *n* is odd $\sqrt[n]{a^n} = a$
    - when *n* is even $\sqrt[n]{a^n} = |a|$
  - We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

### 8.2 Simplify Radical Expressions

- **Simplified Radical Expression**

    - For real numbers *a*, *m* and $n \geq 2$
      $\sqrt[n]{a}$ is considered simplified if *a* has no factors of $m^n$

- **Product Property of n^th Roots**
    - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$
      $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
- **How to simplify a radical expression using the Product Property**
  Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
  Step 2. Use the product rule to rewrite the radical as the product of two radicals.
  Step 3. Simplify the root of the perfect power.
- **Quotient Property of Radical Expressions**
    - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,
      $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- **How to simplify a radical expression using the Quotient Property.**
  Step 1. Simplify the fraction in the radicand, if possible.
  Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
  Step 3. Simplify the radicals in the numerator and the denominator.

## 8.3 Simplify Rational Exponents

- **Rational Exponent** $\boldsymbol{a^{\frac{1}{n}}}$
    - If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- **Rational Exponent** $\boldsymbol{a^{\frac{m}{n}}}$
    - For any positive integers *m* and *n*,
      $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
    - If *a, b* are real numbers and *m, n* are rational numbers, then
        - **Product Property** $a^m \cdot a^n = a^{m+n}$
        - **Power Property** $(a^m)^n = a^{m \cdot n}$
        - **Product to a Power** $(ab)^m = a^m b^m$
        - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n}, \ a \neq 0$
        - **Zero Exponent Definition** $a^0 = 1, \quad a \neq 0$
        - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \ b \neq 0$
        - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n}, \ a \neq 0$

## 8.4 Add, Subtract, and Multiply Radical Expressions

- **Product Property of Roots**
    - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

- **Special Products**

| Binomial Squares | Product of Conjugates |
|---|---|
| $(a+b)^2 = a^2 + 2ab + b^2$ | $(a+b)(a-b) = a^2 - b^2$ |
| $(a-b)^2 = a^2 - 2ab + b^2$ | |

## 8.5 Divide Radical Expressions

- **Quotient Property of Radical Expressions**
    - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,
      $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- **Simplified Radical Expressions**
    - A radical expression is considered simplified if there are:
        - no factors in the radicand that have perfect powers of the index
        - no fractions in the radicand
        - no radicals in the denominator of a fraction

## 8.6 Solve Radical Equations

- **Binomial Squares**
  $(a+b)^2 = a^2 + 2ab + b^2$
  $(a-b)^2 = a^2 - 2ab + b^2$
- **Solve a Radical Equation**
  Step 1. Isolate one of the radical terms on one side of the equation.
  Step 2. Raise both sides of the equation to the power of the index.
  Step 3. Are there any more radicals?
  If yes, repeat Step 1 and Step 2 again.
  If no, solve the new equation.
  Step 4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**
  Step 1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  Step 2. Identify what we are looking for.
  Step 3. Name what we are looking for by choosing a variable to represent it.
  Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  Step 5. Solve the equation using good algebra techniques.
  Step 6. Check the answer in the problem and make sure it makes sense.
  Step 7. Answer the question with a complete sentence.
- **Falling Objects**
    - On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
    - If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

## 8.7 Use Radicals in Functions

- **Properties of $\sqrt[n]{a}$**
  - When *n* is an **even** number and:
    $a \geq 0,$ then $\sqrt[n]{a}$ is a real number.
    $a < 0,$ then $\sqrt[n]{a}$ is not a real number.
  - When *n* is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of *a*.
- **Domain of a Radical Function**
  - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
  - When the **index** of the radical is **odd**, the radicand can be any real number.

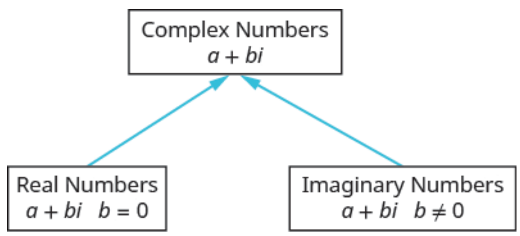
## 8.8 Use the Complex Number System

- **Square Root of a Negative Number**
  - If *b* is a positive real number, then $\sqrt{-b} = \sqrt{b}i$

| | $a + bi$ | |
|---|---|---|
| $b = 0$ | $a + 0 \cdot i$<br>$a$ | Real number |
| $b \neq 0$ | $a + bi$ | Imaginary number |
| $a = 0$ | $0 + bi$<br>$bi$ | Pure imaginary number |

**Table 8.32**

  - A complex number is in **standard form** when written as *a* + *bi*, where *a, b* are real numbers.




- **Product of Complex Conjugates**
  - If *a, b* are real numbers, then
    $(a - bi)(a + bi) = a^2 + b^2$
- **How to Divide Complex Numbers**
  Step 1. Write both the numerator and denominator in standard form.
  Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.
  Step 3. Simplify and write the result in standard form.

# REVIEW EXERCISES

## 8.1 8.1 Simplify Expressions with Roots

### Simplify Expressions with Roots

*In the following exercises, simplify.*

**481.** ⓐ $\sqrt{225}$ ⓑ $-\sqrt{16}$

**482.** ⓐ $-\sqrt{169}$ ⓑ $\sqrt{-8}$

**483.** ⓐ $\sqrt[3]{8}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{243}$

**484.** ⓐ $\sqrt[3]{-512}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1}$

### Estimate and Approximate Roots

*In the following exercises, estimate each root between two consecutive whole numbers.*

**485.** ⓐ $\sqrt{68}$ ⓑ $\sqrt[3]{84}$

*In the following exercises, approximate each root and round to two decimal places.*

**486.** ⓐ $\sqrt{37}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{125}$

### Simplify Variable Expressions with Roots

*In the following exercises, simplify using absolute values as necessary.*

**487.**
ⓐ $\sqrt[3]{a^3}$
ⓑ $\sqrt[7]{b^7}$

**488.**
ⓐ $\sqrt{a^{14}}$
ⓑ $\sqrt{w^{24}}$

**489.**
ⓐ $\sqrt[4]{m^8}$
ⓑ $\sqrt[5]{n^{20}}$

**490.**
ⓐ $\sqrt{121m^{20}}$
ⓑ $-\sqrt{64a^2}$

**491.**
ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

**492.**
ⓐ $\sqrt{144x^2y^2}$
ⓑ $\sqrt{169w^8y^{10}}$
ⓒ $\sqrt[3]{8a^{51}b^6}$

## 8.2 8.2 Simplify Radical Expressions

### Use the Product Property to Simplify Radical Expressions

*In the following exercises, use the Product Property to simplify radical expressions.*

**493.** $\sqrt{125}$

**494.** $\sqrt{675}$

**495.** ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

*In the following exercises, simplify using absolute value signs as needed.*

**496.**
ⓐ $\sqrt{a^{23}}$
ⓑ $\sqrt[3]{b^8}$
ⓒ $\sqrt[8]{c^{13}}$

**497.**
ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
ⓒ $\sqrt[6]{128b^7}$

**498.**
ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
ⓒ $\sqrt[4]{80x^8y^9}$

**499.**
ⓐ $\sqrt[5]{-32}$
ⓑ $\sqrt[8]{-1}$

**500.**
ⓐ $8+\sqrt{96}$
ⓑ $\frac{2+\sqrt{40}}{2}$

### Use the Quotient Property to Simplify Radical Expressions

*In the following exercises, use the Quotient Property to simplify square roots.*

**501.** ⓐ $\sqrt{\frac{72}{98}}$ ⓑ $\sqrt[3]{\frac{24}{81}}$ ⓒ $\sqrt[4]{\frac{6}{96}}$

**502.** ⓐ $\sqrt{\frac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

**503.** $\sqrt{\frac{300m^5}{64}}$

**504.**
ⓐ $\sqrt{\frac{28p^7}{q^2}}$
ⓑ $\sqrt[3]{\frac{81s^8}{t^3}}$
ⓒ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

**505.**
ⓐ $\sqrt{\frac{27p^2q}{108p^4q^3}}$
ⓑ $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$
ⓒ $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

**506.**
ⓐ $\frac{\sqrt{80q^5}}{\sqrt{5q}}$
ⓑ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$
ⓒ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

## 8.3 8.3 Simplify Rational Exponents

### Simplify expressions with $a^{\frac{1}{n}}$

*In the following exercises, write as a radical expression.*

**507.** ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

*In the following exercises, write with a rational exponent.*

**508.** ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $4\sqrt[6]{36r}$

*In the following exercises, simplify.*

**509.**
ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
ⓒ $32^{\frac{1}{5}}$

**510.**
ⓐ $(-1{,}000)^{\frac{1}{3}}$
ⓑ $-1{,}000^{\frac{1}{3}}$
ⓒ $(1{,}000)^{-\frac{1}{3}}$

**511.**
ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

### Simplify Expressions with $a^{\frac{m}{n}}$

*In the following exercises, write with a rational exponent.*

**512.**
ⓐ $\sqrt[4]{r^7}$
ⓑ $\left(\sqrt[5]{2pq}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

*In the following exercises, simplify.*

**513.**

ⓐ $25^{\frac{3}{2}}$

ⓑ $9^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{2}{3}}$

**514.**

ⓐ $-64^{\frac{3}{2}}$

ⓑ $-64^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{3}{2}}$

### Use the Laws of Exponents to Simplify Expressions with Rational Exponents

*In the following exercises, simplify.*

**515.**

ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$

ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$

ⓒ $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

**516.**

ⓐ $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

ⓑ $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

## 8.4 8.4 Add, Subtract and Multiply Radical Expressions

### Add and Subtract Radical Expressions

*In the following exercises, simplify.*

**517.**

ⓐ $7\sqrt{2} - 3\sqrt{2}$

ⓑ $7\sqrt[3]{p} + 2\sqrt[3]{p}$

ⓒ $5\sqrt[3]{x} - 3\sqrt[3]{x}$

**518.**

ⓐ $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$

ⓑ $8\sqrt[4]{11cd} + 5\sqrt[4]{11cd} - 9\sqrt[4]{11cd}$

**519.**

ⓐ $\sqrt{48} + \sqrt{27}$

ⓑ $\sqrt[3]{54} + \sqrt[3]{128}$

ⓒ $6\sqrt[4]{5} - \frac{3}{2}\sqrt[4]{320}$

**520.**

ⓐ $\sqrt{80c^7} - \sqrt{20c^7}$

ⓑ $2\sqrt[4]{162r^{10}} + 4\sqrt[4]{32r^{10}}$

**521.** $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

### Multiply Radical Expressions

*In the following exercises, simplify.*

**522.**

ⓐ $(5\sqrt{6})(-\sqrt{12})$

ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

**523.**

ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$

ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

### Use Polynomial Multiplication to Multiply Radical Expressions

*In the following exercises, multiply.*

**524.**

ⓐ $\sqrt{11}\left(8 + 4\sqrt{11}\right)$

ⓑ $\sqrt[3]{3}\left(\sqrt[3]{9} + \sqrt[3]{18}\right)$

**525.**

ⓐ $(3 - 2\sqrt{7})(5 - 4\sqrt{7})$

ⓑ $\left(\sqrt[3]{x} - 5\right)\left(\sqrt[3]{x} - 3\right)$

**526.** $\left(2\sqrt{7} - 5\sqrt{11}\right)\left(4\sqrt{7} + 9\sqrt{11}\right)$

**527.**
ⓐ $\left(4+\sqrt{11}\right)^2$
ⓑ $\left(3-2\sqrt{5}\right)^2$

**528.** $\left(7+\sqrt{10}\right)\left(7-\sqrt{10}\right)$

**529.** $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

## 8.5 8.5 Divide Radical Expressions

### Divide Square Roots

*In the following exercises, simplify.*

**530.**
ⓐ $\dfrac{\sqrt{48}}{\sqrt{75}}$
ⓑ $\dfrac{\sqrt[3]{81}}{\sqrt[3]{24}}$

**531.**
ⓐ $\dfrac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$
ⓑ $\dfrac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

### Rationalize a One Term Denominator

*In the following exercises, rationalize the denominator.*

**532.** ⓐ $\dfrac{8}{\sqrt{3}}$ ⓑ $\sqrt{\dfrac{7}{40}}$ ⓒ $\dfrac{8}{\sqrt{2y}}$

**533.** ⓐ $\dfrac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\dfrac{7}{54}}$ ⓒ $\dfrac{3}{\sqrt[3]{3x^2}}$

**534.** ⓐ $\dfrac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\dfrac{9}{32}}$ ⓒ $\dfrac{6}{\sqrt[4]{9x^3}}$

### Rationalize a Two Term Denominator

*In the following exercises, simplify.*

**535.** $\dfrac{7}{2-\sqrt{6}}$

**536.** $\dfrac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

**537.** $\dfrac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

## 8.6 8.6 Solve Radical Equations

### Solve Radical Equations

*In the following exercises, solve.*

**538.** $\sqrt{4x-3}=7$

**539.** $\sqrt{5x+1}=-3$

**540.** $\sqrt[3]{4x-1}=3$

**541.** $\sqrt{u-3}+3=u$

**542.** $\sqrt[3]{4x+5}-2=-5$

**543.** $(8x+5)^{\frac{1}{3}}+2=-1$

**544.** $\sqrt{y+4}-y+2=0$

**545.** $2\sqrt{8r+1}-8=2$

### Solve Radical Equations with Two Radicals

*In the following exercises, solve.*

**546.** $\sqrt{10+2c}=\sqrt{4c+16}$

**547.** $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

**548.** $\sqrt{r}+6=\sqrt{r+8}$

**549.** $\sqrt{x+1}-\sqrt{x-2}=1$

### Use Radicals in Applications

*In the following exercises, solve. Round approximations to one decimal place.*

**550. Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

**551. Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

## 8.7 8.7 Radical Functions

### Evaluate a Radical Function

*In the following exercises, evaluate each function.*

**552.** $g(x) = \sqrt{6x+1}$, find

ⓐ $g(4)$

ⓑ $g(8)$

**553.** $G(x) = \sqrt{5x-1}$, find

ⓐ $G(5)$

ⓑ $G(2)$

**554.** $h(x) = \sqrt[3]{x^2-4}$, find

ⓐ $h(-2)$

ⓑ $h(6)$

**555.** For the function $g(x) = \sqrt[4]{4-4x}$, find

ⓐ $g(1)$

ⓑ $g(-3)$

### Find the Domain of a Radical Function

*In the following exercises, find the domain of the function and write the domain in interval notation.*

**556.** $g(x) = \sqrt{2-3x}$

**557.** $F(x) = \sqrt{\dfrac{x+3}{x-2}}$

**558.** $f(x) = \sqrt[3]{4x^2-16}$

**559.** $F(x) = \sqrt[4]{10-7x}$

### Graph Radical Functions

*In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.*

**560.** $g(x) = \sqrt{x+4}$

**561.** $g(x) = 2\sqrt{x}$

**562.** $f(x) = \sqrt[3]{x-1}$

**563.** $f(x) = \sqrt[3]{x}+3$

## 8.8 8.8 The Complex Number System

### Evaluate the Square Root of a Negative Number

*In the following exercises, write each expression in terms of i and simplify if possible.*

**564.**

ⓐ $\sqrt{-100}$

ⓑ $\sqrt{-13}$

ⓒ $\sqrt{-45}$

### Add or Subtract Complex Numbers

*In the following exercises, add or subtract.*

**565.** $\sqrt{-50} + \sqrt{-18}$

**566.** $(8 - i) + (6 + 3i)$

**567.** $(6 + i) - (-2 - 4i)$

**568.** $\left(-7 - \sqrt{-50}\right) - \left(-32 - \sqrt{-18}\right)$

### Multiply Complex Numbers

*In the following exercises, multiply.*

**569.** $(-2 - 5i)(-4 + 3i)$

**570.** $-6i(-3 - 2i)$

**571.** $\sqrt{-4} \cdot \sqrt{-16}$

**572.** $\left(5 - \sqrt{-12}\right)\left(-3 + \sqrt{-75}\right)$

*In the following exercises, multiply using the Product of Binomial Squares Pattern.*

**573.** $(-2 - 3i)^2$

*In the following exercises, multiply using the Product of Complex Conjugates Pattern.*

**574.** $(9 - 2i)(9 + 2i)$

### Divide Complex Numbers

*In the following exercises, divide.*

**575.** $\dfrac{2 + i}{3 - 4i}$

**576.** $\dfrac{-4}{3 - 2i}$

### Simplify Powers of *i*

*In the following exercises, simplify.*

**577.** $i^{48}$

**578.** $i^{255}$

## PRACTICE TEST

*In the following exercises, simplify using absolute values as necessary.*

**579.** $\sqrt[3]{125x^9}$

**580.** $\sqrt{169x^8 y^6}$

**581.** $\sqrt[3]{72x^8 y^4}$

**582.** $\sqrt{\dfrac{45x^3 y^4}{180x^5 y^2}}$

*In the following exercises, simplify. Assume all variables are positive.*

**583.** ⓐ $216^{-\frac{1}{4}}$ ⓑ $-49^{\frac{3}{2}}$

**584.** $\sqrt{-45}$

**585.** $\dfrac{x^{-\frac{1}{4}} \cdot x^{\frac{5}{4}}}{x^{-\frac{3}{4}}}$

**586.** $\left(\dfrac{8x^{\frac{2}{3}} y^{-\frac{5}{2}}}{x^{-\frac{7}{3}} y^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

**587.** $\sqrt{48x^5} - \sqrt{75x^5}$

**588.** $\sqrt{27x^2} - 4x\sqrt{12} + \sqrt{108x^2}$

**589.** $2\sqrt{12x^5} \cdot 3\sqrt{6x^3}$

**590.** $\sqrt[3]{4}\left(\sqrt[3]{16} - \sqrt[3]{6}\right)$

**591.** $(4 - 3\sqrt{3})(5 + 2\sqrt{3})$

**592.** $\dfrac{\sqrt[3]{128}}{\sqrt[3]{54}}$

**593.** $\dfrac{\sqrt{245xy^{-4}}}{\sqrt{45x^{-4} y^3}}$

**594.** $\dfrac{1}{\sqrt[3]{5}}$

**595.** $\dfrac{3}{2 + \sqrt{3}}$

**596.** $\sqrt{-4} \cdot \sqrt{-9}$

**597.** $-4i(-2 - 3i)$

**598.** $\dfrac{4 + i}{3 - 2i}$

**599.** $i^{172}$

*In the following exercises, solve.*

**600.** $\sqrt{2x + 5} + 8 = 6$

**601.** $\sqrt{x + 5} + 1 = x$

**602.** $\sqrt[3]{2x^2 - 6x - 23} = \sqrt[3]{x^2 - 3x + 5}$

*In the following exercise, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.*

**603.** $g(x) = \sqrt{x + 2}$

# 9 QUADRATIC EQUATIONS AND FUNCTIONS

**Figure 9.1** Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology. (credit: “intographics”/Pixabay)

## Chapter Outline



## Introduction

Blink your eyes. You’ve taken a photo. That’s what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

## 9.1 Solve Quadratic Equations Using the Square Root Property

### Learning Objectives

**By the end of this section, you will be able to:**

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

### Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{128}$.
   If you missed this problem, review Example 8.13.
2. Simplify: $\sqrt{\frac{32}{5}}$.

If you missed this problem, review **Example 8.50**.

3. Factor: $9x^2 - 12x + 4$.
If you missed this problem, review **Example 6.23**.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form $ax^2$. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

## Solve Quadratic Equations of the form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

| | |
|---|---|
| | $x^2 = 9$ |
| Put the equation in standard form. | $x^2 - 9 = 0$ |
| Factor the diffe ence of squares. | $(x-3)(x+3) = 0$ |
| Use the Zero Product Property. | $x - 3 = 0 \quad x - 3 = 0$ |
| Solve each equation. | $x = 3 \quad x = -3$ |

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4,\ x = -4$ and $x = 5,\ x = -5$.

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so −13 is also a square root of 169. Therefore, both 13 and −13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of $k$. This leads to the **Square Root Property**.

**Square Root Property**

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}.$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of $k$ and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as $x$ equals positive or negative the square root of $k$.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

| | |
|---|---|
| | $x^2 = 9$ |
| Use the Square Root Property. | $x = \pm\sqrt{9}$ |
| | $x = \pm 3$ |
| | So $x = 3$ or $x = -3$. |

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

| | |
|---|---|
| | $x^2 = 7$ |
| Use the Square Root Property. | $x = \sqrt{7}, \quad x = -\sqrt{7}$ |

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

**EXAMPLE 9.1** HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $AX^2 = K$ USING THE SQUARE ROOT PROPERTY

Solve: $x^2 - 50 = 0$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Isolate the quadratic term and make its coefficient one. | Add 50 to both sides to get $x^2$ by itself. | $x^2 - 50 = 0$<br>$x^2 = 50$ |
| **Step 2.** Use Square Root Property. | Remember to write the ± symbol. | $x = \pm\sqrt{50}$ |
| **Step 3.** Simplify the radical. | <br><br>Rewrite to show two solutions. | $x = \pm\sqrt{25} \cdot \sqrt{2}$<br>$x = \pm 5\sqrt{2}$<br>$x = 5\sqrt{2},\ x = -5\sqrt{2}$ |
| **Step 4.** Check the solutions. | Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$ | $x^2 - 50 = 0$<br>$(5\sqrt{2})^2 - 50 \overset{?}{=} 0$<br>$25 \cdot 2 - 50 \overset{?}{=} 0$<br>$0 = 0$ ✓<br>$x^2 - 50 = 0$<br>$(-5\sqrt{2})^2 - 50 \overset{?}{=} 0$<br>$25 \cdot 2 - 50 \overset{?}{=} 0$<br>$0 = 0$ ✓ |

**TRY IT : :** 9.1 Solve: $x^2 - 48 = 0$.

**TRY IT : :** 9.2 Solve: $y^2 - 27 = 0$.

The steps to take to use the Square Root Property to solve a quadratic equation are listed here.

**HOW TO : :** SOLVE A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY.

Step 1. Isolate the quadratic term and make its coefficient one.
Step 2. Use Square Root Property.
Step 3. Simplify the radical.
Step 4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

**EXAMPLE 9.2**

Solve: $3z^2 = 108$.

## Solution

| | |
|---|---|
| | $3z^2 = 108$ |
| The quadratic term is isolated. Divide by 3 to make its coefficient 1. | $\frac{3z^2}{3} = \frac{108}{3}$ |
| Simplify. | $z^2 = 36$ |
| Use the Square Root Property. | $z = \pm\sqrt{36}$ |
| Simplify the radical. | $z = \pm 6$ |
| Rewrite to show two solutions. | $z = 6, \quad z = -6$ |
| Check the solutions: | |

$$3z^2 = 108 \qquad 3z^2 = 108$$
$$3(6)^2 \stackrel{?}{=} 108 \qquad 3(-6)^2 \stackrel{?}{=} 108$$
$$3(36) \stackrel{?}{=} 108 \qquad 3(36) \stackrel{?}{=} 108$$
$$108 = 108 \checkmark \qquad 108 = 108 \checkmark$$

**TRY IT : :** 9.3 Solve: $2x^2 = 98.$

**TRY IT : :** 9.4 Solve: $5m^2 = 80.$

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

**EXAMPLE 9.3**

Solve: $x^2 + 72 = 0$.

## Solution

| | |
|---|---|
| | $x^2 + 72 = 0$ |
| Isolate the quadratic term. | $x^2 = -72$ |
| Use the Square Root Property. | $x = \pm\sqrt{-72}$ |
| Simplify using complex numbers. | $x = \pm\sqrt{72}\,i$ |
| Simplify the radical. | $x = \pm 6\sqrt{2}\,i$ |
| Rewrite to show two solutions. | $x = 6\sqrt{2}\,i, \quad x = -6\sqrt{2}\,i$ |

Check the solutions:

| | |
|---|---|
| $x^2 + 72 = 0$ | $x^2 + 72 = 0$ |
| $\left(6\sqrt{2}\,i\right)^2 + 72 \stackrel{?}{=} 0$ | $\left(6\sqrt{2}\,i\right)^2 + 72 \stackrel{?}{=} 0$ |
| $6^2\left(\sqrt{2}\right)^2 i^2 + 72 \stackrel{?}{=} 0$ | $(-6)^2\left(\sqrt{2}\right)^2 i^2 + 72 \stackrel{?}{=} 0$ |
| $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ | $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ |
| $0 = 0$ ✓ | $0 = 0$ ✓ |

**TRY IT : :** 9.5 Solve: $c^2 + 12 = 0$.

**TRY IT : :** 9.6 Solve: $q^2 + 24 = 0$.

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

**EXAMPLE 9.4**

Solve: $\frac{2}{3}u^2 + 5 = 17$.

## Solution

| | |
|---|---|
| | $\frac{2}{3}u^2 + 5 = 17$ |
| Isolate the quadratic term. | $\frac{2}{3}u^2 = 12$ |
| Multiply by $\frac{3}{2}$ to make the coefficient 1. | $\frac{3}{2} \cdot \frac{2}{3}u^2 = \frac{3}{2} \cdot 12$ |
| Simplify. | $u^2 = 18$ |
| Use the Square Root Property. | $u = \pm\sqrt{18}$ |
| Simplify the radical. | $u = \pm\sqrt{9 \cdot 2}$ |
| Simplify. | $u = \pm 3\sqrt{2}$ |
| Rewrite to show two solutions. | $u = 3\sqrt{2}, \quad u = -3\sqrt{2}$ |

Check:

| | |
|---|---|
| $\frac{2}{3}u^2 + 5 = 17$ | $\frac{2}{3}u^2 + 5 = 17$ |
| $\frac{2}{3}\left(3\sqrt{2}\right)^2 + 5 \stackrel{?}{=} 17$ | $\frac{2}{3}\left(-3\sqrt{2}\right)^2 + 5 \stackrel{?}{=} 17$ |
| $\frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17$ | $\frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17$ |
| $12 + 5 \stackrel{?}{=} 17$ | $12 + 5 \stackrel{?}{=} 17$ |
| $17 = 17$ ✓ | $17 = 17$ ✓ |

**TRY IT : :** 9.7 Solve: $\frac{1}{2}x^2 + 4 = 24$.

**TRY IT : :** 9.8 Solve: $\frac{3}{4}y^2 - 3 = 18.$

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

**EXAMPLE 9.5**

Solve: $2x^2 - 8 = 41.$

**Solution**

| | |
|---|---|
| | $2x^2 - 8 = 41$ |
| Isolate the quadratic term. | $2x^2 = 49$ |
| Divide by 2 to make the coefficient 1. | $\frac{2x^2}{2} = \frac{49}{2}$ |
| Simplify. | $x^2 = \frac{49}{2}$ |
| Use the Square Root Property. | $x = \pm\sqrt{\frac{49}{2}}$ |
| Rewrite the radical as a fraction of square roots. | $x = \pm\frac{\sqrt{49}}{\sqrt{2}}$ |
| Rationalize the denominator. | $x = \pm\frac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$ |
| Simplify. | $x = \pm\frac{7\sqrt{2}}{2}$ |
| Rewrite to show two solutions. | $x = \frac{7\sqrt{2}}{2}, \quad x = -\frac{7\sqrt{2}}{2}$ |
| Check:<br>We leave the check for you. | |

**TRY IT : :** 9.9 Solve: $5r^2 - 2 = 34.$

**TRY IT : :** 9.10 Solve: $3t^2 + 6 = 70.$

## Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation of the form $a(x - h)^2 = k$ as well. Notice that the quadratic term, $x$, in the original form $ax^2 = k$ is replaced with $(x - h)$.

$$ax^2 = k \qquad a(x-h)^2 = k$$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of $a$, then the Square Root Property can be used on $(x - h)^2$.

**EXAMPLE 9.6**

Solve: $4(y - 7)^2 = 48.$

### Solution

| | |
|---|---|
| | $4(y-7)^2 = 48$ |
| Divide both sides by the coefficient 4. | $(y-7)^2 = 12$ |
| Use the Square Root Property on the binomial | $y - 7 = \pm\sqrt{12}$ |
| Simplify the radical. | $y - 7 = \pm 2\sqrt{3}$ |
| Solve for $y$. | $y = 7 \pm 2\sqrt{3}$ |
| Rewrite to show two solutions. | $y = 7 + 2\sqrt{3}, \quad y = 7 - 2\sqrt{3}$ |

Check:

$$\begin{aligned} 4(y-7)^2 &= 48 \\ 4(7+2\sqrt{3}-7)^2 &\stackrel{?}{=} 48 \\ 4(2\sqrt{3})^2 &\stackrel{?}{=} 48 \\ 4(12) &\stackrel{?}{=} 48 \\ 48 &= 48 \checkmark \end{aligned} \qquad \begin{aligned} 4(y-7)^2 &= 48 \\ 4(7-2\sqrt{3}-7)^2 &\stackrel{?}{=} 48 \\ 4(-2\sqrt{3})^2 &\stackrel{?}{=} 48 \\ 4(12) &\stackrel{?}{=} 48 \\ 48 &= 48 \checkmark \end{aligned}$$

**TRY IT : :** 9.11 Solve: $3(a-3)^2 = 54$.

**TRY IT : :** 9.12 Solve: $2(b+2)^2 = 80$.

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

**EXAMPLE 9.7**

Solve: $\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$.

## Solution

| | |
|---|---|
| | $\left(x-\frac{1}{3}\right)^2 = \frac{5}{9}$ |
| Use the Square Root Property. | $x-\frac{1}{3} = \pm\sqrt{\frac{5}{9}}$ |
| Rewrite the radical as a fraction of square roots. | $x-\frac{1}{3} = \pm\frac{\sqrt{5}}{\sqrt{9}}$ |
| Simplify the radical. | $x-\frac{1}{3} = \pm\frac{\sqrt{5}}{3}$ |
| Solve for $x$. | $x = \frac{1}{3}\pm\frac{\sqrt{5}}{3}$ |
| Rewrite to show two solutions. | $x=\frac{1}{3}+\frac{\sqrt{5}}{3},\quad x=\frac{1}{3}-\frac{\sqrt{5}}{3}$ |

Check:
We leave the check for you.

**TRY IT : :** 9.13 Solve: $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

**TRY IT : :** 9.14 Solve: $\left(y+\frac{3}{4}\right)^2=\frac{7}{16}$.

We will start the solution to the next example by isolating the binomial term.

### EXAMPLE 9.8

Solve: $2(x-2)^2+3=57$.

## Solution

| | |
|---|---|
| | $2(x-2)^2+3 = 57$ |
| Subtract 3 from both sides to isolate the binomial term. | $2(x-2)^2 = 54$ |
| Divide both sides by 2. | $(x-2)^2 = 27$ |
| Use the Square Root Property. | $x-2 = \pm\sqrt{27}$ |
| Simplify the radical. | $x-2 = \pm 3\sqrt{3}$ |
| Solve for $x$. | $x = 2\pm 3\sqrt{3}$ |
| Rewrite to show two solutions. | $x=2+3\sqrt{3},\quad x=2-3\sqrt{3}$ |

Check:
We leave the check for you.

**TRY IT : :** 9.15 Solve: $5(a-5)^2+4=104$.

**TRY IT : :** 9.16 Solve: $3(b+3)^2-8=88$.

Sometimes the solutions are complex numbers.

### EXAMPLE 9.9

Solve: $(2x-3)^2 = -12$.

### Solution

| | |
|---|---|
| | $(2x-3)^2 = -12$ |
| Use the Square Root Property. | $2x-3 = \pm\sqrt{-12}$ |
| Simplify the radical. | $2x-3 = \pm 2\sqrt{3}\,i$ |
| Add 3 to both sides. | $2x = 3 \pm 2\sqrt{3}\,i$ |
| Divide both sides by 2. | $x = \dfrac{3 \pm 2\sqrt{3}\,i}{2}$ |
| Rewrite in standard form. | $x = \dfrac{3}{2} \pm \dfrac{2\sqrt{3}\,i}{2}$ |
| Simplify. | $x = \dfrac{3}{2} \pm \sqrt{3}\,i$ |
| Rewrite to show two solutions. | $x = \dfrac{3}{2} + \sqrt{3}\,i,\quad x = \dfrac{3}{2} - \sqrt{3}\,i$ |

Check:
We leave the check for you.

**TRY IT : :** 9.17 Solve: $(3r+4)^2 = -8$.

**TRY IT : :** 9.18 Solve: $(2t-8)^2 = -10$.

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

**EXAMPLE 9.10**

Solve: $4n^2 + 4n + 1 = 16$.

### Solution

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

| | |
|---|---|
| | $4n^2 + 4n + 1 = 16$ |
| Factor the perfect square trinomial. | $(2n+1)^2 = 16$ |
| Use the Square Root Property. | $2n+1 = \pm\sqrt{16}$ |
| Simplify the radical. | $2n+1 = \pm 4$ |
| Solve for $n$. | $2n = -1 \pm 4$ |
| Divide each side by 2. | $\dfrac{2n}{2} = \dfrac{-1 \pm 4}{2}$<br>$n = \dfrac{-1 \pm 4}{2}$ |
| Rewrite to show two solutions. | $n = \dfrac{-1+4}{2},\quad n = \dfrac{-1-4}{2}$ |
| Simplify each equation. | $n = \dfrac{3}{2},\qquad n = -\dfrac{5}{2}$ |

Check:

$$4n^2 + 4n + 1 = 16 \qquad 4n^2 + 4n + 1 = 16$$

$$4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad 4\left(-\frac{5}{2}\right)^2 + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$

$$4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad 4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$

$$9 + 6 + 1 \stackrel{?}{=} 16 \qquad 25 - 10 + 1 \stackrel{?}{=} 16$$

$$16 = 16 \checkmark \qquad 16 = 16 \checkmark$$

**TRY IT : :** 9.19 Solve: $9m^2 - 12m + 4 = 25$.

**TRY IT : :** 9.20 Solve: $16n^2 + 40n + 25 = 4$.

**MEDIA : :**

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- **Solving Quadratic Equations: The Square Root Property (https://openstax.org/l/37SqRtProp1)**
- **Using the Square Root Property to Solve Quadratic Equations (https://openstax.org/l/37SqRtProp2)**

## 9.1 EXERCISES

### Practice Makes Perfect

**Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property**

*In the following exercises, solve each equation.*

**1.** $a^2 = 49$

**2.** $b^2 = 144$

**3.** $r^2 - 24 = 0$

**4.** $t^2 - 75 = 0$

**5.** $u^2 - 300 = 0$

**6.** $v^2 - 80 = 0$

**7.** $4m^2 = 36$

**8.** $3n^2 = 48$

**9.** $\frac{4}{3}x^2 = 48$

**10.** $\frac{5}{3}y^2 = 60$

**11.** $x^2 + 25 = 0$

**12.** $y^2 + 64 = 0$

**13.** $x^2 + 63 = 0$

**14.** $y^2 + 45 = 0$

**15.** $\frac{4}{3}x^2 + 2 = 110$

**16.** $\frac{2}{3}y^2 - 8 = -2$

**17.** $\frac{2}{5}a^2 + 3 = 11$

**18.** $\frac{3}{2}b^2 - 7 = 41$

**19.** $7p^2 + 10 = 26$

**20.** $2q^2 + 5 = 30$

**21.** $5y^2 - 7 = 25$

**22.** $3x^2 - 8 = 46$

**Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property**

*In the following exercises, solve each equation.*

**23.** $(u - 6)^2 = 64$

**24.** $(v + 10)^2 = 121$

**25.** $(m - 6)^2 = 20$

**26.** $(n + 5)^2 = 32$

**27.** $\left(r - \frac{1}{2}\right)^2 = \frac{3}{4}$

**28.** $\left(x + \frac{1}{5}\right)^2 = \frac{7}{25}$

**29.** $\left(y + \frac{2}{3}\right)^2 = \frac{8}{81}$

**30.** $\left(t - \frac{5}{6}\right)^2 = \frac{11}{25}$

**31.** $(a - 7)^2 + 5 = 55$

**32.** $(b - 1)^2 - 9 = 39$

**33.** $4(x + 3)^2 - 5 = 27$

**34.** $5(x + 3)^2 - 7 = 68$

**35.** $(5c + 1)^2 = -27$

**36.** $(8d - 6)^2 = -24$

**37.** $(4x - 3)^2 + 11 = -17$

**38.** $(2y + 1)^2 - 5 = -23$

**39.** $m^2 - 4m + 4 = 8$

**40.** $n^2 + 8n + 16 = 27$

**41.** $x^2 - 6x + 9 = 12$

**42.** $y^2 + 12y + 36 = 32$

**43.** $25x^2 - 30x + 9 = 36$

**44.** $9y^2 + 12y + 4 = 9$

**45.** $36x^2 - 24x + 4 = 81$

**46.** $64x^2 + 144x + 81 = 25$

## Mixed Practice

*In the following exercises, solve using the Square Root Property.*

**47.** $2r^2 = 32$

**48.** $4t^2 = 16$

**49.** $(a-4)^2 = 28$

**50.** $(b+7)^2 = 8$

**51.** $9w^2 - 24w + 16 = 1$

**52.** $4z^2 + 4z + 1 = 49$

**53.** $a^2 - 18 = 0$

**54.** $b^2 - 108 = 0$

**55.** $\left(p - \frac{1}{3}\right)^2 = \frac{7}{9}$

**56.** $\left(q - \frac{3}{5}\right)^2 = \frac{3}{4}$

**57.** $m^2 + 12 = 0$

**58.** $n^2 + 48 = 0.$

**59.** $u^2 - 14u + 49 = 72$

**60.** $v^2 + 18v + 81 = 50$

**61.** $(m-4)^2 + 3 = 15$

**62.** $(n-7)^2 - 8 = 64$

**63.** $(x+5)^2 = 4$

**64.** $(y-4)^2 = 64$

**65.** $6c^2 + 4 = 29$

**66.** $2d^2 - 4 = 77$

**67.** $(x-6)^2 + 7 = 3$

**68.** $(y-4)^2 + 10 = 9$

## Writing Exercises

**69.** In your own words, explain the Square Root Property.

**70.** In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)^2 = 16$ .

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| solve quadratic equations of the form $ax^2 = k$ using the square root property. | | | |
| solve quadratic equations of the form $a(x-h)^2 = k$ using the square root property. | | | |

*Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."*

ⓑ *If most of your checks were:*

**...confidently.** *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

**...with some help.** *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

**...no - I don't get it!** *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be* *overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

# 9.2 Solve Quadratic Equations by Completing the Square

## Learning Objectives

**By the end of this section, you will be able to:**

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square
- Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square

### Be Prepared!

Before you get started, take this readiness quiz.

1. Expand: $(x + 9)^2$.
   If you missed this problem, review **Example 5.32**.
2. Factor $y^2 - 14y + 49$.
   If you missed this problem, review **Example 6.9**.
3. Factor $5n^2 + 40n + 80$.
   If you missed this problem, review **Example 6.14**.

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

## Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y - 7)^2 = 12$ because the left side was a perfect square.

$$\begin{aligned} (y-7)^2 &= 12 \\ y-7 &= \pm\sqrt{12} \\ y-7 &= \pm 2\sqrt{3} \\ y &= 7 \pm 2\sqrt{3} \end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x - k)^2$ in order to use the Square Root Property.

$$\begin{aligned} x^2 - 10x + 25 &= 18 \\ (x-5)^2 &= 18 \end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let’s look at two examples to help us recognize the patterns.

$$\begin{array}{cc} (x+9)^2 & (y-7)^2 \\ (x+9)(x+9) & (y-7)(y-7) \\ x^2 + 9x + 9x + 81 & y^2 - 7y - 7y + 49 \\ x^2 + 18x + 81 & y^2 - 14y + 49 \end{array}$$

We restate the patterns here for reference.

### Binomial Squares Pattern

If *a* and *b* are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

We can use this pattern to “make” a perfect square.

We will start with the expression $x^2 + 6x$. Since there is a plus sign between the two terms, we will use the $(a + b)^2$ pattern, $a^2 + 2ab + b^2 = (a + b)^2$.

$$\overset{a^2 + 2ab + b^2}{x^2 + 6x + \_\_}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find $b$. But first we start with determining $a$. Notice that the first term of $x^2 + 6x$ is a square, $x^2$. This tells us that $a = x$.

$$\overset{a^2 + \quad 2ab \quad + b^2}{x^2 + 2\cdot x\cdot b + b^2}$$

What number, $b$, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b = 3$.

$$\overset{a^2 + \quad 2ab \quad + b^2}{x^2 + 2\cdot 3\cdot x + \_\_}$$

Now to complete the perfect square trinomial, we will find the last term by squaring $b$, which is $3^2 = 9$.

$$\overset{a^2 + 2ab + b^2}{x^2 + 6x + 9}$$

We can now factor.

$$\overset{(a+b)^2}{(x+3)^2}$$

So we found that adding 9 to $x^2 + 6x$ ‘completes the square’, and we write it as $(x + 3)^2$.

**HOW TO : : COMPLETE A SQUARE OF $x^2 + bx$.**

Step 1. Identify $b$, the coefficient of $x$.

Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

Step 4. Factor the perfect square trinomial, writing it as a binomial squared.

## EXAMPLE 9.11

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $x^2 - 26x$ ⓑ $y^2 - 9y$ ⓒ $n^2 + \frac{1}{2}n$

### Solution

ⓐ

| | $x^2 - bx$<br>$x^2 - 26x$ |
|---|---|
| The coefficient of $x$ is −26. | |
| Find $\left(\frac{1}{2}b\right)^2$.<br>$\left(\frac{1}{2}\cdot(-26)\right)^2$<br>$(13)^2$<br>$169$ | |
| Add 169 to the binomial to complete the square. | $x^2 - 26x + 169$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $(x-13)^2$ |

ⓑ

| | $x^2 - bx$<br>$y^2 - 9y$ |
|---|---|
| The coefficient of $y$ is $-9$. | |
| Find $\left(\frac{1}{2}b\right)^2$.<br>$\left(\frac{1}{2}\cdot(-9)\right)^2$<br>$\left(-\frac{9}{2}\right)^2$<br>$\frac{81}{4}$ | |
| Add $\frac{81}{4}$ to the binomial to complete the square. | $y^2 - 9y + \frac{81}{4}$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $\left(y-\frac{9}{2}\right)^2$ |

ⓒ

| | $x^2 + bx$<br>$n^2 + \frac{1}{2}n$ |
|---|---|
| The coefficient of $n$ is $\frac{1}{2}$. | |

| | |
|---|---|
| Find $\left(\frac{1}{2}b\right)^2$. <br> $\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$ <br> $\left(\frac{1}{4}\right)^2$ <br> $\frac{1}{16}$ | |
| Add $\frac{1}{16}$ to the binomial to complete the square. | $n^2+\frac{1}{2}n+\frac{1}{16}$ |
| Rewrite as a binomial square. | $\left(n+\frac{1}{4}\right)^2$ |

**TRY IT : :** 9.21

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $a^2-20a$ ⓑ $m^2-5m$ ⓒ $p^2+\frac{1}{4}p$

**TRY IT : :** 9.22

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $b^2-4b$ ⓑ $n^2+13n$ ⓒ $q^2-\frac{2}{3}q$

## Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x = 40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

| | |
|---|---|
| | $x^2+6x=40$ |
| | $x^2+6x+\_\_=40+\_\_$ |
| | $x^2+6x+9=40+9$ |
| Add 9 to both sides to complete the square. | $(x+3)^2=49$ |

Now the equation is in the form to solve using the Square Root Property! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

**EXAMPLE 9.12** HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $x^2+bx+c=0$ BY COMPLETING THE SQUARE

Solve by completing the square: $x^2+8x=48$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Isolate the variable terms on one side and the constant terms on the other. | This equation has all the variables on the left. | $\underset{x^2+bx}{x^2+8x}=\underset{c}{48}$ |
| **Step 2.** Find $\left(\frac{1}{2}\cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation. | Take half of 8 and square it. $4^2=16$ Add 16 to BOTH sides of the equation. | $x^2+8x+\underset{\left(\frac{1}{2}\cdot 8\right)^2}{\_\_\_\_}=48$ <br> $x^2+8x+16=48+16$ |
| **Step 3.** Factor the perfect square trinomial as a binomial square. | $x^2+8x+16=(x+4)^2$ Add the terms on the right. | $(x+4)^2=64$ |
| **Step 4.** Use the Square Root Property. | | $x+4=\pm\sqrt{64}$ |
| **Step 5.** Simplify the radical and then solve the two resulting equations. | | $x+4=\pm 8$ <br> $x+4=8 \quad x+4=-8$ <br> $x=4 \quad x=-12$ |
| **Step 6.** Check the solutions. | Put each answer in the original equation to check. Substitute $x=4$. <br> Substitute $x=-12$. | $x^2+8x=48$ <br> $(4)^2+8(4)\stackrel{?}{=}48$ <br> $16+32\stackrel{?}{=}48$ <br> $48=48\checkmark$ <br> $x^2+8x=48$ <br> $(-12)^2+8(-12)\stackrel{?}{=}48$ <br> $144-96\stackrel{?}{=}48$ <br> $48=48\checkmark$ |

**TRY IT : :** 9.23 Solve by completing the square: $x^2+4x=5$.

**TRY IT : :** 9.24 Solve by completing the square: $y^2-10y=-9$.

The steps to solve a quadratic equation by completing the square are listed here.

**HOW TO : :** SOLVE A QUADRATIC EQUATION OF THE FORM $x^2+bx+c=0$ BY COMPLETING THE SQUARE.

Step 1. Isolate the variable terms on one side and the constant terms on the other.

Step 2. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

Step 3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right

Step 4. Use the Square Root Property.

Step 5. Simplify the radical and then solve the two resulting equations.

Step 6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

**EXAMPLE 9.13**

Solve by completing the square: $x^2 + 4x = -21$.

**Solution**

| | $\overset{x^2\ +\ bx\qquad c}{x^2 + 4x = -21}$ |
|---|---|
| The variable terms are on the left side. Take half of 4 and square it. | $x^2 + 4x + \underset{\left(\frac{1}{2}\cdot 4\right)^2}{\underline{\qquad}} = -21$ |
| $\left(\frac{1}{2}(4)\right)^2 = 4$ | |
| Add 4 to both sides. | $x^2 + 4x + 4 = -21 + 4$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $(x+2)^2 = -17$ |
| Use the Square Root Property. | $x + 2 = \pm\sqrt{-17}$ |
| Simplify using complex numbers. | $x + 2 = \pm\sqrt{17}\,i$ |
| Subtract 2 from each side. | $x = -2 \pm \sqrt{17}\,i$ |
| Rewrite to show two solutions. | $x = -2 + \sqrt{17}\,i, \quad x = -2 - \sqrt{17}\,i$ |
| We leave the check to you. | |

**TRY IT : :** 9.25 Solve by completing the square: $y^2 - 10y = -35$.

**TRY IT : :** 9.26 Solve by completing the square: $z^2 + 8z = -19$.

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

**EXAMPLE 9.14**

Solve by completing the square: $y^2 - 18y = -6$.

**Solution**

| | $\overset{x^2\ -\ bx\qquad c}{y^2 - 18y = -6}$ |
|---|---|
| The variable terms are on the left side. Take half of $-18$ and square it. | |
| $\left(\frac{1}{2}(-18)\right)^2 = 81$ | $y^2 - 18y + \underset{\left(\frac{1}{2}\cdot(-18)\right)^2}{\underline{\qquad}} = -6$ |
| Add 81 to both sides. | $y^2 - 18y + 81 = -6 + 81$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $(y-9)^2 = 75$ |

| | |
|---|---|
| Use the Square Root Property. | $y-9=\pm\sqrt{75}$ |
| Simplify the radical. | $y-9=\pm 5\sqrt{3}$ |
| Solve for $y$. | $y=9\pm 5\sqrt{3}$ |
| Check. $y^2-18y=-6$; $(9+5\sqrt{3})^2-18(9+5\sqrt{3})\stackrel{?}{=}-6$; $81+90\sqrt{3}+75-162+90\sqrt{3}\stackrel{?}{=}-6$; $-6=-6$ ✓ | $y^2-18y=-6$; $(9-5\sqrt{3})^2-18(9-5\sqrt{3})\stackrel{?}{=}-6$; $81+90\sqrt{3}+75-162+90\sqrt{3}\stackrel{?}{=}-6$; $-6=-6$ ✓ |

Another way to check this would be to use a calculator. Evaluate $y^2-18y$ for both of the solutions. The answer should be $-6$.

**TRY IT : :** 9.27 Solve by completing the square: $x^2-16x=-16$.

**TRY IT : :** 9.28 Solve by completing the square: $y^2+8y=11$.

We will start the next example by isolating the variable terms on the left side of the equation.

**EXAMPLE 9.15**

Solve by completing the square: $x^2+10x+4=15$.

**Solution**

| | |
|---|---|
| | $x^2+10x+4=15$ |
| Isolate the variable terms on the left side.<br>Subtract 4 to get the constant terms on the right side. | $x^2+10x=11$ |
| Take half of 10 and square it. | |
| $\left(\frac{1}{2}(10)\right)^2=25$ | $x^2-10x+\underline{\quad\left(\frac{1}{2}\cdot(10)\right)^2\quad}=11$ |
| Add 25 to both sides. | $x^2+10x+25=11+25$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $(x+5)^2=36$ |
| Use the Square Root Property. | $x+5=\pm\sqrt{36}$ |
| Simplify the radical. | $x+5=\pm 6$ |
| Solve for $x$. | $x=-5\pm 6$ |
| Rewrite to show two solutions. | $x=-5+6,\quad x=-5-6$ |
| Solve the equations. | $x=1,\quad x=-11$ |

Check:

| | |
|---|---|
| $x^2 + 10x + 4 = 15$ | $x^2 + 10x + 4 = 15$ |
| $(1)^2 + 10(1) + 4 \stackrel{?}{=} 15$ | $(-11)^2 + 10(-11) + 4 \stackrel{?}{=} 15$ |
| $1 + 10 + 4 \stackrel{?}{=} 15$ | $121 + 110 + 4 \stackrel{?}{=} 15$ |
| $15 = 15 ✓$ | $15 = 15 ✓$ |

**TRY IT : : 9.29** Solve by completing the square: $a^2 + 4a + 9 = 30$.

**TRY IT : : 9.30** Solve by completing the square: $b^2 + 8b - 4 = 16$.

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

**EXAMPLE 9.16**

Solve by completing the square: $n^2 = 3n + 11$.

## Solution

| | |
|---|---|
| | $n^2 = 3n + 11$ |
| Subtract $3n$ to get the variable terms on the left side. | $n^2 - 3n = 11$ |
| Take half of $-3$ and square it. | |
| $\left(\frac{1}{2}(-3)\right)^2 = \frac{9}{4}$ | $n^2 - 3n + \underline{\quad\left(\frac{1}{2}\cdot(-3)\right)^2\quad} = 11$ |
| Add $\frac{9}{4}$ to both sides. | $n^2 - 3n + \frac{9}{4} = 11 + \frac{9}{4}$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $\left(n - \frac{3}{2}\right)^2 = \frac{44}{4} + \frac{9}{4}$ |
| Add the fractions on the right side. | $\left(n - \frac{3}{2}\right)^2 = \frac{53}{4}$ |
| Use the Square Root Property. | $n - \frac{3}{2} = \pm\sqrt{\frac{53}{4}}$ |
| Simplify the radical. | $n - \frac{3}{2} = \pm\frac{\sqrt{53}}{2}$ |
| Solve for $n$. | $n = \frac{3}{2} \pm \frac{\sqrt{53}}{2}$ |
| Rewrite to show two solutions. | $n = \frac{3}{2} + \frac{\sqrt{53}}{2}, \quad n = \frac{3}{2} - \frac{\sqrt{53}}{2}$ |
| Check:<br>We leave the check for you! | |

**TRY IT : : 9.31** Solve by completing the square: $p^2 = 5p + 9$.

> **TRY IT : :** 9.32 Solve by completing the square: $q^2 = 7q - 3.$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the Zero Product Property since it says "If $a \cdot b = 0,$ then $a = 0$ or $b = 0$." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

**EXAMPLE 9.17**

Solve by completing the square: $(x-3)(x+5) = 9.$

### Solution

| | |
|---|---|
| | $(x-3)(x+5) = 9$ |
| We multiply the binomials on the left. | $x^2 + 2x - 15 = 9$ |
| Add 15 to isolate the constant terms on the right. | $x^2 + 2x \quad = 24$ |
| Take half of 2 and square it. | |
| $\left(\frac{1}{2} \cdot (2)\right)^2 = 1$ | $x^2 + 2x + \underline{\quad\left(\frac{1}{2} \cdot (2)\right)^2\quad} = 24$ |
| Add 1 to both sides. | $x^2 + 2x + 1 = 24 + 1$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $(x+1)^2 = 25$ |
| Use the Square Root Property. | $x + 1 = \pm\sqrt{25}$ |
| Solve for $x$. | $x = -1 \pm 5$ |
| Rewrite to show two solutions. | $x = -1 + 5, x = -1 - 5$ |
| Simplify. | $x = 4, \quad x = -6$ |
| Check:<br>We leave the check for you! | |

> **TRY IT : :** 9.33 Solve by completing the square: $(c-2)(c+8) = 11.$

> **TRY IT : :** 9.34 Solve by completing the square: $(d-7)(d+3) = 56.$

## Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of completing the square works best when the coefficient of $x^2$ is 1, so the left side of the equation is of the form $x^2 + bx + c$. If the $x^2$ term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

**EXAMPLE 9.18**

Solve by completing the square: $3x^2 - 12x - 15 = 0.$

### Solution

To complete the square, we need the coefficient of $x^2$ to be one. If we factor out the coefficient of $x^2$ as a common factor, we can continue with solving the equation by completing the square.

| | |
|---|---|
| | $3x^2 - 12x - 15 = 0$ |
| Factor out the greatest common factor. | $3(x^2 - 4x - 5) = 0$ |
| Divide both sides by 3 to isolate the trinomial with coefficient 1. | $\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$ |
| Simplify. | $x^2 - 4x - 5 = 0$ |
| Add 5 to get the constant terms on the right side. | $x^2 - 4x \quad = 5$ |
| Take half of 4 and square it. | |
| $\left(\frac{1}{2}(-4)\right)^2 = 4$ | $x^2 - 4x + \underline{\quad\quad} = 5$ <br> $\left(\frac{1}{2} \cdot (4)\right)^2$ |
| Add 4 to both sides. | $x^2 - 4x + 4 = 5 + 4$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $(x - 2)^2 = 9$ |
| Use the Square Root Property. | $x - 2 = \pm\sqrt{9}$ |
| Solve for $x$. | $x - 2 = \pm 3$ |
| Rewrite to show two solutions. | $x = 2 + 3, \quad x = 2 - 3$ |
| Simplify. | $x = 5, \quad x = -1$ |
| Check: | |

$$\begin{array}{rr} x = 5 & x = -1 \\ 3x^2 - 12x - 15 = 0 & 3x^2 - 12x - 15 = 0 \\ 3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0 & 3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0 \\ 75 - 60 - 15 \stackrel{?}{=} 0 & 3 + 12 - 15 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$

**TRY IT : :** 9.35 Solve by completing the square: $2m^2 + 16m + 14 = 0$.

**TRY IT : :** 9.36 Solve by completing the square: $4n^2 - 24n - 56 = 8$.

To complete the square, the coefficient of the $x^2$ must be 1. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

**EXAMPLE 9.19**

Solve by completing the square: $2x^2 - 3x = 20$.

## Solution

To complete the square we need the coefficient of $x^2$ to be one. We will divide both sides of the equation by the coefficient of $x^2$. Then we can continue with solving the equation by completing the square.

| | |
|---|---|
| | $2x^2 - 3x = 20$ |
| Divide both sides by 2 to get the coefficient of $x^2$ to be 1. | $\frac{2x^2 - 3x}{2} = \frac{20}{2}$ |
| Simplify. | $x^2 - \frac{3}{2}x = 10$ |
| Take half of $-\frac{3}{2}$ and square it. | |
| $\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$ | $x^2 - \frac{3}{2}x + \underline{\quad\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2\quad} = 10$ |
| Add $\frac{9}{16}$ to both sides. | $x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$ |
| Add the fractions on the right side. | $\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$ |
| Use the Square Root Property. | $x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$ |
| Simplify the radical. | $x - \frac{3}{4} = \pm\frac{13}{4}$ |
| Solve for $x$. | $x = \frac{3}{4} \pm \frac{13}{4}$ |
| Rewrite to show two solutions. | $x = \frac{3}{4} + \frac{13}{4},\quad x = \frac{3}{4} - \frac{13}{4}$ |
| Simplify. | $x = 4,\quad x = -\frac{5}{2}$ |
| Check: We leave the check for you! | |

**TRY IT : :** 9.37 Solve by completing the square: $3r^2 - 2r = 21$.

**TRY IT : :** 9.38 Solve by completing the square: $4t^2 + 2t = 20$.

Now that we have seen that the coefficient of $x^2$ must be 1 for us to complete the square, we update our procedure for solving a quadratic equation by completing the square to include equations of the form $ax^2 + bx + c = 0$.

**HOW TO : :** SOLVE A QUADRATIC EQUATION OF THE FORM $ax^2 + bx + c = 0$ BY COMPLETING THE SQUARE.

Step 1. Divide by $a$ to make the coefficient of $x^2$ term 1.

Step 2. Isolate the variable terms on one side and the constant terms on the other.

Step 3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

Step 4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right

Step 5. Use the Square Root Property.

Step 6. Simplify the radical and then solve the two resulting equations.

Step 7. Check the solutions.

**EXAMPLE 9.20**

Solve by completing the square: $3x^2 + 2x = 4$.

## Solution

Again, our first step will be to make the coefficient of $x^2$ one. By dividing both sides of the equation by the coefficient of $x^2$, we can then continue with solving the equation by completing the square.

| | |
|---|---|
| | $3x^2 + 2x = 4$ |
| Divide both sides by 3 to make the coefficient of $x^2$ equal 1. | $\frac{3x^2 + 2x}{3} = \frac{4}{3}$ |
| Simplify. | $x^2 + \frac{2}{3}x = \frac{4}{3}$ |
| Take half of $\frac{2}{3}$ and square it. | |
| $\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2 = \frac{1}{9}$ | $x^2 + \frac{2}{3}x + \underset{\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2}{\underline{\quad}} = \frac{4}{3}$ |
| Add $\frac{1}{9}$ to both sides. | $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$ |
| Factor the perfect square trinomial, writing it as a binomial squared. | $\left(x + \frac{1}{3}\right)^2 = \frac{12}{9} + \frac{1}{9}$ |
| Use the Square Root Property. | $x + \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$ |
| Simplify the radical. | $x + \frac{1}{3} = \pm\frac{\sqrt{13}}{3}$ |
| Solve for $x$. | $x = -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$ |
| Rewrite to show two solutions. | $x = -\frac{1}{3} + \frac{\sqrt{13}}{3}, \quad x = -\frac{1}{3} - \frac{\sqrt{13}}{3}$ |
| Check:<br>We leave the check for you! | |

**TRY IT : :** 9.39 Solve by completing the square: $4x^2 + 3x = 2$.

**TRY IT : :** 9.40 Solve by completing the square: $3y^2 - 10y = -5$.

**MEDIA : :**

Access these online resources for additional instruction and practice with completing the square.

- **Completing Perfect Square Trinomials (https://openstax.org/l/37CompTheSq1)**
- **Completing the Square 1 (https://openstax.org/l/37CompTheSq2)**
- **Completing the Square to Solve Quadratic Equations (https://openstax.org/l/37CompTheSq3)**
- **Completing the Square to Solve Quadratic Equations: More Examples (https://openstax.org/l/37CompTheSq4)**
- **Completing the Square 4 (https://openstax.org/l/37CompTheSq5)**

## 9.2 EXERCISES

### Practice Makes Perfect

#### Complete the Square of a Binomial Expression

*In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.*

**71.**
ⓐ $m^2 - 24m$
ⓑ $x^2 - 11x$
ⓒ $p^2 - \frac{1}{3}p$

**72.**
ⓐ $n^2 - 16n$
ⓑ $y^2 + 15y$
ⓒ $q^2 + \frac{3}{4}q$

**73.**
ⓐ $p^2 - 22p$
ⓑ $y^2 + 5y$
ⓒ $m^2 + \frac{2}{5}m$

**74.**
ⓐ $q^2 - 6q$
ⓑ $x^2 - 7x$
ⓒ $n^2 - \frac{2}{3}n$

#### Solve Quadratic Equations of the form $x^2 + bx + c = 0$ by Completing the Square

*In the following exercises, solve by completing the square.*

**75.** 5. $u^2 + 2u = 3$

**76.** $z^2 + 12z = -11$

**77.** $x^2 - 20x = 21$

**78.** $y^2 - 2y = 8$

**79.** $m^2 + 4m = -44$

**80.** $n^2 - 2n = -3$

**81.** $r^2 + 6r = -11$

**82.** $t^2 - 14t = -50$

**83.** $a^2 - 10a = -5$

**84.** $b^2 + 6b = 41$

**85.** $x^2 + 5x = 2$

**86.** $y^2 - 3y = 2$

**87.** $u^2 - 14u + 12 = 1$

**88.** $z^2 + 2z - 5 = 2$

**89.** $r^2 - 4r - 3 = 9$

**90.** $t^2 - 10t - 6 = 5$

**91.** $v^2 = 9v + 2$

**92.** $w^2 = 5w - 1$

**93.** $x^2 - 5 = 10x$

**94.** $y^2 - 14 = 6y$

**95.** $(x + 6)(x - 2) = 9$

**96.** $(y + 9)(y + 7) = 80$

**97.** $(x + 2)(x + 4) = 3$

**98.** $(x - 2)(x - 6) = 5$

#### Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

*In the following exercises, solve by completing the square.*

**99.** $3m^2 + 30m - 27 = 6$

**100.** $2x^2 - 14x + 12 = 0$

**101.** $2n^2 + 4n = 26$

**102.** $5x^2 + 20x = 15$

**103.** $2c^2 + c = 6$

**104.** $3d^2 - 4d = 15$

**105.** $2x^2 + 7x - 15 = 0$

**106.** $3x^2 - 14x + 8 = 0$

**107.** $2p^2 + 7p = 14$

**108.** $3q^2 - 5q = 9$

**109.** $5x^2 - 3x = -10$

**110.** $7x^2 + 4x = -3$

## Writing Exercises

**111.** Solve the equation $x^2 + 10x = -25$

ⓐ by using the Square Root Property

ⓑ by Completing the Square

ⓒ Which method do you prefer? Why?

**112.** Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can… | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| complete the square of a binomial expression. | | | |
| solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square. | | | |
| solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square. | | | |

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

# 9.3 Solve Quadratic Equations Using the Quadratic Formula

## Learning Objectives

**By the end of this section, you will be able to:**

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

### Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $b^2 - 4ab$ when $a = 3$ and $b = -2$.
   If you missed this problem, review **Example 1.21**.
2. Simplify: $\sqrt{108}$.
   If you missed this problem, review **Example 8.13**.
3. Simplify: $\sqrt{50}$.
   If you missed this problem, review **Example 8.76**.

## Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for *x*.

We start with the standard form of a quadratic equation and solve it for *x* by completing the square.

| | |
|---|---|
| | $ax^2 + bx + c = 0 \quad a \neq 0$ |
| Isolate the variable terms on one side. | $ax^2 + bx = -c$ |
| Make the coefficient of $x^2$ equal to 1, by dividing by *a*. | $\frac{ax^2}{a} + \frac{b}{a}x = -\frac{c}{a}$ |
| Simplify. | $x^2 + \frac{b}{a}x = -\frac{c}{a}$ |
| To complete the square, find $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ and add it to both sides of the equation. | |
| $\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$ | $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + = -\frac{c}{a} + \frac{b^2}{4a^2}$ |
| The left side is a perfect square, factor it. | $\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$ |
| Find the common denominator of the right side and write equivalent fractions with the common denominator. | $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c \cdot 4a}{a \cdot 4a}$ |

| | |
|---|---|
| Simplify. | $\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$ |
| Combine to one fraction. | $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$ |
| Use the square root property. | $x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$ |
| Simplify the radical. | $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$ |
| Add $-\frac{b}{2a}$ to both sides of the equation. | $x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$ |
| Combine the terms on the right side. | $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ |
| | This equation is the Quadratic Formula. |

**Quadratic Formula**

The solutions to a quadratic equation of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of *a*, *b*, and *c* from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the formula is an equation. Make sure you use both sides of the equation.

**EXAMPLE 9.21** HOW TO SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values. | This equation is in standard form. | $ax^2+bx+c=0$<br>$2x^2+9x-5=0$<br>$a=2,\ b=9,\ c=-5$ |
| **Step 2.** Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*. | Substitute in $a=2$, $b=9$, $c=-5$ | $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$<br>$x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$ |

| | | |
|---|---|---|
| **Step 3.** Simplify the fraction, and solve for *x*. | | $x = \frac{-9 \pm \sqrt{81-(-40)}}{4}$<br>$x = \frac{-9 \pm \sqrt{121}}{4}$<br>$x = \frac{-9 \pm 11}{4}$<br>$x = \frac{-9+11}{4}$ $\quad x = \frac{-9-11}{4}$<br>$x = \frac{2}{4}$ $\quad x = \frac{-20}{4}$<br>$x = \frac{1}{2}$ $\quad x = -5$ |
| **Step 4.** Check the solutions. | Put each answer in the original equation to check.<br>Substitute $x = \frac{1}{2}$. | $2x^2 + 9x - 5 = 0$<br>$2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$<br>$2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$<br>$2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$<br>$\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$<br>$\frac{10}{2} - 5 \stackrel{?}{=} 0$<br>$5 - 5 \stackrel{?}{=} 0$<br>$0 = 0$ ✓ |
| | Substitute $x = -5$. | $2x^2 + 9x - 5 = 0$<br>$2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$<br>$2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$<br>$50 - 45 - 5 \stackrel{?}{=} 0$<br>$0 = 0$ ✓ |

**TRY IT : :** 9.41 Solve by using the Quadratic Formula: $3y^2 - 5y + 2 = 0$.

**TRY IT : :** 9.42 Solve by using the Quadratic Formula: $4z^2 + 2z - 6 = 0$.

**HOW TO : :** SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA.

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of *a*, *b*, and *c*.

Step 2. Write the Quadratic Formula. Then substitute in the values of *a*, *b*, and *c*.

Step 3. Simplify.

Step 4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic

Formula is an EQUATION. Be sure you start with "$x$ =".

**EXAMPLE 9.22**

Solve by using the Quadratic Formula: $x^2 - 6x = -5$.

## Solution

| | $x^2 - 6x = -5$ |
|---|---|
| Write the equation in standard form by adding 5 to each side. | $x^2 - 6x + 5 = 0$ |
| This equation is now in standard form. | $ax^2 + bx + c = 0$<br>$x^2 - 6x + 5 = 0$ |
| Identify the values of $a$, $b$, $c$. | $a = 1,\ b = -6,\ c = 5$ |
| Write the Quadratic Formula. | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of $a$, $b$, $c$. | $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$ |
| Simplify. | $x = \frac{6 \pm \sqrt{36 - 20}}{2}$<br>$x = \frac{6 \pm \sqrt{16}}{2}$<br>$x = \frac{6 \pm 4}{2}$ |
| Rewrite to show two solutions. | $x = \frac{6+4}{2}, \quad x = \frac{6-4}{2}$ |
| Simplify. | $x = \frac{10}{2}, \quad x = \frac{2}{2}$ |
| | $x = 5, \quad x = 1$ |

Check:

$$\begin{array}{rr} x^2 - 6x + 5 = 0 & x^2 - 6x + 5 = 0 \\ 5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0 & 1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0 \\ 25 - 30 + 5 \stackrel{?}{=} 0 & 1 - 6 + 5 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$

**TRY IT : :** 9.43 Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

**TRY IT : :** 9.44 Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

**EXAMPLE 9.23**

Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

## Solution

| | $2x^2 + 10x + 11 = 0$ |
|---|---|
| This equation is in standard form. | $\begin{array}{l} ax^2 + bx + c = 0 \\ 2x^2 + 10x + 11 = 0 \end{array}$ |
| Identify the values of *a*, *b*, and *c*. | $a = 2,\ b = 10,\ c = 11$ |
| Write the Quadratic Formula. | $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of *a*, *b*, and *c*. | $x = \dfrac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$ |
| Simplify. | $x = \dfrac{-10 \pm \sqrt{100 - 88}}{4}$ |
| | $x = \dfrac{-10 \pm \sqrt{12}}{4}$ |
| Simplify the radical. | $x = \dfrac{-10 \pm 2\sqrt{3}}{4}$ |
| Factor out the common factor in the numerator. | $x = \dfrac{2(-5 \pm \sqrt{3})}{4}$ |
| Remove the common factors. | $x = \dfrac{-5 \pm \sqrt{3}}{2}$ |
| Rewrite to show two solutions. | $x = \dfrac{-5 + \sqrt{3}}{2},\quad x = \dfrac{-5 - \sqrt{3}}{2}$ |
| Check:<br>We leave the check for you! | |

**TRY IT : :** 9.45 Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

**TRY IT : :** 9.46 Solve by using the Quadratic Formula: $5n^2 + 4n - 4 = 0$.

When we substitute *a*, *b*, and *c* into the Quadratic Formula and the radicand is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

**EXAMPLE 9.24**

Solve by using the Quadratic Formula: $3p^2 + 2p + 9 = 0$.

## Solution

| | $3p^2 + 2p + 9 = 0$ |
|---|---|
| This equation is in standard form | $\begin{array}{l} ax^2 + bx + c = 0 \\ 3p^2 + 2p + 9 = 0 \end{array}$ |
| Identify the values of $a, b, c$. | $a = 3,\ b = 2,\ c = 9$ |
| Write the Quadratic Formula. | $p = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of $a, b, c$. | $p = \dfrac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$ |

| | |
|---|---|
| Simplify. | $p=\frac{-2\pm\sqrt{4-108}}{6}$ |
| | $p=\frac{-2\pm\sqrt{-104}}{6}$ |
| Simplify the radical using complex numbers. | $p=\frac{-2\pm\sqrt{104}\,i}{6}$ |
| Simplify the radical. | $p=\frac{-2\pm 2\sqrt{26}\,i}{6}$ |
| Factor the common factor in the numerator. | $p=\frac{2\left(-1\pm\sqrt{26}\,i\right)}{6}$ |
| Remove the common factors. | $p=\frac{-1\pm\sqrt{26}\,i}{3}$ |
| Rewrite in standard $a+bi$ form. | $p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$ |
| Write as two solutions. | $p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$ |

**TRY IT : :** 9.47 Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

**TRY IT : :** 9.48 Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2+bx+c=0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

**EXAMPLE 9.25**

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

## Solution

Our first step is to get the equation in standard form.

| | |
|---|---|
| | $x(x+6)+4=0$ |
| Distribute to get the equation in standard form. | $x^2+6x+4=0$ |
| This equation is now in standard form | $\begin{array}{c} ax^2+bx+c=0 \\ x^2+6x+4=0 \end{array}$ |
| Identify the values of $a, b, c$. | $a=1, b=6, c=4$ |
| Write the Quadratic Formula. | $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ |
| Then substitute in the values of $a, b, c$. | $x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$ |
| Simplify. | $x=\frac{-6\pm\sqrt{36-16}}{2}$ |
| | $x=\frac{-6\pm\sqrt{20}}{2}$ |
| Simplify the radical. | $x=\frac{-6\pm 2\sqrt{5}}{2}$ |

| | |
|---|---|
| Factor the common factor in the numerator. | $x=\frac{2(-3\pm 2\sqrt{5})}{2}$ |
| Remove the common factors. | $x=-3\pm 2\sqrt{5}$ |
| Write as two solutions. | $x=-3+2\sqrt{5},\quad x=-3-2\sqrt{5}$ |
| Check:<br>We leave the check for you! | |

**TRY IT : :** 9.49 Solve by using the Quadratic Formula: $x(x+2)-5=0$.

**TRY IT : :** 9.50 Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

**EXAMPLE 9.26**

Solve by using the Quadratic Formula: $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$.

## Solution

Our first step is to clear the fractions.

| | |
|---|---|
| | $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$ |
| Multiply both sides by the LCD, 6, to clear the fractions. | $6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$ |
| Multiply. | $3u^2+4u=2$ |
| Subtract 2 to get the equation in standard form. | $ax^2+bx+c=0$<br>$3u^2+4u-2=0$ |
| Identify the values of *a*, *b*, and *c*. | $a=3,\ b=4,\ c=-2$ |
| Write the Quadratic Formula. | $u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ |
| Then substitute in the values of *a*, *b*, and *c*. | $u=\frac{-(4)\pm\sqrt{(4)^2-4\cdot 3\cdot(-2)}}{2\cdot 3}$ |
| Simplify. | $u=\frac{-4\pm\sqrt{16+24}}{6}$ |
| | $u=\frac{-4\pm\sqrt{40}}{6}$ |
| Simplify the radical. | $u=\frac{-4\pm 2\sqrt{10}}{6}$ |
| Factor the common factor in the numerator. | $u=\frac{2(-2\pm\sqrt{10})}{6}$ |
| Remove the common factors. | $u=\frac{-2\pm\sqrt{10}}{3}$ |
| Rewrite to show two solutions. | $u=\frac{-2+\sqrt{10}}{3},\qquad u=\frac{-2-\sqrt{10}}{3}$ |

Check:
We leave the check for you!

**TRY IT : :** 9.51 Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

**TRY IT : :** 9.52 Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Think about the equation $(x - 3)^2 = 0$. We know from the Zero Product Property that this equation has only one solution, $x = 3$.

We will see in the next example how using the Quadratic Formula to solve an equation whose standard form is a perfect square trinomial equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0 , which leads to only one solution.

**EXAMPLE 9.27**

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

## Solution

| | |
|---|---|
| | $4x^2 - 20x = -25$ |
| Add 25 to get the equation in standard form. | $ax^2 + bx + c = 0$<br>$4x^2 - 20x + 25 = 0$ |
| Identify the values of *a*, *b*, and *c*. | $a = 4,\ b = -20,\ c = 25$ |
| Write the quadratic formula. | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of *a*, *b*, and *c*. | $x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$ |
| Simplify. | $x = \frac{20 \pm \sqrt{400 - 400}}{8}$ |
| | $x = \frac{20 \pm \sqrt{0}}{8}$ |
| Simplify the radical. | $x = \frac{20}{8}$ |
| Simplify the fraction. | $x = \frac{5}{2}$ |
| Check:<br>We leave the check for you! | |

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x - 5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

**TRY IT : :** 9.53 Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

**TRY IT : :** 9.54 Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

## Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

### Discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

the quantity $b^2 - 4ac$ is called the discriminant.

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

| Quadratic Equation (in standard form) | Discriminant $b^2 - 4ac$ | Value of the Discriminant | Number and Type of solutions |
|---|---|---|---|
| $2x^2 + 9x - 5 = 0$ | $9^2 - 4 \cdot 2(-5)$<br>$121$ | + | 2 real |
| $4x^2 - 20x + 25 = 0$ | $(-20)^2 - 4 \cdot 4 \cdot 25$<br>$0$ | 0 | 1 real |
| $3p^2 + 2p + 9 = 0$ | $2^2 - 4 \cdot 3 \cdot 9$<br>$-104$ | – | 2 complex |

When the discriminant is positive, the quadratic equation has **2 real solutions**. $x = \frac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x = \frac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x = \frac{-b \pm \sqrt{-}}{2a}$

### Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

**EXAMPLE 9.28**

Determine the number of solutions to each quadratic equation.

ⓐ $3x^2 + 7x - 9 = 0$ ⓑ $5n^2 + n + 4 = 0$ ⓒ $9y^2 - 6y + 1 = 0$.

## Solution

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

| | $3x^2 + 7x - 9 = 0$ |
|---|---|
| The equation is in standard form, identify $a, b,$ and $c$. | $a = 3, \quad b = 7, \quad c = -9$ |
| Write the discriminant. | $b^2 - 4ac$ |
| Substitute in the values of $a, b,$ and $c$. | $(7)^2 - 4 \cdot 3 \cdot (-9)$ |
| Simplify. | $49 + 108$ |
| | $157$ |

Since the discriminant is positive, there are 2 real solutions to the equation.

ⓑ

| | $5n^2 + n + 4 = 0$ |
|---|---|
| The equation is in standard form, identify $a, b,$ and $c$. | $a = 5, \quad b = 1, \quad c = 4$ |
| Write the discriminant. | $b^2 - 4ac$ |
| Substitute in the values of $a, b,$ and $c$. | $(1)^2 - 4 \cdot 5 \cdot 4$ |
| Simplify. | $1 - 80$ |
| | $-79$ |

Since the discriminant is negative, there are 2 complex solutions to the equation.

ⓒ

| | $9y^2 - 6y + 1 = 0$ |
|---|---|
| The equation is in standard form, identify $a, b,$ and $c$. | $a = 9, b = -6, c = 1$ |
| Write the discriminant. | $b^2 - 4ac$ |
| Substitute in the values of $a, b,$ and $c$. | $(-6)^2 - 4 \cdot 9 \cdot 1$ |
| Simplify. | $36 - 36$ |
| | $0$ |

Since the discriminant is 0, there is 1 real solution to the equation.

**TRY IT : :** 9.55 Determine the numberand type of solutions to each quadratic equation.

ⓐ $8m^2 - 3m + 6 = 0$ ⓑ $5z^2 + 6z - 2 = 0$ ⓒ $9w^2 + 24w + 16 = 0.$

**TRY IT : :** 9.56 Determine the number and type of solutions to each quadratic equation.

ⓐ $b^2 + 7b - 13 = 0$ ⓑ $5a^2 - 6a + 10 = 0$ ⓒ $4r^2 - 20r + 25 = 0.$

## Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

**Methods for Solving Quadratic Equations**

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2 = k$ or $a(x - h)^2 = k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

**HOW TO : :** IDENTIFY THE MOST APPROPRIATE METHOD TO SOLVE A QUADRATIC EQUATION.

Step 1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.

Step 2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.

Step 3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

**EXAMPLE 9.29**

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $5z^2 = 17$ ⓑ $4x^2 - 12x + 9 = 0$ ⓒ $8u^2 + 6u = 11$.

## Solution

ⓐ

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

ⓑ

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

ⓒ

| | |
|---|---|
| | $8u^2 + 6u = 11$ |
| Put the equation in standard form. | $8u^2 + 6u - 11 = 0$ |

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

**TRY IT : :** 9.57 Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $x^2 + 6x + 8 = 0$ ⓑ $(n-3)^2 = 16$ ⓒ $5p^2 - 6p = 9$.

**TRY IT : :** 9.58 Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $8a^2 + 3a - 9 = 0$ ⓑ $4b^2 + 4b + 1 = 0$ ⓒ $5c^2 = 125$.

**MEDIA : :**

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula (https://openstax.org/l/37QuadForm1)
- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions (https://openstax.org/l/37QuadForm2)
- Discriminant in Quadratic Formula (https://openstax.org/l/37QuadForm3)

# 9.3 EXERCISES

## Practice Makes Perfect

### Solve Quadratic Equations Using the Quadratic Formula

*In the following exercises, solve by using the Quadratic Formula.*

**113.** $4m^2 + m - 3 = 0$

**114.** $4n^2 - 9n + 5 = 0$

**115.** $2p^2 - 7p + 3 = 0$

**116.** $3q^2 + 8q - 3 = 0$

**117.** $p^2 + 7p + 12 = 0$

**118.** $q^2 + 3q - 18 = 0$

**119.** $r^2 - 8r = 33$

**120.** $t^2 + 13t = -40$

**121.** $3u^2 + 7u - 2 = 0$

**122.** $2p^2 + 8p + 5 = 0$

**123.** $2a^2 - 6a + 3 = 0$

**124.** $5b^2 + 2b - 4 = 0$

**125.** $x^2 + 8x - 4 = 0$

**126.** $y^2 + 4y - 4 = 0$

**127.** $3y^2 + 5y - 2 = 0$

**128.** $6x^2 + 2x - 20 = 0$

**129.** $2x^2 + 3x + 3 = 0$

**130.** $2x^2 - x + 1 = 0$

**131.** $8x^2 - 6x + 2 = 0$

**132.** $8x^2 - 4x + 1 = 0$

**133.** $(v + 1)(v - 5) - 4 = 0$

**134.** $(x + 1)(x - 3) = 2$

**135.** $(y + 4)(y - 7) = 18$

**136.** $(x + 2)(x + 6) = 21$

**137.** $\frac{1}{3}m^2 + \frac{1}{12}m = \frac{1}{4}$

**138.** $\frac{1}{3}n^2 + n = -\frac{1}{2}$

**139.** $\frac{3}{4}b^2 + \frac{1}{2}b = \frac{3}{8}$

**140.** $\frac{1}{9}c^2 + \frac{2}{3}c = 3$

**141.** $16c^2 + 24c + 9 = 0$

**142.** $25d^2 - 60d + 36 = 0$

**143.** $25q^2 + 30q + 9 = 0$

**144.** $16y^2 + 8y + 1 = 0$

### Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

*In the following exercises, determine the number of solutions for each quadratic equation.*

**145.**
ⓐ $4x^2 - 5x + 16 = 0$
ⓑ $36y^2 + 36y + 9 = 0$
ⓒ $6m^2 + 3m - 5 = 0$

**146.**
ⓐ $9v^2 - 15v + 25 = 0$
ⓑ $100w^2 + 60w + 9 = 0$
ⓒ $5c^2 + 7c - 10 = 0$

**147.**
ⓐ $r^2 + 12r + 36 = 0$
ⓑ $8t^2 - 11t + 5 = 0$
ⓒ $3v^2 - 5v - 1 = 0$

**148.**
ⓐ $25p^2 + 10p + 1 = 0$
ⓑ $7q^2 - 3q - 6 = 0$
ⓒ $7y^2 + 2y + 8 = 0$

### Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

*In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.*

**149.**

ⓐ $x^2 - 5x - 24 = 0$

ⓑ $(y + 5)^2 = 12$

ⓒ $14m^2 + 3m = 11$

**150.**

ⓐ $(8v + 3)^2 = 81$

ⓑ $w^2 - 9w - 22 = 0$

ⓒ $4n^2 - 10 = 6$

**151.**

ⓐ $6a^2 + 14 = 20$

ⓑ $\left(x - \frac{1}{4}\right)^2 = \frac{5}{16}$

ⓒ $y^2 - 2y = 8$

**152.**

ⓐ $8b^2 + 15b = 4$

ⓑ $\frac{5}{9}v^2 - \frac{2}{3}v = 1$

ⓒ $\left(w + \frac{4}{3}\right)^2 = \frac{2}{9}$

## Writing Exercises

**153.** Solve the equation $x^2 + 10x = 120$

ⓐ by completing the square

ⓑ using the Quadratic Formula

ⓒ Which method do you prefer? Why?

**154.** Solve the equation $12y^2 + 23y = 24$

ⓐ by completing the square

ⓑ using the Quadratic Formula

ⓒ Which method do you prefer? Why?

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| solve quadratic equations using the quadratic formula. | | | |
| use the discriminant to predict the number of solutions of a quadratic equation. | | | |
| identify the most appropriate method to use to solve a quadratic equation. | | | |

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

# 9.4 Solve Quadratic Equations in Quadratic Form

## Learning Objectives

**By the end of this section, you will be able to:**

- Solve equations in quadratic form

### Be Prepared!

Before you get started, take this readiness quiz.

1. Factor by substitution: $y^4 - y^2 - 20$.
   If you missed this problem, review Example 6.21.
2. Factor by substitution: $(y-4)^2 + 8(y-4) + 15$.
   If you missed this problem, review Example 6.22.
3. Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$ ⓑ $\left(x^{\frac{1}{3}}\right)^2$ ⓒ $\left(x^{-1}\right)^2$.
   If you missed this problem, review Example 8.33.

## Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard $u$ for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is $x^2$ and its square, $x^4$, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$.) So we let $u = x^2$ and factored.

| | |
|---|---|
| | $x^4 - 4x^2 - 5$ |
| | $(x^2)^2 - 4(x^2) - 5$ |
| Let $u = x^2$ and substitute. | $u^2 - 4u - 5$ |
| Factor the trinomial. | $(u+1)(u-5)$ |
| Replace $u$ with $x^2$. | $(x^2+1)(x^2-5)$ |

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, $x$, and its square, $x^2$, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard $u$ to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax^2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

**EXAMPLE 9.30** HOW TO SOLVE EQUATIONS IN QUADRATIC FORM

Solve: $6x^4 - 7x^2 + 2 = 0$

## Solution

| | | |
|---|---|---|
| **Step 1.** Identify a substitution that will put the equation in quadratic form. | Since $(x^2)^2 = x^4$, we let $u = x^2$. | $6x^4 - 7x^2 + 2 = 0$ |
| **Step 2.** Rewrite the equation with the substitution to put it in quadratic form. | Rewrite to prepare for the substitution.<br>Substitute $u = x^2$. | $6(x^2)^2 - 7x^2 + 2 = 0$<br>$6u^2 - 7u + 2 = 0$ |
| **Step 3.** Solve the quadratic equation for *u*. | We can solve by factoring.<br>Use the Zero Product Property. | $(2u - 1)(3u - 2) = 0$<br>$2u - 1 = 0,\ 3u - 2 = 0$<br>$2u = 1,\ 3u = 2$<br>$u = \frac{1}{2} \quad u = \frac{2}{3}$ |
| **Step 4.** Substitute the original variable back into the results, using the substitution. | Replace *u* with $x^2$. | $x^2 = \frac{1}{2} \qquad x^2 = \frac{2}{3}$ |
| **Step 5.** Solve for the original variable. | Solve for *x*, using the Square Root Property. | $x = \pm\sqrt{\frac{1}{2}} \qquad x = \pm\sqrt{\frac{2}{3}}$<br>$x = \pm\frac{\sqrt{2}}{2} \qquad x = \pm\frac{\sqrt{6}}{3}$<br>There are four solutions.<br>$x = \frac{\sqrt{2}}{2} \qquad x = \frac{\sqrt{6}}{3}$<br>$x = -\frac{\sqrt{2}}{2} \qquad x = -\frac{\sqrt{6}}{3}$ |
| **Step 6.** Check the solutions. | Check all four solutions.<br>We will show one check here. | $x = \frac{\sqrt{2}}{2}$<br>$6x^4 - 7x^2 + 2 = 0$<br>$6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$<br>$6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right)^2 + 2 \stackrel{?}{=} 0$<br>$\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$<br>$0 = 0 \checkmark$<br>We leave the other checks to you! |

**TRY IT : :** 9.59 Solve: $x^4 - 6x^2 + 8 = 0$.

**TRY IT : :** 9.60 Solve: $x^4 - 11x^2 + 28 = 0$.

We summarize the steps to solve an equation in quadratic form.

**HOW TO : :** SOLVE EQUATIONS IN QUADRATIC FORM.

Step 1. Identify a substitution that will put the equation in quadratic form.
Step 2. Rewrite the equation with the substitution to put it in quadratic form.
Step 3. Solve the quadratic equation for *u*.
Step 4. Substitute the original variable back into the results, using the substitution.
Step 5. Solve for the original variable.
Step 6. Check the solutions.

In the next example, the binomial in the middle term, $(x - 2)$ is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

**EXAMPLE 9.31**

Solve: $(x-2)^2 + 7(x-2) + 12 = 0$.

**Solution**

| | |
|---|---|
| | $(x-2)^2 + 7(x-2) + 12 = 0$ |
| Prepare for the substitution. | $(x-2)^2 + 7(x-2) + 12 = 0$ |
| Let $u = x - 2$ and substitute. | $u^2 + 7u + 12 = 0$ |
| Solve by factoring. | $(u+3)(u+4) = 0$<br>$u + 3 = 0, \quad u + 4 = 0$<br>$u = -3, \quad u = -4$ |
| Replace $u$ with $x - 2$. | $x - 2 = -3, \quad x - 2 = -4$ |
| Solve for $x$. | $x = -1, \quad x = -2$ |

Check:

$$x = -1$$
$$(x-2)^2 + 7(x-2) + 12 = 0$$
$$(-1-2)^2 + 7(-1-2) + 12 \stackrel{?}{=} 0$$
$$(-3)^2 + 7(-3) + 12 \stackrel{?}{=} 0$$
$$9 - 21 + 12 \stackrel{?}{=} 0$$
$$0 = 0 \checkmark$$

$$x = -2$$
$$(x-2)^2 + 7(x-2) + 12 = 0$$
$$(-2-2)^2 + 7(-2-2) + 12 \stackrel{?}{=} 0$$
$$(-4)^2 + 7(-4) + 12 \stackrel{?}{=} 0$$
$$16 - 28 + 12 \stackrel{?}{=} 0$$
$$0 = 0 \checkmark$$

**TRY IT : :** 9.61 Solve: $(x-5)^2 + 6(x-5) + 8 = 0$.

**TRY IT : :** 9.62 Solve: $(y-4)^2 + 8(y-4) + 15 = 0$.

In the next example, we notice that $(\sqrt{x})^2 = x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

**EXAMPLE 9.32**

Solve: $x - 3\sqrt{x} + 2 = 0.$

## Solution

The $\sqrt{x}$ in the middle term, is squared in the first term $(\sqrt{x})^2 = x.$ If we let $u = \sqrt{x}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

| | |
|---|---|
| | $x - 3\sqrt{x} + 2 = 0$ |
| Rewrite the trinomial to prepare for the substitution. | $(\sqrt{x})^2 - 3\sqrt{x} + 2 = 0$ |
| Let $u = \sqrt{x}$ and substitute. | $u^2 - 3u + 2 = 0$ |
| Solve by factoring. | $(u-2)(u-1) = 0$ |
| | $u - 2 = 0, \quad u - 1 = 0$ |
| | $u = 2, \quad u = 1$ |
| Replace $u$ with $\sqrt{x}$. | $\sqrt{x} = 2, \quad \sqrt{x} = 1$ |
| Solve for $x$, by squaring both sides. | $x = 4, \quad x = 1$ |
| Check: | |

| $x = 4$ | $x = 1$ |
|---|---|
| $x - 3\sqrt{x} + 2 = 0$ | $x - 3\sqrt{x} + 2 = 0$ |
| $4 - 3\sqrt{4} + 2 \stackrel{?}{=} 0$ | $1 - 3\sqrt{1} + 2 \stackrel{?}{=} 0$ |
| $4 - 6 + 2 \stackrel{?}{=} 0$ | $1 - 3 + 2 \stackrel{?}{=} 0$ |
| $0 = 0$ ✓ | $0 = 0$ ✓ |

**TRY IT : :** 9.63 Solve: $x - 7\sqrt{x} + 12 = 0.$

**TRY IT : :** 9.64 Solve: $x - 6\sqrt{x} + 8 = 0.$

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

### EXAMPLE 9.33

Solve: $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0.$

## Solution

The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}.$ If we let $u = x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

| | |
|---|---|
| | $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$ |
| Rewrite the trinomial to prepare for the substitution. | $\left(x^{\frac{1}{3}}\right)^2 - 2\left(x^{\frac{1}{3}}\right) - 24 = 0$ |
| Let $u = x^{\frac{1}{3}}$ and substitute. | $u^2 - 2u - 24 = 0$ |

| | |
|---|---|
| Solve by factoring. | $(u-6)(u+4)=0$<br>$u-6=0,\quad u+4=0$<br>$u=6,\quad u=-4$ |
| Replace $u$ with $x^{\frac{1}{3}}$. | $x^{\frac{1}{3}}=6,\quad x^{\frac{1}{3}}=-4$ |
| Solve for $x$ by cubing both sides. | $\left(x^{\frac{1}{3}}\right)^3=(6)^3,\quad \left(x^{\frac{1}{3}}\right)^3=(-4)^3$<br>$x=216,\quad x=-64$ |
| Check:<br>$x=216$<br>$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$<br>$(216)^{\frac{2}{3}}-2(216)^{\frac{1}{3}}-24\stackrel{?}{=}0$<br>$36-12-24\stackrel{?}{=}0$<br>$0=0✓$<br><br>$x=-64$<br>$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$<br>$(-64)^{\frac{2}{3}}-2(-64)^{\frac{1}{3}}-24\stackrel{?}{=}0$<br>$16+8-24\stackrel{?}{=}0$<br>$0=0✓$ | |

**TRY IT : :** 9.65 Solve: $x^{\frac{2}{3}}-5x^{\frac{1}{3}}-14=0.$

**TRY IT : :** 9.66 Solve: $x^{\frac{1}{2}}+8x^{\frac{1}{4}}+15=0.$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

**EXAMPLE 9.34**

Solve: $3x^{-2}-7x^{-1}+2=0.$

## Solution

The $x^{-1}$ in the middle term is squared in the first term $\left(x^{-1}\right)^2=x^{-2}$. If we let $u=x^{-1}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

| | |
|---|---|
| | $3x^{-2}-7x^{-1}+2=0$ |
| Rewrite the trinomial to prepare for the substitution. | $3(x^{-1})^2-7(x^{-1})+2=0$ |
| Let $u=x^{-1}$ and substitute. | $3u^2-7u+2=0$ |
| Solve by factoring. | $(3u-1)(u-2)=0$ |
| | $3u-1=0,\quad u-2=0$ |
| | $u=\frac{1}{3},\quad u=2$ |
| Replace $u$ with $x^{-1}$. | $x^{-1}=\frac{1}{3},\quad x^{-1}=2$ |
| Solve for $x$ by taking the reciprocal since $x^{-1}=\frac{1}{x}$. | $x=3,\quad x=\frac{1}{2}$ |

Check:

$$\begin{array}{rr} x=3 & x=\frac{1}{2} \\ 3x^{-2}-7x^{-1}+2=0 & 3x^{-2}-7x^{-1}+2=0 \\ 3(3)^{-2}-7(3)^{-1}+2\stackrel{?}{=}0 & 3\left(\frac{1}{2}\right)^{-2}-7\left(\frac{1}{2}\right)^{-1}+2\stackrel{?}{=}0 \\ 3\left(\frac{1}{9}\right)-7\left(\frac{1}{3}\right)+2\stackrel{?}{=}0 & 3(4)-7(2)+2\stackrel{?}{=}0 \\ \frac{1}{3}-\left(\frac{7}{3}\right)+\frac{6}{3}\stackrel{?}{=}0 & 12-14+2\stackrel{?}{=}0 \\ 0=0\checkmark & 0=0\checkmark \end{array}$$

**TRY IT : :** 9.67 Solve: $8x^{-2}-10x^{-1}+3=0$.

**TRY IT : :** 9.68 Solve: $6x^{-2}-23x^{-1}+20=0$.

**MEDIA : :**

Access this online resource for additional instruction and practice with solving quadratic equations.

- **Solving Equations in Quadratic Form (https://openstax.org/l/37QuadForm4)**

# 9.4 EXERCISES

## Practice Makes Perfect

### Solve Equations in Quadratic Form

*In the following exercises, solve.*

**155.** $x^4 - 7x^2 + 12 = 0$

**156.** $x^4 - 9x^2 + 18 = 0$

**157.** $x^4 - 13x^2 - 30 = 0$

**158.** $x^4 + 5x^2 - 36 = 0$

**159.** $2x^4 - 5x^2 + 3 = 0$

**160.** $4x^4 - 5x^2 + 1 = 0$

**161.** $2x^4 - 7x^2 + 3 = 0$

**162.** $3x^4 - 14x^2 + 8 = 0$

**163.** $(x - 3)^2 - 5(x - 3) - 36 = 0$

**164.** $(x + 2)^2 - 3(x + 2) - 54 = 0$

**165.** $(3y + 2)^2 + (3y + 2) - 6 = 0$

**166.** $(5y - 1)^2 + 3(5y - 1) - 28 = 0$

**167.** $\left(x^2 + 1\right)^2 - 5\left(x^2 + 1\right) + 4 = 0$

**168.** $\left(x^2 - 4\right)^2 - 4\left(x^2 - 4\right) + 3 = 0$

**169.** $2\left(x^2 - 5\right)^2 - 5\left(x^2 - 5\right) + 2 = 0$

**170.** $2\left(x^2 - 5\right)^2 - 7\left(x^2 - 5\right) + 6 = 0$

**171.** $x - \sqrt{x} - 20 = 0$

**172.** $x - 8\sqrt{x} + 15 = 0$

**173.** $x + 6\sqrt{x} - 16 = 0$

**174.** $x + 4\sqrt{x} - 21 = 0$

**175.** $6x + \sqrt{x} - 2 = 0$

**176.** $6x + \sqrt{x} - 1 = 0$

**177.** $10x - 17\sqrt{x} + 3 = 0$

**178.** $12x + 5\sqrt{x} - 3 = 0$

**179.** $x^{\frac{2}{3}} + 9x^{\frac{1}{3}} + 8 = 0$

**180.** $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} = 28$

**181.** $x^{\frac{2}{3}} + 4x^{\frac{1}{3}} = 12$

**182.** $x^{\frac{2}{3}} - 11x^{\frac{1}{3}} + 30 = 0$

**183.** $6x^{\frac{2}{3}} - x^{\frac{1}{3}} = 12$

**184.** $3x^{\frac{2}{3}} - 10x^{\frac{1}{3}} = 8$

**185.** $8x^{\frac{2}{3}} - 43x^{\frac{1}{3}} + 15 = 0$

**186.** $20x^{\frac{2}{3}} - 23x^{\frac{1}{3}} + 6 = 0$

**187.** $x + 8x^{\frac{1}{2}} + 7 = 0$

**188.** $2x - 7x^{\frac{1}{2}} = 15$

**189.** $6x^{-2} + 13x^{-1} + 5 = 0$

**190.** $15x^{-2} - 26x^{-1} + 8 = 0$

**191.** $8x^{-2} - 2x^{-1} - 3 = 0$

**192.** $15x^{-2} - 4x^{-1} - 4 = 0$

## Writing Exercises

**193.** Explain how to recognize an equation in quadratic form.

**194.** Explain the procedure for solving an equation in quadratic form.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| solve equations in quadratic form. | | | |

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

# 9.5 Solve Applications of Quadratic Equations

## Learning Objectives

**By the end of this section, you will be able to:**

- Solve applications modeled by quadratic equations

### Be Prepared!

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is −100. Find the numbers.
   If you missed this problem, review Example 2.18.
2. Solve: $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.
   If you missed this problem, review Example 7.35.
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review Example 2.34.

## Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let's first summarize the methods we now have to solve quadratic equations.

### Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As you solve each equation, choose the method that is most convenient for you to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

**HOW TO : :** USE A PROBLEM-SOLVING STRATEGY.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.

Step 5. **Solve** the equation using algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence

We have solved number applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one $n$, then the next one is $n + 2$. The next one would be $n + 2 + 2$ or $n + 4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

| | **Consecutive even integers** | | **Consecutive odd integers** |
|---|---|---|---|
| | 64, 66, 68 | | 77, 79, 81 |
| $n$ | $1^{st}$ even integer | $n$ | $1^{st}$ odd integer |
| $n+2$ | $2^{nd}$ consecutive even integer | $n+2$ | $2^{nd}$ consecutive odd integer |
| $n+4$ | $3^{rd}$ consecutive even integer | $n+4$ | $3^{rd}$ consecutive odd integer |

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

**EXAMPLE 9.35**

The product of two consecutive odd integers is 195. Find the integers.

## Solution

**Step 1. Read** the problem.

**Step 2. Identify** what we are looking for. — We are looking for two consecutive odd integers.

**Step 3. Name** what we are looking for. — Let $n =$ the fir t odd integer.
$n + 2 =$ the next odd integer

**Step 4. Translate** into an equation. State the problem in one sentence. — "The product of two consecutive odd integers is 195."

The product of the fir t odd integer and the second odd integer is 195.

Translate into an equation.

$$n(n+2) = 195$$

**Step 5. Solve** the equation. Distribute.

$$n^2 + 2n = 195$$

Write the equation in standard form.

$$n^2 + 2n - 195 = 0$$

Factor.

$$(n+15)(n-13) = 0$$

Use the Zero Product Property.

$$n + 15 = 0 \quad n - 13 = 0$$

Solve each equation.

$$n = -15, \quad n = 13$$

There are two values of $n$ that are solutions. This will give us two pairs of consecutive odd integers for our solution.

| First odd integer $n = 13$ | First odd integer $n = -15$ |
|---|---|
| next odd integer $n + 2$ | next odd integer $n + 2$ |
| $13 + 2$ | $-15 + 2$ |
| 15 | $-13$ |

**Step 6. Check** the answer.

Do these pairs work?

Are they consecutive odd integers?

$13, 15$ yes
$-13, -15$ yes

Is their product 195?

$13 \cdot 15 = 195$ yes
$-13(-15) = 195$ yes

**Step 7. Answer** the question. — Two consecutive odd integers whose product is 195 are 13, 15 and $-13, -15$.

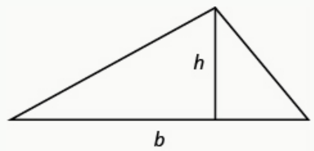
**TRY IT : :** 9.69 The product of two consecutive odd integers is 99. Find the integers.

**TRY IT : :** 9.70 The product of two consecutive even integers is 168. Find the integers.

We will use the formula for the area of a triangle to solve the next example.

### Area of a Triangle

For a triangle with base, *b*, and height, *h*, the area, *A*, is given by the formula $A = \frac{1}{2}bh$.

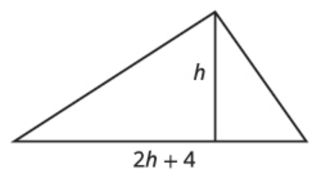



Recall that when we solve geometric applications, it is helpful to draw the figure.

### EXAMPLE 9.36

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

#### Solution

| | |
|---|---|
| **Step 1. Read** the problem.<br>Draw a picture. |   |
| **Step 2. Identify** what we are looking for. | We are looking for the base and height. |
| **Step 3. Name** what we are looking for. | Let *h* = the height of the triangle.<br>2*h* + 4 = the base of the triangle |
| **Step 4. Translate** into an equation.<br>We know the area. Write the formula for the area of a triangle. | $A = \frac{1}{2}bh$ |
| **Step 5. Solve** the equation.<br>Substitute in the values. | $120 = \frac{1}{2}(2h + 4)h$ |
| Distribute. | $120 = h^2 + 2h$ |
| This is a quadratic equation, rewrite it in standard form. | $h^2 + 2h - 120 = 0$ |
| Factor. | $(h - 10)(h + 12) = 0$ |
| Use the Zero Product Property. | $h - 10 = 0 \quad h + 12 = 0$ |
| Simplify. | $h = 10, \quad \cancel{h = -12}$ |

Since *h* is the height of a window, a value of *h* = −12 does not make sense.

The height of the triangle $h = 10$.

| | |
|---|---|
| | The base of the triangle $2h + 4.$<br>$2 \cdot 10 + 4$<br>$24$ |
| **Step 6. Check** the answer.<br>Does a triangle with height 10 and base 24 have area 120? Yes. | |
| **Step 7. Answer** the question. | The height of the triangular window is 10 feet and the base is 24 feet. |

**TRY IT : :** 9.71

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

**TRY IT : :** 9.72

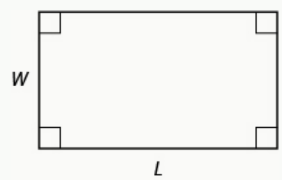
If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

**Area of a Rectangle**

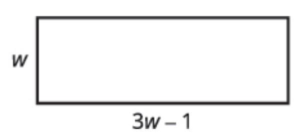
For a rectangle with length, *L*, and width, *W*, the area, *A*, is given by the formula *A* = *LW*.




**EXAMPLE 9.37**

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

## Solution

| | |
|---|---|
| Step 1. **Read** the problem.<br>Draw a picture. |  |
| Step 2. **Identify** what we are looking for. | We are looking for the length and width. |
| Step 3. **Name** what we are looking for. | Let $w =$ the width of the rectangle.<br>$3w - 1 =$ the length of the rectangle |

| | |
|---|---|
| Step 4. **Translate** into an equation. We know the area. Write the formula for the area of a rectangle. | $A = L \cdot W$ |
| Step 5. **Solve** the equation. Substitute in the values. | $150 = (3w - 1)w$ |
| Distribute. | $150 = 3w^2 - w$ |
| This is a quadratic equation; rewrite it in standard form. Solve the equation using the Quadratic Formula. | $ax^2 + bx + c = 0$ <br> $3w^2 - w - 150 = 0$ |
| Identify the $a, b, c$ values. | $a = 3, b = -1, c = -150$ |
| Write the Quadratic Formula. | $w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of $a, b, c$. | $w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$ |
| Simplify. | $w = \frac{1 \pm \sqrt{1 + 1800}}{6}$ <br> $w = \frac{1 \pm \sqrt{1801}}{6}$ |
| Rewrite to show two solutions. | $w = \frac{1 + \sqrt{1801}}{6}, \quad w = \frac{1 - \sqrt{1801}}{6}$ |
| Approximate the answers using a calculator. We eliminate the negative solution for the width. | $w \approx 7.2, \quad$ ~~$w \approx -6.9$~~ <br> Width $w \approx 7.2$ <br> Length $\approx 3w - 1$ <br> $\approx 3(7.2) - 1$ <br> $\approx 20.6$ |
| Step 6. **Check** the answer. Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150. | |
| Step 7. **Answer** the question. | The width of the rectangle is approximately 7.2 feet and the length is approximately 20.6 feet. |

**TRY IT : :** 9.73

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

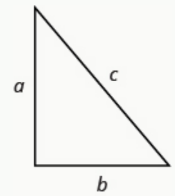
**TRY IT : :** 9.74

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot.?

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

**Pythagorean Theorem**

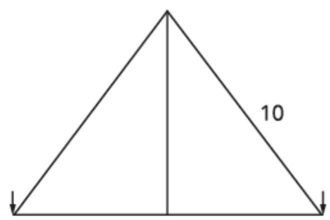
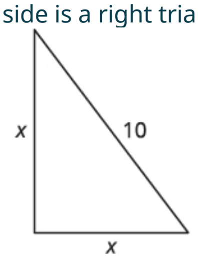
In any right triangle, where $a$ and $b$ are the lengths of the legs, and $c$ is the length of the hypotenuse, $a^2 + b^2 = c^2$.




**EXAMPLE 9.38**

Rene is setting up a holiday light display. He wants to make a ‘tree’ in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

**Solution**

| | |
|---|---|
| **Step 1. Read** the problem. Draw a picture. |  |
| **Step 2. Identify** what we are looking for. | We are looking for the height of the pole. |
| **Step 3. Name** what we are looking for. | The distance from the base of the pole to either stake is the same as the height of the pole.<br>Let $x =$ the height of the pole.<br>$x =$ the distance from pole to stake<br>Each side is a right triangle. We draw a picture of one of them.<br> |
| **Step 4. Translate** into an equation.<br>We can use the Pythagorean Theorem to solve for *x*.<br>Write the Pythagorean Theorem. | $a^2 + b^2 = c^2$ |
| **Step 5. Solve** the equation.<br>Substitute. | $x^2 + x^2 = 10^2$ |
| Simplify. | $2x^2 = 100$ |
| Divide by 2 to isolate the variable. | $\frac{2x^2}{2} = \frac{100}{2}$ |
| Simplify. | $x^2 = 50$ |

| | |
|---|---|
| Use the Square Root Property. | $x = \pm\sqrt{50}$ |
| Simplify the radical. | $x = \pm 5\sqrt{2}$ |
| Rewrite to show two solutions. | $x = 5\sqrt{2}$, ~~$x = -5\sqrt{2}$~~ |
| | If we approximate this number to the nearest tenth with a calculator, we find $x \approx 7.1$. |
| **Step 6. Check** the answer. Check on your own in the Pythagorean Theorem. | |
| **Step 7. Answer** the question. | The pole should be about 7.1 feet tall. |

**TRY IT : :** 9.75

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

**TRY IT : :** 9.76

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, $v_0$, propels the object up until gravity causes the object to fall back down.

**Projectile motion**

The height in feet, $h$, of an object shot upwards into the air with initial velocity, $v_0$, after $t$ seconds is given by the formula

$$h = -16t^2 + v_0 t$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

**EXAMPLE 9.39**

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

**Solution**

| | |
|---|---|
| **Step 1. Read** the problem. | |
| **Step 2. Identify** what we are looking for. | We are looking for the number of seconds, which is time. |
| **Step 3. Name** what we are looking for. | Let $t =$ the number of seconds. |
| **Step 4. Translate** into an equation. Use the formula. | $h = -16t^2 + v_0 t$ |
| **Step 5. Solve** the equation. We know the velocity $v_0$ is 130 feet per second. The height is 260 feet. Substitute the values. | $260 = -16t^2 + 130t$ |

| | |
|---|---|
| This is a quadratic equation, rewrite it in standard form. Solve the equation using the Quadratic Formula. | $\begin{array}{l} ax^2 + bx + c = 0 \\ 16t^2 - 130t + 260 = 0 \end{array}$ |
| Identify the values of $a$, $b$, $c$. | $a = 16,\ b = -130,\ c = 260$ |
| Write the Quadratic Formula. | $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of $a$, $b$, $c$ . | $t = \dfrac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$ |
| Simplify. | $t = \dfrac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ <br> $t = \dfrac{130 \pm \sqrt{260}}{32}$ |
| Rewrite to show two solutions. | $t = \dfrac{130 + \sqrt{260}}{32},\quad t = \dfrac{130 - \sqrt{260}}{32}$ |
| Approximate the answer with a calculator. | $t \approx 4.6$ seconds, $t \approx 3.6$ seconds |
| **Step 6. Check** the answer. The check is left to you. | |
| **Step 7. Answer** the question. | The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds. |

**TRY IT : :** 9.77

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h = -16t^2 + v_0t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

**TRY IT : :** 9.78

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h = -16t^2 + v_0t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

| | Rate | • Time | = Distance |
|---|---|---|---|
| | | | |
| | | | |
| | | | |

The formula $D = rt$ assumes we know $r$ and $t$ and use them to find $D$. If we know $D$ and $r$ and need to find $t$, we would solve the equation for $t$ and get the formula $t = \frac{D}{r}$.

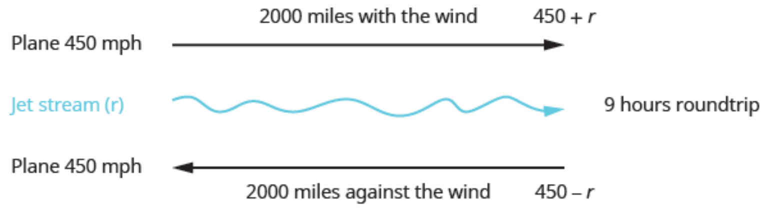
Some uniform motion problems are also modeled by quadratic equations.

**EXAMPLE 9.40**

Professor Smith just returned from a conference that was 2,000 miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

## Solution

This is a uniform motion situation. A diagram will help us visualize the situation.




We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is 450 + *r*.

When the plane flies against the wind, the wind decreases its speed and the rate is 450 − *r*.

Write in the rates.
Write in the distances.
Since $D = r \cdot t$, we solve for $t$ and get $t = \frac{D}{r}$.
We divide the distance by the rate in each row, and place the expression in the time column.

| Type | Rate | • | Time | = | Distance |
|---|---|---|---|---|---|
| Headwind | 450 – *r* | | $\frac{2000}{450 - r}$ | | 2000 |
| Tailwind | 450 + *r* | | $\frac{2000}{450 + r}$ | | 2000 |
| | | | 9 | | |

We know the times add to 9 and so we write our equation.

$$\frac{2000}{450 - r} + \frac{2000}{450 + r} = 9$$

We multiply both sides by the LCD.

$$(450 - r)(450 + r)\left(\frac{2000}{450 - r} + \frac{2000}{450 + r}\right) = 9(450 - r)(450 + r)$$

Simplify.

$$2000(450 + r) + 2000(450 - r) = 9(450 - r)(450 + r)$$

Factor the 2,000.

$$2000(450 + r + 450 - r) = 9\left(450^2 - r^2\right)$$

Solve.

$$2000(900) = 9\left(450^2 - r^2\right)$$

Divide by 9.

$$2000(100) = 450^2 - r^2$$

Simplify.

$$200000 = 202500 - r^2$$
$$-2500 = -r^2$$
$$50 = r \quad \text{The speed of the jet stream.}$$

Check:
Is 50 mph a reasonable speed for the jet stream? Yes.
If the plane is traveling 450 mph and the wind is 50 mph,
Tailwind

$450 + 50 = 500 \text{ mph} \qquad \frac{2000}{500} = 4 \text{ hours}$

Headwind

$450 - 50 = 400 \text{ mph} \qquad \frac{2000}{400} = 5 \text{ hours}$

The times add to 9 hours, so it checks.

The speed of the jet stream was 50 mph.

**TRY IT : :** 9.79

MaryAnne just returned from a visit with her grandchildren back east . The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

**TRY IT : :** 9.80

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We'll use a similar scenario now.

**EXAMPLE 9.41**

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

## Solution

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

Let $x =$ the number of hours for Press #2 to complete the job.
Enter the hours per job for Press #1, Press #2, and when they work together.

| | Number of hours needed to complete the job. | Part of job completed/hour |
|---|---|---|
| Press #1 | $x + 12$ | $\frac{1}{x+12}$ |
| Press #2 | $x$ | $\frac{1}{x}$ |
| Together | 8 | $\frac{1}{8}$ |

The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.
Translate to an equation.

Work completed by
Press #1 + Press #2 = Together

$$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$$

Solve.

$$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$$

Multiply by the LCD, $8x(x + 12)$.

$$8x(x+12)\left(\frac{1}{x+12} + \frac{1}{x}\right) = \left(\frac{1}{8}\right)8x(x+12)$$

Simplify.

$$8x + 8(x + 12) = x(x + 12)$$

$$8x + 8x + 96 = x^2 + 12x$$

$$0 = x^2 - 4x - 96$$

Solve.

$$0 = (x - 12)(x + 8)$$

$$x - 12 = 0,\ x + 8 = 0$$

$$x = 12,\ \cancel{x = -8} \text{ hours}$$

Since the idea of negative hours does not make sense, we use the value $x = 12$.

| | |
|---|---|
| 12 + 12 | 12 |
| 24 hours | 12 hours |

Write our sentence answer.

Press #1 would take 24 hours and
Press #2 would take 12 hours to do the job alone.

**TRY IT : :** 9.81

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

**TRY IT : :** 9.82

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

**MEDIA : :**

Access these online resources for additional instruction and practice with solving applications modeled by quadratic equations.

- **Word Problems Involving Quadratic Equations (https://openstax.org/l/37QuadForm5)**
- **Quadratic Equation Word Problems (https://openstax.org/l/37QuadForm6)**
- **Applying the Quadratic Formula (https://openstax.org/l/37QuadForm7)**

# 9.5 EXERCISES

## Practice Makes Pefect

### Solve Applications Modeled by Quadratic Equations

*In the following exercises, solve using any method.*

**195.** The product of two consecutive odd numbers is 255. Find the numbers.

**196.** The product of two consecutive even numbers is 360. Find the numbers.

**197.** The product of two consecutive even numbers is 624. Find the numbers.

**198.** The product of two consecutive odd numbers is 1,023. Find the numbers.

**199.** The product of two consecutive odd numbers is 483. Find the numbers.

**200.** The product of two consecutive even numbers is 528. Find the numbers.

*In the following exercises, solve using any method. Round your answers to the nearest tenth, if needed.*

**201.** A triangle with area 45 square inches has a height that is two less than four times the base Find the base and height of the triangle.

**202.** The base of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the base and height of the triangle.

**203.** The area of a triangular flower bed in the park has an area of 120 square feet. The base is 4 feet longer that twice the height. What are the base and height of the triangle?

**204.** A triangular banner for the basketball championship hangs in the gym. It has an area of 75 square feet. What is the length of the base and height , if the base is two-thirds of the height?

**205.** The length of a rectangular driveway is five feet more than three times the width. The area is 50 square feet. Find the length and width of the driveway.

**206.** A rectangular lawn has area 140 square yards. Its width that is six less than twice the length. What are the length and width of the lawn?

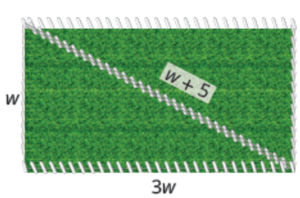
**207.** A rectangular table for the dining room has a surface area of 24 square feet. The length is two more feet than twice the width of the table. Find the length and width of the table.

**208.** The new computer has a surface area of 168 square inches. If the the width is 5.5 inches less that the length, what are the dimensions of the computer?

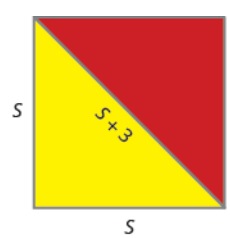
**209.** The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.

**210.** The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times as the length of the other leg . Round to the nearest tenth. Find the lengths of the three sides of the triangle.

**211.** A rectangular garden will be divided into two plots by fencing it on the diagonal. The diagonal distance from one corner of the garden to the opposite corner is five yards longer than the width of the garden. The length of the garden is three times the width. Find the length of the diagonal of the garden.




**212.** Nautical flags are used to represent letters of the alphabet. The flag for the letter, O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The hypotenuse of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.




**213.** Gerry plans to place a 25-foot ladder against the side of his house to clean his gutters. The bottom of the ladder will be 5 feet from the house.How for up the side of the house will the ladder reach?

**214.** John has a 10-foot piece of rope that he wants to use to support his 8-foot tree. How far from the base of the tree should he secure the rope?

**215.** A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when the height of the firework rocket will be 1200 feet.

**216.** An arrow is shot vertically upward at a rate of 220 feet per second. Use the projectile formula $h = -16t^2 + v_0 t$, to determine when height of the arrow will be 400 feet.

**217.** A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0 t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

**218.** A stone is dropped from a 196-foot platform. Use the formula $h = -16t^2 + v_0 t + 196$ to determine how many seconds it will take for the stone to hit the ground. (Since the stone is dropped, $v_0 = 0$.)

**219.** The businessman took a small airplane for a quick flight up the coast for a lunch meeting and then returned home. The plane flew a total of 4 hours and each way the trip was 200 miles. What was the speed of the wind that affected the plane which was flying at a speed of 120 mph?

**220.** The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 300 miles. If the plane was flying at 125 mph, what was the speed of the wind that affected the plane?

**221.** Roy kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

**222.** Rick paddled up the river, spent the night camping, and and then paddled back. He spent 10 hours paddling and the campground was 24 miles away. If Rick kayaked at a speed of 5 mph, what was the speed of the current?

**223.** Two painters can paint a room in 2 hours if they work together. The less experienced painter takes 3 hours more than the more experienced painter to finish the job. How long does it take for each painter to paint the room individually?

**224.** Two gardeners can do the weekly yard maintenance in 8 minutes if they work together. The older gardener takes 12 minutes more than the younger gardener to finish the job by himself. How long does it take for each gardener to do the weekly yard maintainence individually?

**225.** It takes two hours for two machines to manufacture 10,000 parts. If Machine #1 can do the job alone in one hour less than Machine #2 can do the job, how long does it take for each machine to manufacture 10,000 parts alone?

**226.** Sully is having a party and wants to fill his swimming pool. If he only uses his hose it takes 2 hours more than if he only uses his neighbor's hose. If he uses both hoses together, the pool fills in 4 hours. How long does it take for each hose to fill the hot tub?

## Writing Exercises

**227.** Make up a problem involving the product of two consecutive odd integers.

ⓐ Start by choosing two consecutive odd integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where $p$ is the product you found in part (b).

ⓓ Did you get the numbers you started with?

**228.** Make up a problem involving the product of two consecutive even integers.

ⓐ Start by choosing two consecutive even integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where $p$ is the product you found in part (b).

ⓓ Did you get the numbers you started with?

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| solve applications of the quadratic formula. | | | |

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

# 9.6 Graph Quadratic Functions Using Properties

## Learning Objectives

**By the end of this section, you will be able to:**

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

### Be Prepared!

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
   If you missed this problem, review **Example 3.54**.
2. Solve: $2x^2 + 3x - 2 = 0.$
   If you missed this problem, review **Example 6.45**.
3. Evaluate $-\frac{b}{2a}$ when *a* = 3 and *b* = −6.
   If you missed this problem, review **Example 1.21**.

## Recognize the Graph of a Quadratic Function

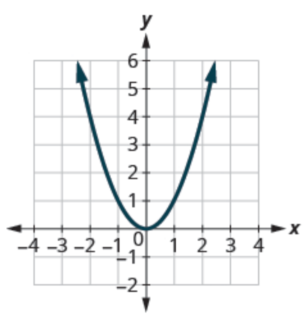
Previously we very briefly looked at the function $f(x) = x^2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f(x) = ax^2 + bx + c$ if $a \neq 0.$ We call this kind of function a quadratic function.

### Quadratic Function

A **quadratic function**, where *a*, *b*, and *c* are real numbers and $a \neq 0,$ is a function of the form

$$f(x) = ax^2 + bx + c$$

We graphed the quadratic function $f(x) = x^2$ by plotting points.




| $x$ | $f(x) = x^2$ | $(x, f(x))$ |
|---|---|---|
| −3 | 9 | (−3, 9) |
| −2 | 4 | (−2, 4) |
| −1 | 1 | (−1, 1) |
| 0 | 0 | (0, 0) |
| 1 | 1 | (1, 1) |
| 2 | 4 | (2, 4) |
| 3 | 9 | (3, 9) |

Every quadratic function has a graph that looks like this. We call this figure a **parabola**.

Let's practice graphing a parabola by plotting a few points.

**EXAMPLE 9.42**

Graph $f(x) = x^2 - 1.$

### Solution

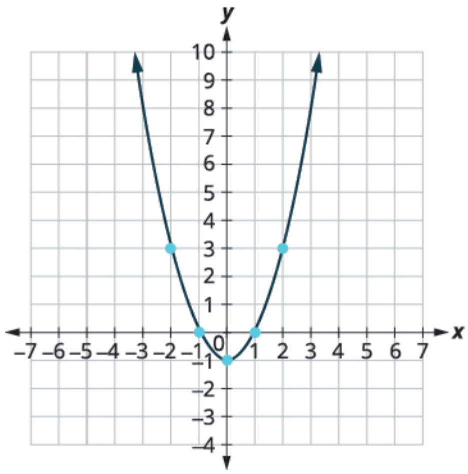
We will graph the function by plotting points.

Choose integer values for *x*, substitute them into the equation and simplify to find $f(x)$.

Record the values of the ordered pairs in the chart.

| $f(x) = x^2 - 1$ | |
|---|---|
| $x$ | $f(x)$ |
| 0 | –1 |
| 1 | 0 |
| –1 | 0 |
| 2 | 3 |
| –2 | 3 |

Plot the points, and then connect them with a smooth curve. The result will be the graph of the function $f(x) = x^2 - 1$.




**TRY IT : :** 9.83 Graph $f(x) = -x^2$..

**TRY IT : :** 9.84 Graph $f(x) = x^2 + 1$.

All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward. See **Figure 9.2**.

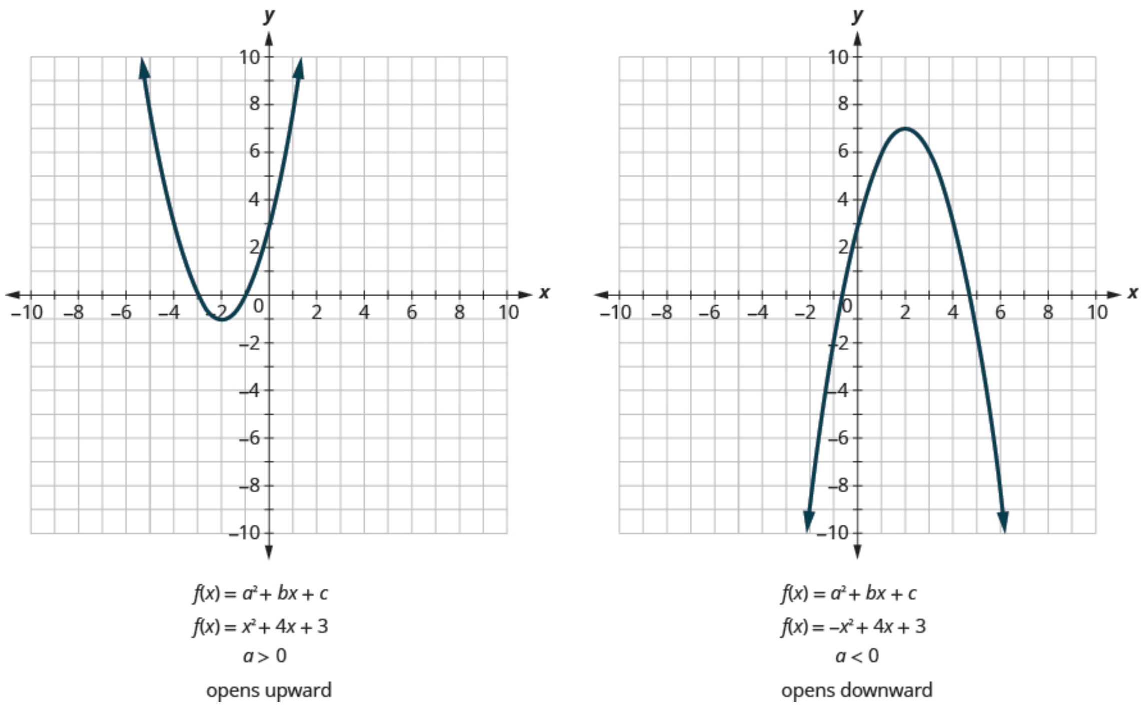



**Figure 9.2**

Notice that the only difference in the two functions is the negative sign before the quadratic term ($x^2$ in the equation of the graph in Figure 9.2). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

### Parabola Orientation

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward
- $a < 0$, the parabola opens downward

## EXAMPLE 9.43

Determine whether each parabola opens upward or downward:

ⓐ $f(x) = -3x^2 + 2x - 4$ ⓑ $f(x) = 6x^2 + 7x - 9$.

### Solution

ⓐ

| | |
|---|---|
| Find the value of "*a*". | $f(x) = ax^2 + bx + c$<br>$f(x) = -3x^2 + 2x - 4$<br>$a = -3$ |
| | Since the "*a*" is negative, the parabola will open downward. |

ⓑ

| | |
|---|---|
| Find the value of "*a*". | $f(x) = ax^2 + bx + c$<br>$f(x) = 6x^2 + 7x - 9$<br>$a = 6$ |
| | Since the "*a*" is positive, the parabola will open upward. |

**TRY IT : :** 9.85

Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ $f(x) = 2x^2 + 5x - 2$ ⓑ $f(x) = -3x^2 - 4x + 7$.

**TRY IT : :** 9.86

Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ $f(x) = -2x^2 - 2x - 3$ ⓑ $f(x) = 5x^2 - 2x - 1$.

## Find the Axis of Symmetry and Vertex of a Parabola

Look again at Figure 9.2. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry. See Figure 9.3.

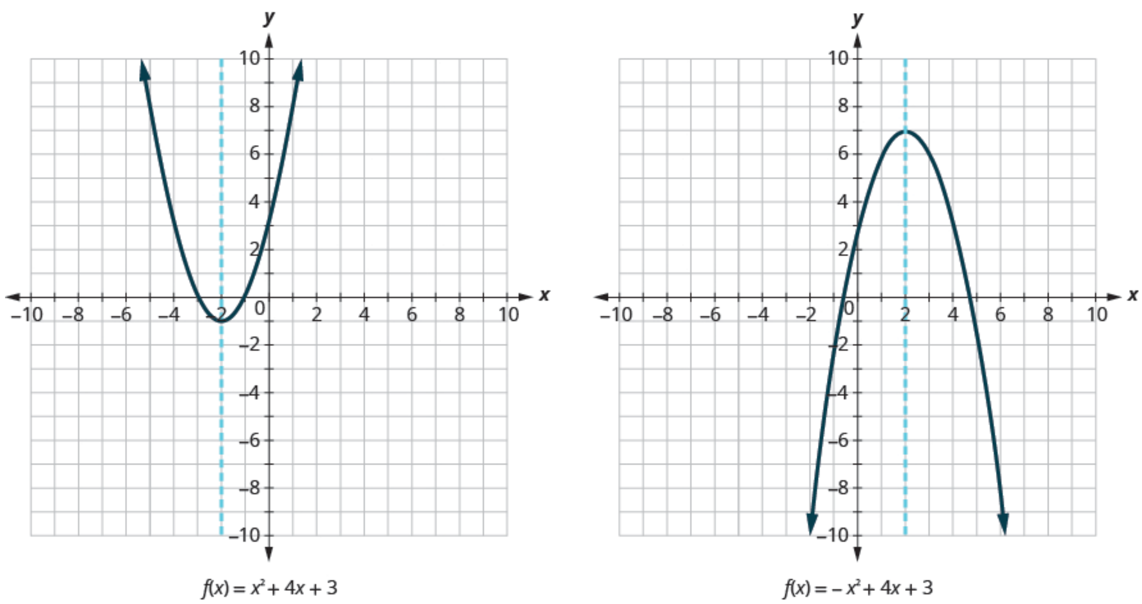



**Figure 9.3**

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

| $f(x) = ax^2 + bx + c$ | $f(x) = ax^2 + bx + c$ |
|---|---|
| $f(x) = x^2 + 4x + 3$ | $f(x) = -x^2 + 4x + 3$ |
| axis of symmetry | axis of symmetry |
| $x = -\frac{b}{2a}$ | $x = -\frac{b}{2a}$ |
| $x = -\frac{4}{2 \cdot 1}$ | $x = -\frac{4}{2(-1)}$ |
| $x = -2$ | $x = 2$ |

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its *x*-coordinate is $-\frac{b}{2a}$. To find the *y*-coordinate of the vertex we substitute the value of the *x*-coordinate into the quadratic function.

| $f(x) = x^2 + 4x + 3$ | $f(x) = -x^2 + 4x + 3$ |
|---|---|
| axis of symmetry is $x = -2$ | axis of symmetry is $x = 2$ |
| vertex is $(-2, \_\_)$ | vertex is $(2, \_\_)$ |
| $f(x) = x^2 + 4x + 3$ | $f(x) = -x^2 + 4x + 3$ |
| $f(x) = (-2)^2 + 4(-2) + 3$ | $f(x) = -(2)^2 + 4(2) + 3$ |
| $f(x) = -1$ | $f(x) = 7$ |
| vertex is $(-2, -1)$ | vertex is $(2, 7)$ |

### Axis of Symmetry and Vertex of a Parabola

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
- the *y*-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

### EXAMPLE 9.44

For the graph of $f(x) = 3x^2 - 6x + 2$ find:

ⓐ the axis of symmetry ⓑ the vertex.

#### Solution

ⓐ

| | $f(x) = ax^2 + bx + c$ <br> $f(x) = 3x^2 - 6x + 2$ |
|---|---|
| The axis of symmetry is the vertical line $x = -\frac{b}{2a}$. | |
| Substitute the values of $a, b$ into the equation. | $x = -\frac{-6}{2 \cdot 3}$ |
| Simplify. | $x = 1$ |

The axis of symmetry is the line $x = 1$.

ⓑ

| | |
|---|---|
| | $f(x) = 3x^2 - 6x + 2$ |
| The vertex is a point on the line of symmetry, so its *x*-coordinate will be $x = 1$. Find $f(1)$. | $f(1) = 3(1)^2 - 6(1) + 2$ |
| Simplify. | $f(1) = 3 \cdot 1 - 6 + 2$ |
| The result is the *y*-coordinate. | $f(1) = -1$ |
| | The vertex is $(1, -1)$. |

**TRY IT : :** 9.87 For the graph of $f(x) = 2x^2 - 8x + 1$ find:

ⓐ the axis of symmetry ⓑ the vertex.

**TRY IT : :** 9.88 For the graph of $f(x) = 2x^2 - 4x - 3$ find:

ⓐ the axis of symmetry ⓑ the vertex.

## Find the Intercepts of a Parabola

When we graphed linear equations, we often used the *x*- and *y*-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the *y*-intercept the value of *x* is zero. So to find the *y*-intercept, we substitute *x* = 0 into the function.

Let's find the *y*-intercepts of the two parabolas shown in **Figure 9.4**.

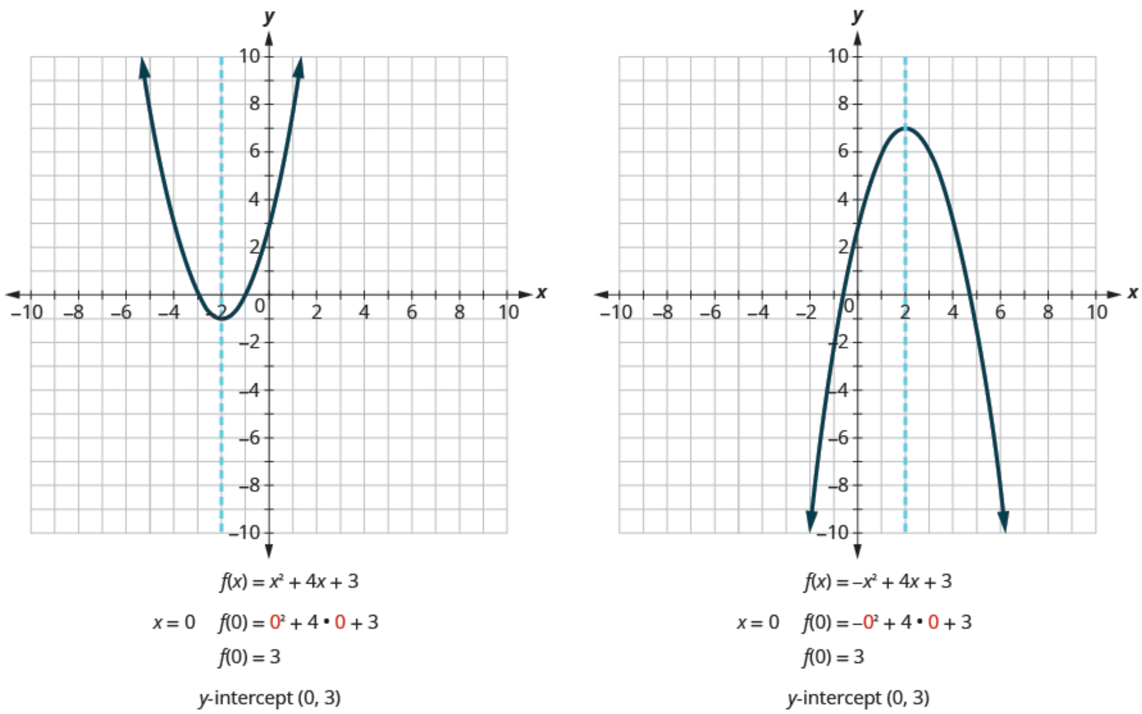



**Figure 9.4**

An $x$-intercept results when the value of $f(x)$ is zero. To find an $x$-intercept, we let $f(x) = 0$. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for $x$.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ 0 &= ax^2 + bx + c \end{aligned}$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the $x$-intercepts of the two parabolas we looked at. First we will find the $x$-intercepts of the parabola whose function is $f(x) = x^2 + 4x + 3$.

| | |
|---|---|
| | $f(x) = x^2 + 4x + 3$ |
| Let $f(x) = 0$. | $0 = x^2 + 4x + 3$ |
| Factor. | $0 = (x + 1)(x + 3)$ |
| Use the Zero Product Property. | $x + 1 = 0 \quad x + 3 = 0$ |
| Solve. | $x = -1 \qquad x = -3$ |
| | The $x$-intercepts are $(-1, 0)$ and $(-3, 0)$. |

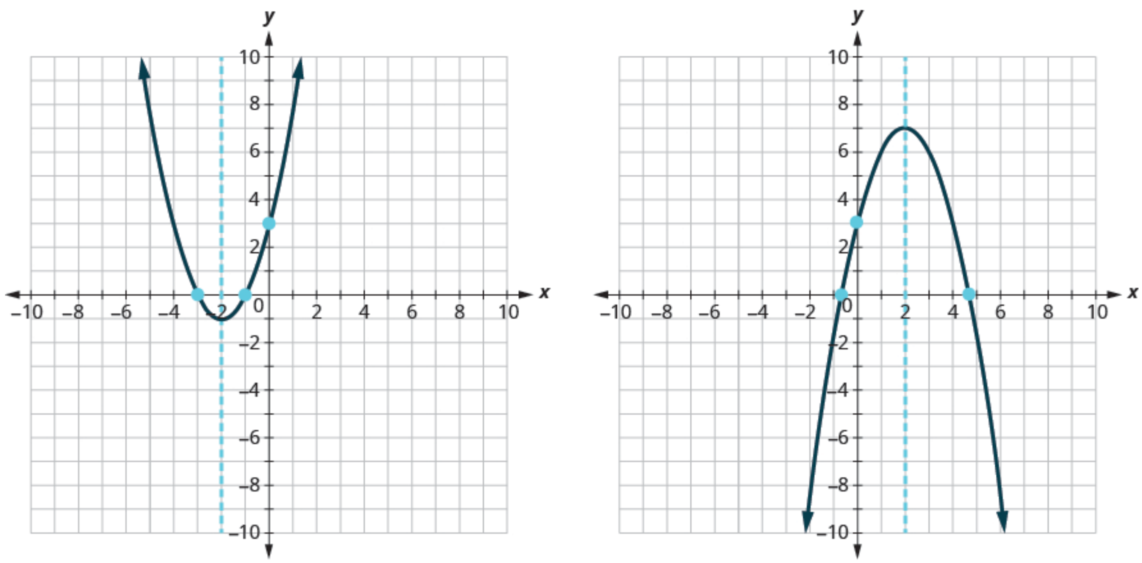
Now we will find the $x$-intercepts of the parabola whose function is $f(x) = -x^2 + 4x + 3$.

| | $f(x) = -x^2 + 4x + 3$ |
|---|---|
| Let $f(x) = 0$. | $0 = -x^2 + 4x + 3$ |
| This quadratic does not factor, so we use the Quadratic Formula. | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| $a = -1,\ b = 4,\ c = 3$ | $x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(3)}}{2(-1)}$ |
| Simplify. | $x = \frac{-4 \pm \sqrt{28}}{-2}$ |
| | $x = \frac{-4 \pm 2\sqrt{7}}{-2}$ |
| | $x = \frac{-2\left(2 \pm \sqrt{7}\right)}{-2}$ |
| | $x = 2 \pm \sqrt{7}$ |
| | The *x*-intercepts are $\left(2 + \sqrt{7},\ 0\right)$ and $\left(2 - \sqrt{7},\ 0\right)$. |

We will use the decimal approximations of the *x*-intercepts, so that we can locate these points on the graph,

$$\left(2 + \sqrt{7},\ 0\right) \approx (4.6,\ 0) \qquad \left(2 - \sqrt{7},\ 0\right) \approx (-0.6,\ 0)$$

Do these results agree with our graphs? See **Figure 9.5**.



$f(x) = x^2 + 4x + 3$
$y$-intercept (0, 3)
$x$-intercepts (–1, 0) and (–3, 0)

$f(x) = -x^2 + 4x + 3$
$y$-intercept (0, 3)
$x$-intercepts $(2 + \sqrt{7}, 0) \approx (4.6, 0)$
$(2 - \sqrt{7}, 0) \approx (-0.6, 0)$

**Figure 9.5**

### Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$ :

| $y$-intercept | $x$-intercepts |
|---|---|
| Let $x = 0$ and solve for $f(x)$. | Let $f(x) = 0$ and solve for $x$. |

**EXAMPLE 9.45**

Find the intercepts of the parabola whose function is $f(x) = x^2 - 2x - 8$.

## Solution

| | |
|---|---|
| To find the ***y***-intercept, let $x = 0$ and solve for $f(x)$. | $f(x) = x^2 - 2x - 8$ |
| | $f(0) = 0^2 - 2 \cdot 0 - 8$ |
| | $f(0) = -8$ |
| | When $x = 0$, then $f(0) = -8$. The $y$-intercept is the point $(0, -8)$. |
| To find the ***x***-intercept, let $f(x) = 0$ and solve for $x$. | $f(x) = x^2 - 2x - 8$ |
| | $0 = x^2 - 2x - 8$ |
| Solve by factoring. | $0 = (x - 4)(x + 2)$ |
| | $0 = x - 4 \quad 0 = x + 2$ |

| | |
|---|---|
| | $4 = x \qquad -2 = x$ |
| | When $f(x) = 0$, then $x = 4$ or $x = -2$. The *x*-intercepts are the points $(4,\ 0)$ and $(-2,\ 0)$. |

**TRY IT : :** 9.89 Find the intercepts of the parabola whose function is $f(x) = x^2 + 2x - 8$.

**TRY IT : :** 9.90 Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 12$.

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for *x* and the results were the solutions to the equation.

We are now looking at quadratic functions of the form $f(x) = ax^2 + bx + c$. The graphs of these functions are parabolas. The *x*-intercepts of the parabolas occur where $f(x) = 0$.

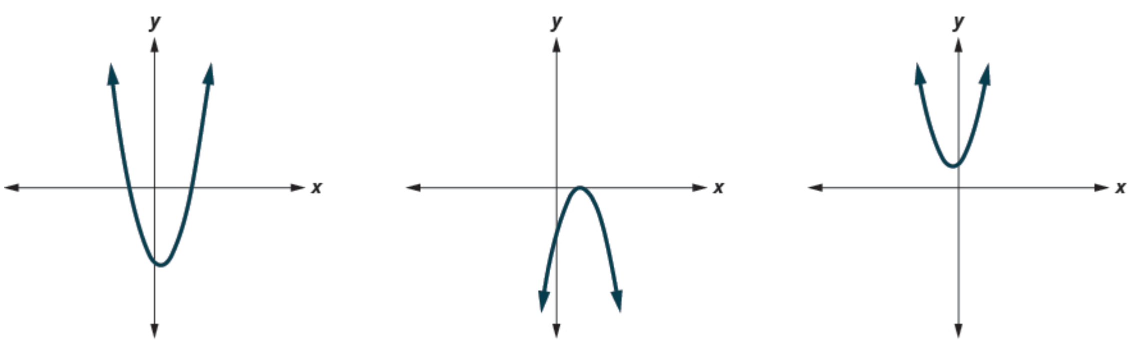
For example:

| Quadratic equation | | Quadratic function |
|---|---|---|
| | | $f(x) = x^2 - 2x - 15$ |
| $x^2 - 2x - 15 = 0$ | Let $f(x) = 0$. | $0 = x^2 - 2x - 15$ |
| $(x-5)(x+3) = 0$ | | $0 = (x-5)(x+3)$ |
| $x - 5 = 0 \quad x + 3 = 0$ | | $x - 5 = 0 \quad x + 3 = 0$ |
| $x = 5 \quad x = -3$ | | $x = 5 \quad x = -3$ |
| | | $(5,\ 0)$ and $(-3,\ 0)$ |
| | | *x*-intercepts |

The solutions of the quadratic function are the *x* values of the *x*-intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the *x*-intercepts of the graphs, the number of *x*-intercepts is the same as the number of solutions.

Previously, we used the discriminant to determine the number of solutions of a quadratic function of the form $ax^2 + bx + c = 0$. Now we can use the discriminant to tell us how many *x*-intercepts there are on the graph.




| $b^2 - 4ac > 0$ | $b^2 - 4ac = 0$ | $b^2 - 4ac < 0$ |
|---|---|---|
| Two solutions | One solution | No real solution |
| Two *x*-intercepts | One *x*-intercept | No *x*-intercept |

Before you to find the values of the *x*-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

**EXAMPLE 9.46**

Find the intercepts of the parabola for the function $f(x) = 5x^2 + x + 4$.

### Solution

| | |
|---|---|
| | $f(x) = 5x^2 + x + 4$ |
| To find the *y*-intercept, let $x = 0$ and solve for $f(x)$. | $f(0) = 5 \cdot 0^2 + 0 + 4$ |
| | $f(0) = 4$ |
| | When $x = 0$, then $f(0) = 4$. The *y*-intercept is the point $(0, 4)$. |
| To find the *x*-intercept, let $f(x) = 0$ and solve for $x$. | $f(x) = 5x^2 + x + 4$ |
| | $0 = 5x^2 + x + 4$ |
| Find the value of the discriminant to predict the number of solutions which is also the number of *x*-intercepts. | |
| $b^2 - 4ac$<br>$1^2 - 4 \cdot 5 \cdot 4$<br>$1 - 80$<br>$-79$ | |
| | Since the value of the discriminant is negative, there is no real solution to the equation. There are no *x*-intercepts. |

**TRY IT : :** 9.91 Find the intercepts of the parabola whose function is $f(x) = 3x^2 + 4x + 4$.

**TRY IT : :** 9.92 Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 5$.

## Graph Quadratic Functions Using Properties

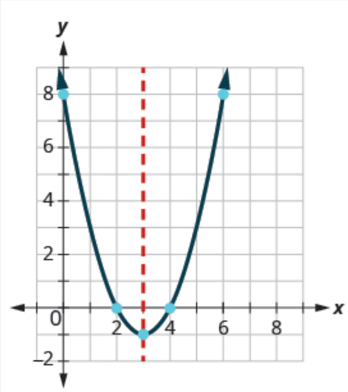
Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

**EXAMPLE 9.47** HOW TO GRAPH A QUADRATIC FUNCTION USING PROPERTIES

Graph $f(x) = x^2 - 6x + 8$ by using its properties.

### Solution

| | | |
|---|---|---|
| **Step 1.** Determine whether the parabola opens upward or downward. | Look at *a* in the equation.<br>$f(x) = x^2 - 6x + 8$<br>Since *a* is positive, the parabola opens upward.  | $f(x) = x^2 - 6x + 8$<br>$a = 1, b = -6, c = 8$<br>**The parabola opens upward.** |

| | | |
|---|---|---|
| **Step 2.** Find the axis of symmetry. | $f(x) = x^2 - 6x + 8$<br>The axis of symmetry is the line $x = -\frac{b}{2a}$. | Axis of Symmetry<br>$x = -\frac{b}{2a}$<br>$x = -\frac{(-6)}{2 \cdot 1}$<br>$x = 3$<br>**The axis of symmetry is the line $x = 3$.** |
| **Step 3.** Find the vertex. | The vertex is on the axis of symmetry. Substitute $x = 3$ into the function. | Vertex<br>$f(x) = x^2 - 6x + 8$<br>$f(3) = (3)^2 - 6(3) + 8$<br>$f(3) = -1$<br>**The vertex is (3, −1).** |
| **Step 4.** Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry. | We find $f(0)$.<br><br>We use the axis of symmetry to find a point symmetric to the *y*-intercept. The *y*-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$. | *y*-intercept<br>$f(x) = x^2 - 6x + 8$<br>$f(0) = (0)^2 - 6(0) + 8$<br>$f(0) = 8$<br>**The *y*-intercept is (0, 8).**<br><br>Point symmetric to *y*-intercept:<br>**The point is (6, 8).** |
| **Step 5.** Find the *x*-intercepts. Find additional points if needed. | We solve $f(x) = 0$.<br>We can solve this quadratic equation by factoring. | *x*-intercepts<br>$f(x) = x^2 - 6x + 8$<br>$0 = x^2 - 6x + 8$<br>$0 = (x - 2)(x - 4)$<br>$x = 2$ or $x = 4$<br>**The *x*-intercepts are (2, 0) and (4, 0).** |
| **Step 6.** Graph the parabola. | We graph the vertex, intercepts, and the point symmetric to the *y*-intercept. We connect these 5 points to sketch the parabola. |  |

**TRY IT : :** 9.93 Graph $f(x) = x^2 + 2x - 8$ by using its properties.

**TRY IT : :** 9.94 Graph $f(x) = x^2 - 8x + 12$ by using its properties.

We list the steps to take in order to graph a quadratic function here.

**HOW TO : : TO GRAPH A QUADRATIC FUNCTION USING PROPERTIES.**

Step 1. Determine whether the parabola opens upward or downward.
Step 2. Find the equation of the axis of symmetry.
Step 3. Find the vertex.
Step 4. Find the $y$-intercept. Find the point symmetric to the $y$-intercept across the axis of symmetry.
Step 5. Find the $x$-intercepts. Find additional points if needed.
Step 6. Graph the parabola.

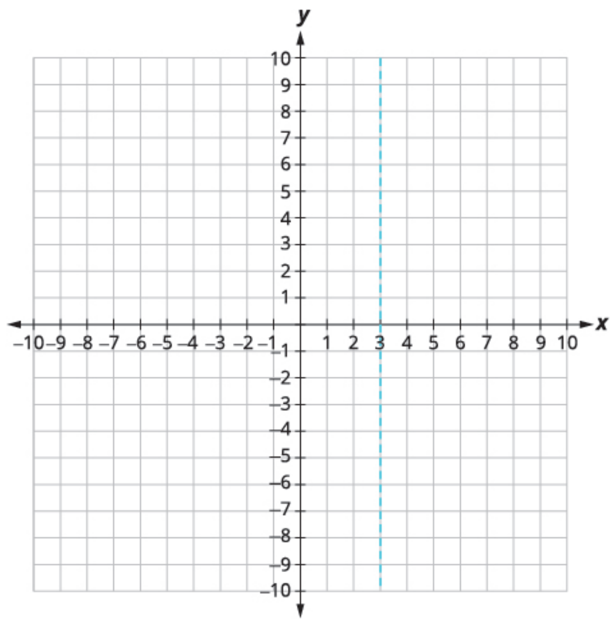
We were able to find the $x$-intercepts in the last example by factoring. We find the $x$-intercepts in the next example by factoring, too.

**EXAMPLE 9.48**

Graph $f(x) = x^2 + 6x - 9$ by using its properties.

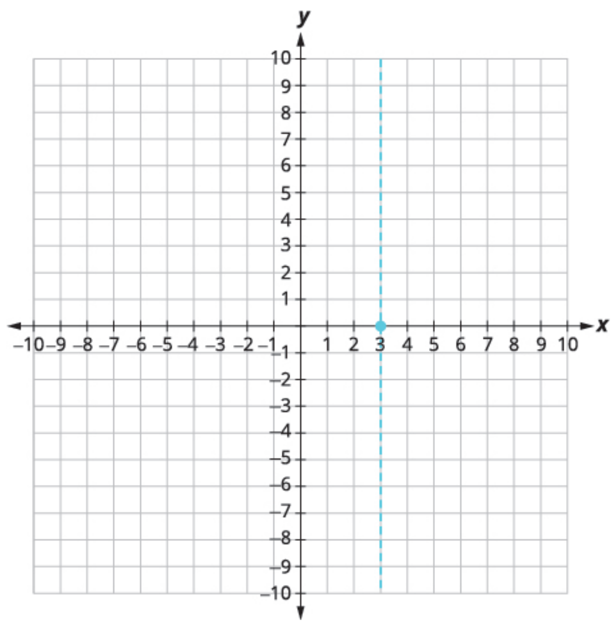
### Solution

| | |
|---|---|
| | $f(x) = ax^2 + bx + c$<br>$f(x) = -x^2 + 6x - 9$ |
| Since $a$ is $-1$, the parabola opens downward. | |
|  | |
| To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$. | $x = -\frac{b}{2a}$ |
| | $x = -\frac{6}{2(-1)}$ |
| | $x = 3$ |
| | The axis of symmetry is $x = 3$.<br>The vertex is on the line $x = 3$. |

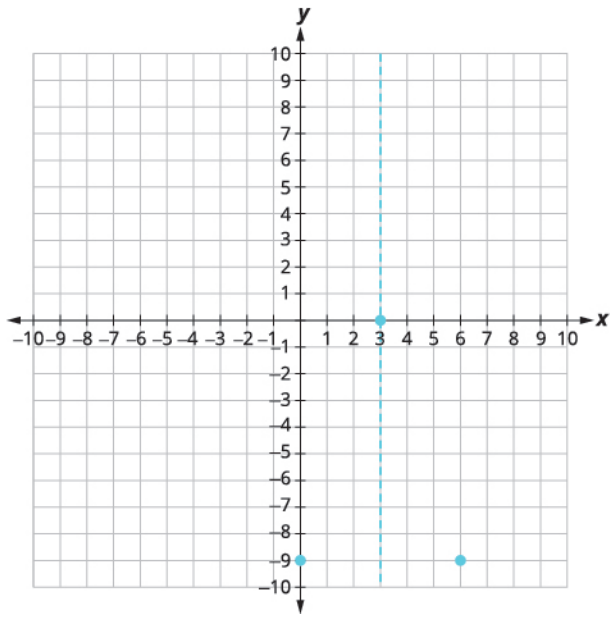
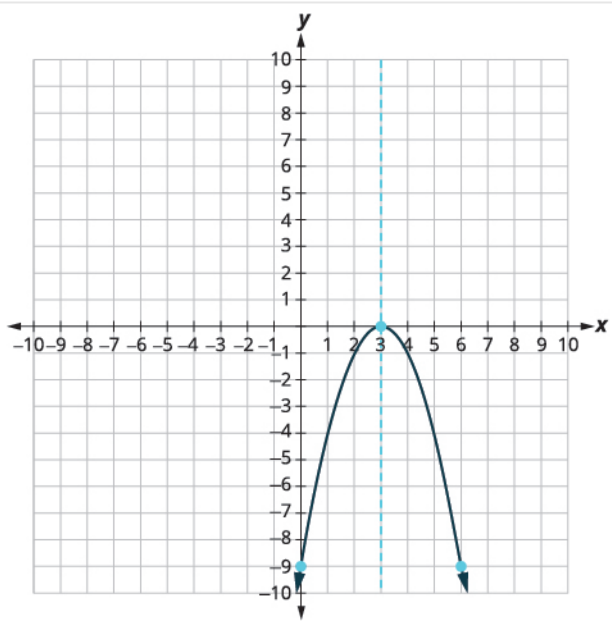



| | |
|---|---|
| Find $f(3)$. | $f(x) = -x^2 + 6x - 9$ |
| | $f(3) = -3^2 + 6 \cdot 3 - 9$ |
| | $f(3) = -9 + 18 - 9$ |
| | $f(3) = 0$ |

The vertex is $(3, 0)$.




| | |
|---|---|
| The *y*-intercept occurs when $x = 0$. Find $f(0)$. | $f(x) = -x^2 + 6x - 9$ |
| Substitute $x = 0$. | $f(0) = -0^2 + 6 \cdot 0 - 9$ |
| Simplify. | $f(0) = -9$ |

| | |
|---|---|
| | The $y$-intercept is $(0, -9)$. |
| The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$. |  |
| | Point symmetric to the $y$-intercept is $(6, -9)$ |
| The $x$-intercept occurs when $f(x) = 0$. | $f(x) = -x^2 + 6x - 9$ |
| Find $f(x) = 0$. | $0 = -x^2 + 6x - 9$ |
| Factor the GCF. | $0 = -(x^2 - 6x + 9)$ |
| Factor the trinomial. | $0 = -(x - 3)^2$ |
| Solve for $x$. | $0 = 3$ |
| Connect the points to graph the parabola. |  |

**TRY IT : :** 9.95 Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.

**TRY IT : :** 9.96 Graph $f(x) = 4x^2 + 24x + 36$ by using its properties.

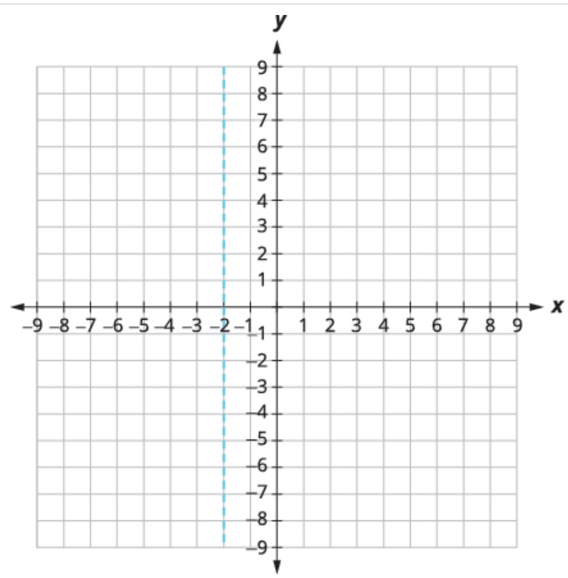
For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the $x$-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one $x$-intercept, and it is the vertex of the parabola.

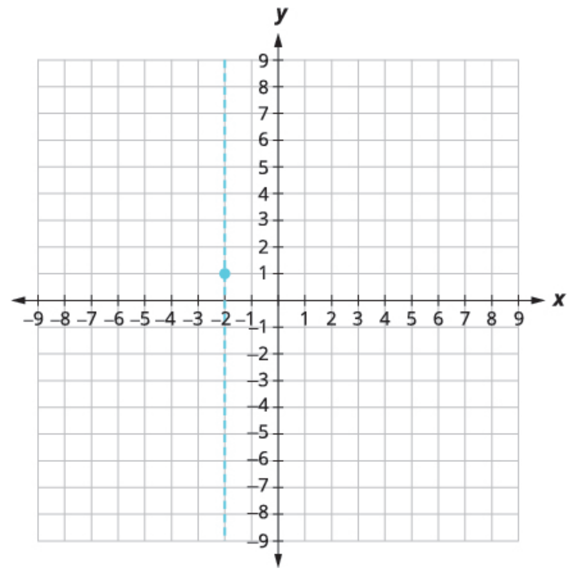
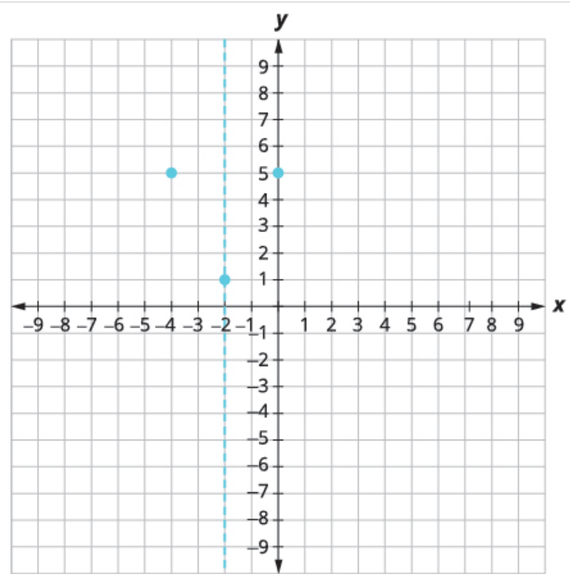
How many $x$-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

**EXAMPLE 9.49**

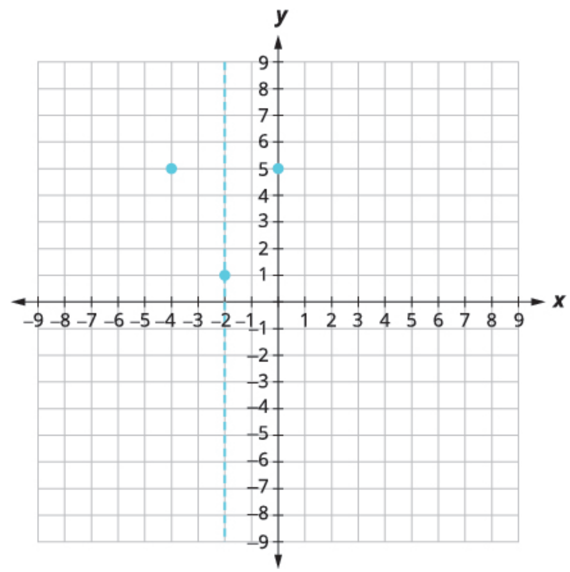
Graph $f(x) = x^2 + 4x + 5$ by using its properties.

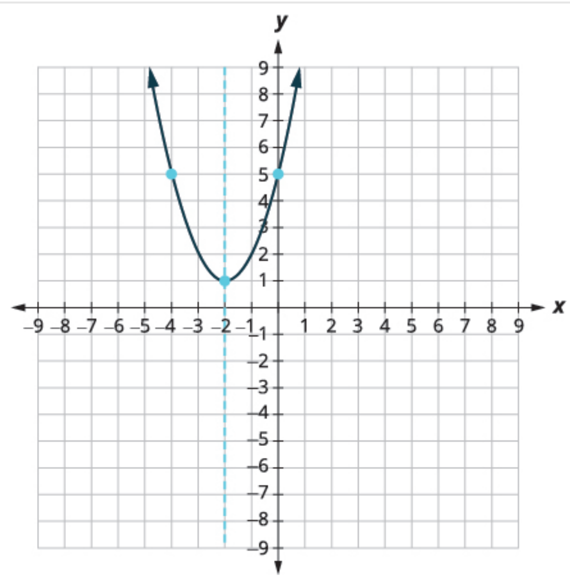
### Solution

| | |
|---|---|
| | $f(x) = ax^2 + bx + c$<br>$f(x) = x^2 + 4x + 5$ |
| Since $a$ is 1, the parabola opens upward. | |
|  | |
| To find the axis of symmetry, find $x = -\frac{b}{2a}$. | $x = -\frac{b}{2a}$ |
| | $x = -\frac{4}{(2)1}$ |
| | $x = -2$ |
| | The equation of the axis of symmetry is $x = -2$. |
| |  |
| The vertex is on the line $x = -2$. | |
| Find $f(x)$ when $x = -2$. | $f(x) = x^2 + 4x + 5$ |
| | $f(-2) = (-2)^2 + 4(-2) + 5$ |
| | $f(-2) = 4 - 8 + 5$ |

| | |
|---|---|
| | $f(-2) = 1$ |
| | The vertex is $(-2,\ 1)$. |
| |  |
| The *y*-intercept occurs when $x = 0$. | $f(x) = x^2 + 4x - 5$ |
| Find $f(0)$. | $f(0) = 5$ |
| Simplify. | $f(0) = 5$ |
| | The *y*-intercept is $(0,\ 5)$. |
| The point $(-4,\ 5)$ is two units to the left of the line of symmetry.<br>The point two units to the right of the line of symmetry is $(0,\ 5)$. |  |
| | Point symmetric to the *y*-intercept is $(-4,\ 5)$. |

The $x$-intercept occurs when $f(x) = 0$.




| | |
|---|---|
| Find $f(x) = 0$. | $0 = x^2 + 4x + 5$ |
| Test the discriminant. | |
| | $b^2 - 4ac$ |
| | $4^2 - 4 \cdot 1 \cdot 5$ |
| | $16 - 20$ |
| | $-4$ |
| Since the value of the discriminant is negative, there is no real solution and so no $x$-intercept. | |
| Connect the points to graph the parabola. You may want to choose two more points for greater accuracy. |  |

**TRY IT : :** 9.97 Graph $f(x) = x^2 - 2x + 3$ by using its properties.

**TRY IT : :** 9.98 Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.

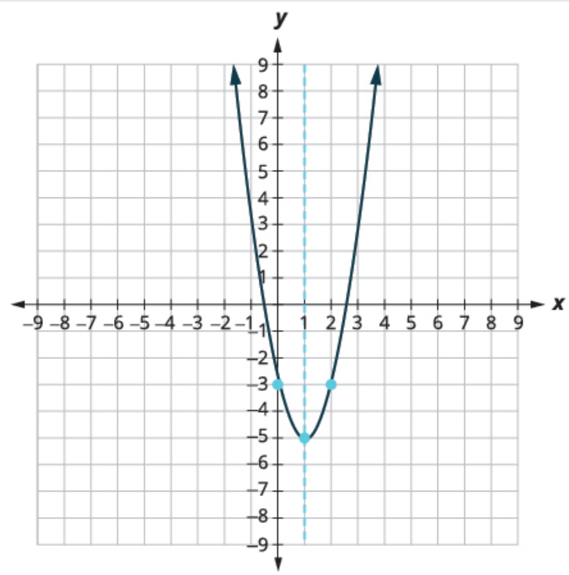
Finding the $y$-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the Quadratic Formula to find the $x$-intercepts.

**EXAMPLE 9.50**

Graph $f(x) = 2x^2 - 4x - 3$ by using its properties.

**Solution**

| | |
|---|---|
| | $f(x) = ax^2 + bx + c$<br>$f(x) = 2x^2 - 4x - 3$ |
| Since $a$ is 2, the parabola opens upward.  | |
| To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$. | $x = -\frac{b}{2a}$ |
| | $x = -\frac{-4}{2 \cdot 2}$ |
| | $x = 1$ |
| | **The equation of the axis of symmetry is $x = 1$.** |
| The vertex is on the line $x = 1$. | $f(x) = 2x^2 - 4x - 3$ |
| Find $f(1)$. | $f(x) = 2(1)^2 - 4(1) - 3$ |
| | $f(1) = 2 - 4 - 3$ |
| | $f(1) = -5$ |
| | **The vertex is $(1, -5)$.** |
| The $y$-intercept occurs when $x = 0$. | $f(x) = 2x^2 - 4x - 3$ |
| Find $f(0)$. | $f(0) = 2(0)^2 - 4(0) - 3$ |
| Simplify. | $f(0) = -3$ |
| | **The $y$-intercept is $(0, -3)$.** |
| The point $(0, -3)$ is one unit to the left of the line of symmetry. | **Point symmetric to the $y$-intercept is $(2, -3)$** |
| The point one unit to the right of the line of symmetry is $(2, -3)$. | |
| The $x$-intercept occurs when $y = 0$. | $f(x) = 2x^2 - 4x - 3$ |
| Find $f(x) = 0$. | $0 = 2x^2 - 4x - 3$ |
| Use the Quadratic Formula. | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Substitute in the values of $a, b,$ and $c$. | $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(3)}}{2(2)}$ |
| Simplify. | $x = \frac{-4 \pm \sqrt{16 + 24}}{4}$ |

| | |
|---|---|
| Simplify inside the radical. | $x = \dfrac{4 \pm \sqrt{40}}{4}$ |
| Simplify the radical. | $x = \dfrac{4 \pm 2\sqrt{10}}{4}$ |
| Factor the GCF. | $x = \dfrac{2\left(2 \pm \sqrt{10}\right)}{4}$ |
| Remove common factors. | $x = \dfrac{2 \pm \sqrt{10}}{2}$ |
| Write as two equations. | $x = \dfrac{2 + \sqrt{10}}{2}, \quad x = \dfrac{2 - \sqrt{10}}{2}$ |
| Approximate the values. | $x \approx 2.5, \quad x \approx -0.6$ |
| | **The approximate values of the *x*-intercepts are $(2.5, 0)$ and $(-0.6, 0)$.** |
| Graph the parabola using the points found. |  |



> **TRY IT : :** 9.99 Graph $f(x) = 5x^2 + 10x + 3$ by using its properties.

> **TRY IT : :** 9.100 Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.

## Solve Maximum and Minimum Applications

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The *y*-coordinate of the vertex is the minimum value of a parabola that opens upward. It is the maximum value of a parabola that opens downward. See **Figure 9.6**.

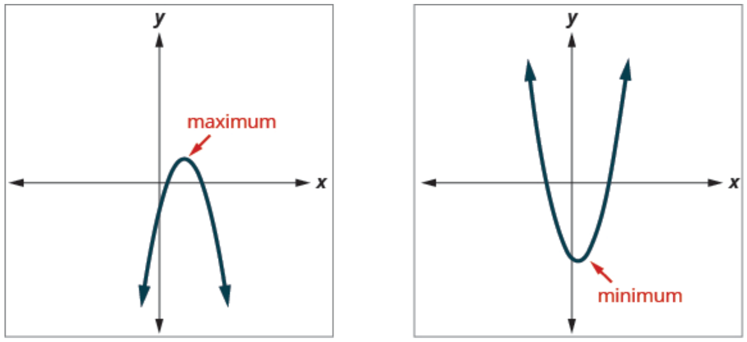



**Figure 9.6**

### Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

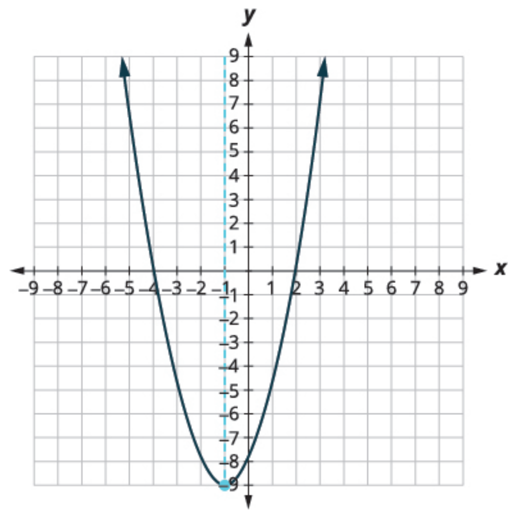
- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

## EXAMPLE 9.51

Find the minimum or maximum value of the quadratic function $f(x) = x^2 + 2x - 8$.

### Solution

| | |
|---|---|
| | $f(x) = x^2 + 2x - 8$ |
| Since *a* is positive, the parabola opens upward.<br>The quadratic equation has a minimum. | |
| Find the equation of the axis of symmetry. | $x = -\frac{b}{2a}$ |
| | $x = -\frac{2}{2 \times 1}$ |
| | $x = -1$ |
| | The equation of the axis of symmetry is $x = -1$. |
| The vertex is on the line $x = -1$. | $f(x) = x^2 + 2x - 8$ |
| Find $f(-1)$. | $f(-1) = (-1)^2 + 2(-1) - 8$ |
| | $f(-1) = 1 - 2 - 8$ |
| | $f(-1) = -9$ |
| | The vertex is $(-1, -9)$. |
| Since the parabola has a minimum, the *y*-coordinate of the vertex is the minimum *y*-value of the quadratic equation.<br>The minimum value of the quadratic is $-9$ and it occurs when $x = -1$. | |




Show the graph to verify the result.

**TRY IT : :** 9.101 Find the maximum or minimum value of the quadratic function $f(x) = x^2 - 8x + 12$.

**TRY IT : :** 9.102 Find the maximum or minimum value of the quadratic function $f(x) = -4x^2 + 16x - 11$.

We have used the formula

$$h(t) = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, $h$ , of an object shot upwards into the air with initial velocity, $v_0$, after $t$ seconds .

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex ($t$, $h$), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

**EXAMPLE 9.52**

The quadratic equation $h(x) = -16t^2 + 176t + 4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

ⓐ How many seconds will it take the volleyball to reach its maximum height? ⓑ Find the maximum height of the volleyball.

## Solution

$$h(t) = -16t^2 + 176t + 4$$

Since $a$ is negative, the parabola opens downward.
The quadratic function has a maximum.

ⓐ

| | |
|---|---|
| Find the equation of the axis of symmetry. | $t=-\frac{b}{2a}$ |
| | $t=-\frac{176}{2(-16)}$ |
| | $t=5.5$ |
| | The equation of the axis of symmetry is $t=5.5$. |
| The vertex is on the line $t=5.5$. | The maximum occurs when $t=5.5$ seconds. |

ⓑ

| | |
|---|---|
| Find $h(5.5)$. | $h(t)=-16t^2+176t+4$ |
| | $h(t)=-16(5.5)^2+176(5.5)+4$ |
| Use a calculator to simplify. | $h(t)=488$ |
| | The vertex is (5.5, 488). |

Since the parabola has a maximum, the *h*-coordinate of the vertex is the maximum value of the quadratic function.

The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

**TRY IT : :** 9.103

Solve, rounding answers to the nearest tenth.

The quadratic function $h(x)=-16t^2+128t+32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

**TRY IT : :** 9.104

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of. $h(x)=-16t^2+208t$. When will the rocket reach its maximum height? What will be the maximum height?

**MEDIA : :**

Access these online resources for additional instruction and practice with graphing quadratic functions using properties.

- **Quadratic Functions: Axis of Symmetry and Vertex (https://openstax.org/l/37QuadFunct1)**
- **Finding x- and y-intercepts of a Quadratic Function (https://openstax.org/l/37QuadFunct2)**
- **Graphing Quadratic Functions (https://openstax.org/l/37QuadFunct3)**
- **Solve Maxiumum or Minimum Applications (https://openstax.org/l/37QuadFunct4)**
- **Quadratic Applications: Minimum and Maximum (https://openstax.org/l/37QuadFunct5)**

## 9.6 EXERCISES

### Practice Makes Perfect

#### Recognize the Graph of a Quadratic Function

*In the following exercises, graph the functions by plotting points.*

**229.** $f(x) = x^2 + 3$

**230.** $f(x) = x^2 - 3$

**231.** $y = -x^2 + 1$

**232.** $f(x) = -x^2 - 1$

*For each of the following exercises, determine if the parabola opens up or down.*

**233.**
ⓐ $f(x) = -2x^2 - 6x - 7$
ⓑ $f(x) = 6x^2 + 2x + 3$

**234.**
ⓐ $f(x) = 4x^2 + x - 4$
ⓑ $f(x) = -9x^2 - 24x - 16$

**235.**
ⓐ $f(x) = -3x^2 + 5x - 1$
ⓑ $f(x) = 2x^2 - 4x + 5$

**236.**
ⓐ $f(x) = x^2 + 3x - 4$
ⓑ $f(x) = -4x^2 - 12x - 9$

#### Find the Axis of Symmetry and Vertex of a Parabola

*In the following functions, find ⓐ the equation of the axis of symmetry and ⓑ the vertex of its graph.*

**237.** $f(x) = x^2 + 8x - 1$

**238.** $f(x) = x^2 + 10x + 25$

**239.** $f(x) = -x^2 + 2x + 5$

**240.** $f(x) = -2x^2 - 8x - 3$

#### Find the Intercepts of a Parabola

*In the following exercises, find the intercepts of the parabola whose function is given.*

**241.** $f(x) = x^2 + 7x + 6$

**242.** $f(x) = x^2 + 10x - 11$

**243.** $f(x) = x^2 + 8x + 12$

**244.** $f(x) = x^2 + 5x + 6$

**245.** $f(x) = -x^2 + 8x - 19$

**246.** $f(x) = -3x^2 + x - 1$

**247.** $f(x) = x^2 + 6x + 13$

**248.** $f(x) = x^2 + 8x + 12$

**249.** $f(x) = 4x^2 - 20x + 25$

**250.** $f(x) = -x^2 - 14x - 49$

**251.** $f(x) = -x^2 - 6x - 9$

**252.** $f(x) = 4x^2 + 4x + 1$

#### Graph Quadratic Functions Using Properties

*In the following exercises, graph the function by using its properties.*

**253.** $f(x) = x^2 + 6x + 5$

**254.** $f(x) = x^2 + 4x - 12$

**255.** $f(x) = x^2 + 4x + 3$

**256.** $f(x) = x^2 - 6x + 8$

**257.** $f(x) = 9x^2 + 12x + 4$

**258.** $f(x) = -x^2 + 8x - 16$

**259.** $f(x) = -x^2 + 2x - 7$

**260.** $f(x) = 5x^2 + 2$

**261.** $f(x) = 2x^2 - 4x + 1$

**262.** $f(x) = 3x^2 - 6x - 1$

**263.** $f(x) = 2x^2 - 4x + 2$

**264.** $f(x) = -4x^2 - 6x - 2$

**265.** $f(x) = -x^2 - 4x + 2$

**266.** $f(x) = x^2 + 6x + 8$

**267.** $f(x) = 5x^2 - 10x + 8$

**268.** $f(x) = -16x^2 + 24x - 9$

**269.** $f(x) = 3x^2 + 18x + 20$

**270.** $f(x) = -2x^2 + 8x - 10$

### Solve Maximum and Minimum Applications

*In the following exercises, find the maximum or minimum value of each function.*

**271.** $f(x) = 2x^2 + x - 1$

**272.** $y = -4x^2 + 12x - 5$

**273.** $y = x^2 - 6x + 15$

**274.** $y = -x^2 + 4x - 5$

**275.** $y = -9x^2 + 16$

**276.** $y = 4x^2 - 49$

*In the following exercises, solve. Round answers to the nearest tenth.*

**277.** An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

**278.** A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic function $h(t) = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

**279.** A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

**280.** A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic function $h(t) = -16t^2 + 122t + 0$ to find how long it will take for the ball to reach its maxiumum height, and then find the maximum height.

**281.** A computer store owner estimates that by charging $x$ dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, $R$, received when the selling price of a computer is $x$, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

**282.** A retailer who sells backpacks estimates that by selling them for $x$ dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic function $R(x) = -x^2 + 100x$ is used to find the $R$, received when the selling price of a backpack is $x$. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

**283.** A retailer who sells fashion boots estimates that by selling them for $x$ dollars each, he will be able to sell $70 - x$ boots a week. Use the quadratic function $R(x) = -x^2 + 70x$ to find the revenue received when the average selling price of a pair of fashion boots is $x$. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

**284.** A cell phone company estimates that by charging $x$ dollars each for a certain cell phone, they can sell $8 - x$ cell phones per day. Use the quadratic function $R(x) = -x^2 + 8x$ to find the revenue received when the selling price of a cell phone is $x$. Find the selling price that will give them the maximum revenue, and then find the amount of the maximum revenue.

**285.** A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A(x) = x(240 - 2x)$ gives the area of the corral, $A$, for the length, $x$, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

**286.** A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic function $A(x) = x(100 - 2x)$ gives the area, $A$, of the dog run for the length, $x$, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

**287.** A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the "walls." He wants to maximize the area using 80 feet of fencing. The quadratic function $A(x) = x(80 - 2x)$ gives the area of the patio, where $x$ is the width of one side. Find the maximum area of the patio.

**288.** A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic function $A(x) = x(300 - 2x)$ to determine the maximum area of the fenced in yard.

## Writing Exercise

**289.** How do the graphs of the functions $f(x)=x^2$ and $f(x)=x^2-1$ differ? We graphed them at the start of this section. What is the difference between their graphs? How are their graphs the same?

**290.** Explain the process of finding the vertex of a parabola.

**291.** Explain how to find the intercepts of a parabola.

**292.** How can you use the discriminant when you are graphing a quadratic function?

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| recognize the graph of a quadratic equation. | | | |
| find the axis of symmetry and vertex of a parabola. | | | |
| find the intercepts of a parabola. | | | |
| graph quadratic equations in two variables. | | | |
| solve maximum and minimum applications. | | | |

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

# 9.7 Graph Quadratic Functions Using Transformations

## Learning Objectives

**By the end of this section, you will be able to:**

- Graph quadratic functions of the form $f(x) = x^2 + k$
- Graph quadratic functions of the form $f(x) = (x - h)^2$
- Graph quadratic functions of the form $f(x) = ax^2$
- Graph quadratic functions using transformations
- Find a quadratic function from its graph

### Be Prepared!

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
   If you missed this problem, review **Example 3.54**.
2. Factor completely: $y^2 - 14y + 49$.
   If you missed this problem, review **Example 6.24**.
3. Factor completely: $2x^2 - 16x + 32$.
   If you missed this problem, review **Example 6.26**.

## Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the last section, we learned how to graph quadratic functions using their properties. Another method involves starting with the basic graph of $f(x) = x^2$ and 'moving' it according to information given in the function equation. We call this graphing quadratic functions using transformations.

In the first example, we will graph the quadratic function $f(x) = x^2$ by plotting points. Then we will see what effect adding a constant, *k*, to the equation will have on the graph of the new function $f(x) = x^2 + k$.

### EXAMPLE 9.53

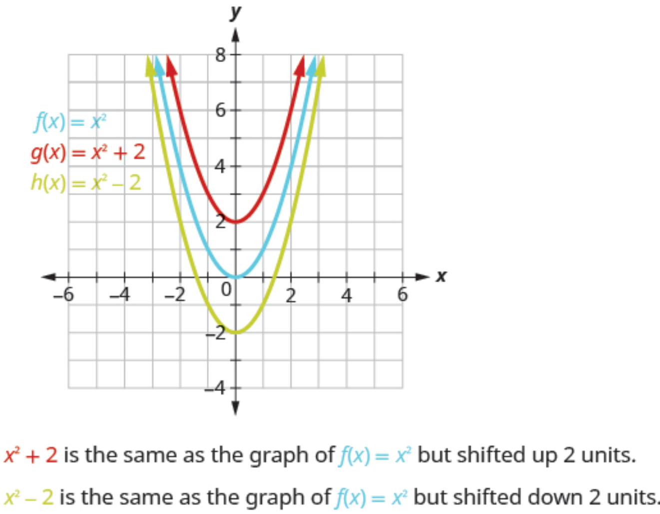
Graph $f(x) = x^2$, $g(x) = x^2 + 2$, and $h(x) = x^2 - 2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

#### Solution

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

| $x$ | $f(x) = x^2$ | $(x, f(x))$ | $g(x) = x^2 + 2$ | $(x, g(x))$ | $h(x) = x^2 - 2$ | $(x, h(x))$ |
|---|---|---|---|---|---|---|
| –3 | 9 | (–3, 9) | 9 + 2 | (–3, 11) | 9 – 2 | (–3, 7) |
| –2 | 4 | (–2, 4) | 4 + 2 | (–2, 6) | 4 – 2 | (–2, 2) |
| –1 | 1 | (–1, 1) | 1 + 2 | (–1, 3) | 1 – 2 | (–1, 1) |
| 0 | 0 | (0, 0) | 0 + 2 | (0, 2) | 0 – 2 | (0, –2) |
| 1 | 1 | (1, 1) | 1 + 2 | (1, 3) | 1 – 2 | (1, –1) |
| 2 | 4 | (2, 4) | 4 + 2 | (2, 6) | 4 – 2 | (2, 2) |
| 3 | 9 | (3, 9) | 9 + 2 | (3, 11) | 9 – 2 | (3, 7) |

The $g(x)$ values are two more than the $f(x)$ values. Also, the $h(x)$ values are two less than the $f(x)$ values. Now we will graph all three functions on the same rectangular coordinate system.




The graph of $g(x) = x^2 + 2$ is the same as the graph of $f(x) = x^2$ but shifted up 2 units.

The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ but shifted down 2 units.

---

**TRY IT : :** 9.105

ⓐ Graph $f(x) = x^2,\ g(x) = x^2 + 1,$ and $h(x) = x^2 - 1$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

**TRY IT : :** 9.106

ⓐ Graph $f(x) = x^2,\ g(x) = x^2 + 6,$ and $h(x) = x^2 - 6$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

The last example shows us that to graph a quadratic function of the form $f(x) = x^2 + k,$ we take the basic parabola graph of $f(x) = x^2$ and vertically shift it up $(k > 0)$ or shift it down $(k < 0)$.

*This transformation is called a vertical shift.*

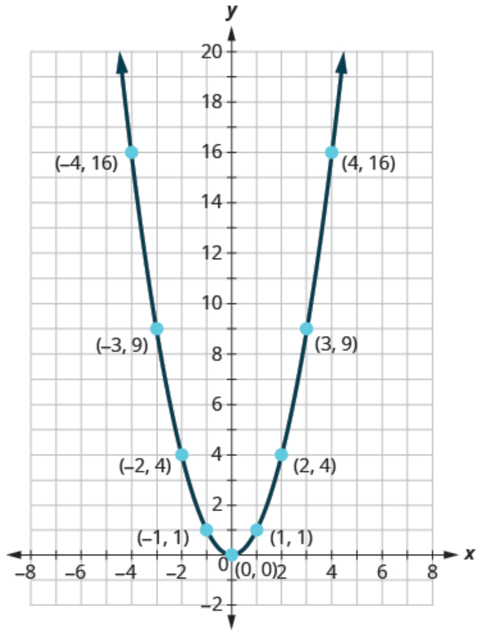
**Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift**

The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically $k$ units.

- If $k > 0$, shift the parabola vertically up $k$ units.
- If $k < 0$, shift the parabola vertically down $|k|$ units.

Now that we have seen the effect of the constant, $k$, it is easy to graph functions of the form $f(x) = x^2 + k.$ We just start with the basic parabola of $f(x) = x^2$ and then shift it up or down.

It may be helpful to practice sketching $f(x) = x^2$ quickly. We know the values and can sketch the graph from there.

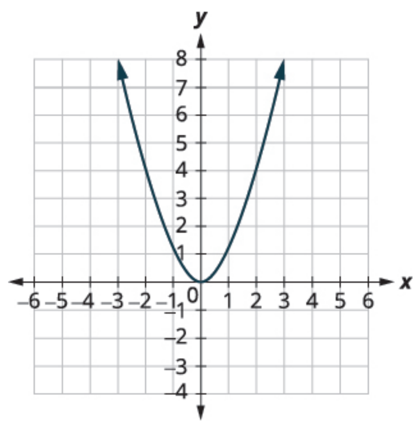



Once we know this parabola, it will be easy to apply the transformations. The next example will require a vertical shift.

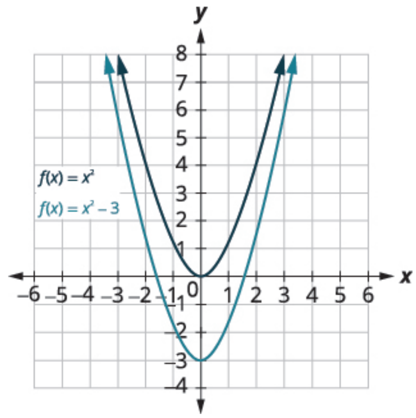
**EXAMPLE 9.54**

Graph $f(x) = x^2 - 3$ using a vertical shift.

### Solution

| | |
|---|---|
| We first draw the graph of $f(x) = x^2$ on the grid. |  |
| Determine $k$. | $f(x) = x^2 + k$<br>$f(x) = x^2 - 3$ |
| | $k = -3$ |

Shift the graph $f(x) = x^2$ down 3.




**TRY IT : :** 9.107 Graph $f(x) = x^2 - 5$ using a vertical shift.

**TRY IT : :** 9.108 Graph $f(x) = x^2 + 7$ using a vertical shift.

## Graph Quadratic Functions of the form $f(x) = (x - h)^2$

In the first example, we graphed the quadratic function $f(x) = x^2$ by plotting points and then saw the effect of adding a constant *k* to the function had on the resulting graph of the new function $f(x) = x^2 + k$.

We will now explore the effect of subtracting a constant, *h*, from *x* has on the resulting graph of the new function $f(x) = (x - h)^2$.

**EXAMPLE 9.55**

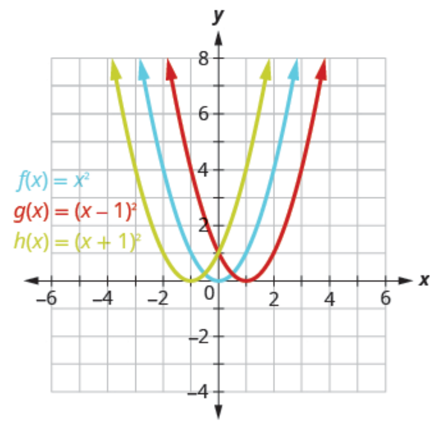
Graph $f(x) = x^2$, $g(x) = (x - 1)^2$, and $h(x) = (x + 1)^2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

### Solution

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

| $x$ | $f(x) = x^2$ | $(x, f(x))$ | $g(x) = (x - 1)^2$ | $(x, g(x))$ | $h(x) = (x + 1)^2$ | $(x, h(x))$ |
|---|---|---|---|---|---|---|
| –3 | 9 | (–3, 9) | 16 | (–3, 16) | 4 | (–3, 4) |
| –2 | 4 | (–2, 4) | 9 | (–2, 9) | 1 | (–2, 1) |
| –1 | 1 | (–1, 1) | 4 | (–1, 4) | 0 | (–1, 0) |
| 0 | 0 | (0, 0) | 1 | (0, 1) | 1 | (0, 1) |
| 1 | 1 | (1, 1) | 0 | (1, 0) | 4 | (1, 4) |
| 2 | 4 | (2, 4) | 1 | (2, 1) | 9 | (2, 9) |
| 3 | 9 | (3, 9) | 4 | (3, 4) | 16 | (3, 16) |

The $g(x)$ values and the $h(x)$ values share the common numbers 0, 1, 4, 9, and 16, but are shifted.




The graph of $g(x) = (x - 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 1 unit.

The graph of $h(x) = (x + 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 1 unit.


**TRY IT : :** 9.109

ⓐ Graph $f(x) = x^2$, $g(x) = (x + 2)^2$, and $h(x) = (x - 2)^2$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

**TRY IT : :** 9.110

ⓐ Graph $f(x) = x^2$, $g(x) = x^2 + 5$, and $h(x) = x^2 - 5$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

The last example shows us that to graph a quadratic function of the form $f(x) = (x - h)^2$, we take the basic parabola graph of $f(x) = x^2$ and shift it left ($h > 0$) or shift it right ($h < 0$).

*This transformation is called a horizontal shift*.

### Graph a Quadratic Function of the form $f(x) = (x - h)^2$ Using a Horizontal Shift

The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally $h$ units.

- If $h > 0$, shift the parabola horizontally left $h$ units.
- If $h < 0$, shift the parabola horizontally right $|h|$ units.

Now that we have seen the effect of the constant, $h$, it is easy to graph functions of the form $f(x) = (x - h)^2$. We just start with the basic parabola of $f(x) = x^2$ and then shift it left or right.

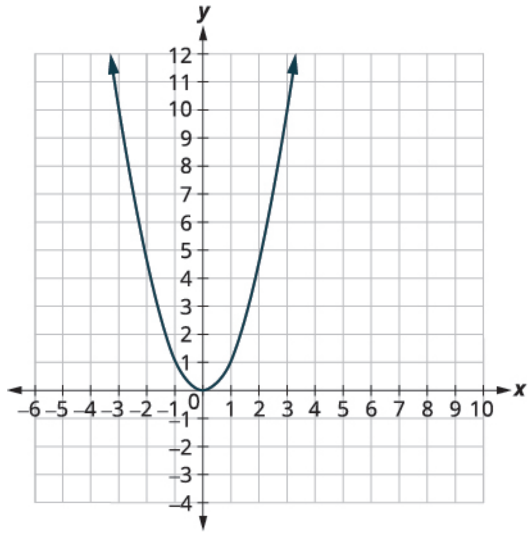
The next example will require a horizontal shift.

**EXAMPLE 9.56**

Graph $f(x) = (x - 6)^2$ using a horizontal shift.

## Solution

We first draw the graph of $f(x) = x^2$ on the grid.

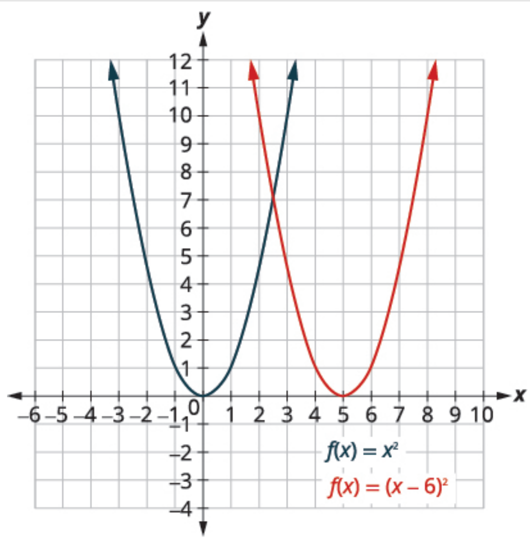



Determine *h*.

$f(x) = (x - h)^2$
$f(x) = (x - 6)^2$

$h = 6$

Shift the graph $f(x) = x^2$ to the right 6 units.




**TRY IT : :** 9.111 Graph $f(x) = (x - 4)^2$ using a horizontal shift.

**TRY IT : :** 9.112 Graph $f(x) = (x + 6)^2$ using a horizontal shift.

Now that we know the effect of the constants *h* and *k*, we will graph a quadratic function of the form $f(x) = (x - h)^2 + k$ by first drawing the basic parabola and then making a horizontal shift followed by a vertical shift. We could do the vertical shift followed by the horizontal shift, but most students prefer the horizontal shift followed by the vertical.

**EXAMPLE 9.57**

Graph $f(x) = (x + 1)^2 - 2$ using transformations.

**Solution**

This function will involve two transformations and we need a plan.

Let's first identify the constants *h*, *k*.

$$f(x) = (x + 1)^2 - 2$$

$$f(x) = \quad (x - h)^2 \quad + \quad k$$

$$f(x) = (x - (-1))^2 + (-2)$$

$$h = -1 \quad k = -2$$

The *h* constant gives us a horizontal shift and the *k* gives us a vertical shift.

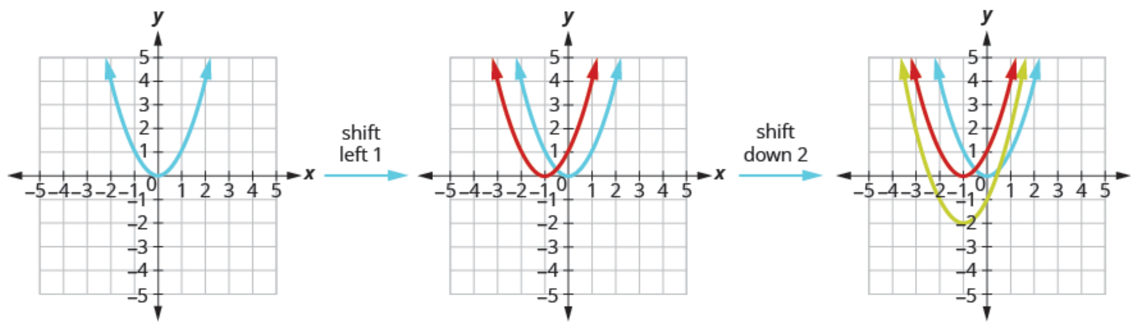
$$f(x) = x^2 \longrightarrow f(x) = (x + 1)^2 \longrightarrow f(x) = (x + 1)^2 - 2$$

$h = -1$ Shift left 1 unit; $k = -2$ Shift down 2 units

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = (x + 1)^2$, shift the graph $f(x) = x^2$ to the left 1 unit.

To graph $f(x) = (x + 1)^2 - 2$, shift the graph $f(x) = (x + 1)^2$ down 2 units.




$f(x) = x^2$ $\quad$ $f(x) = (x + 1)^2$ $\quad$ $f(x) = (x + 1)^2 - 2$

**TRY IT : :** 9.113 Graph $f(x) = (x + 2)^2 - 3$ using transformations.

**TRY IT : :** 9.114 Graph $f(x) = (x - 3)^2 + 1$ using transformations.

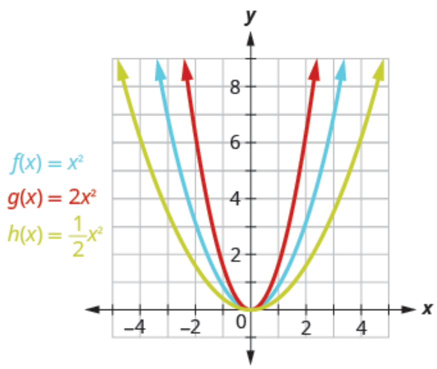
## Graph Quadratic Functions of the Form $f(x) = ax^2$

So far we graphed the quadratic function $f(x) = x^2$ and then saw the effect of including a constant *h* or *k* in the equation had on the resulting graph of the new function. We will now explore the effect of the coefficient *a* on the resulting graph of the new function $f(x) = ax^2$.

Let's look at the quadratic functions $f(x) = x^2$, $g(x) = 2x^2$ and $h(x) = \frac{1}{2}x^2$.

| $x$ | $f(x) = x^2$ | $(x, f(x))$ | $g(x) = 2x^2$ | $(x, g(x))$ | $h(x) = \frac{1}{2}x^2$ | $(x, h(x))$ |
|---|---|---|---|---|---|---|
| –2 | 4 | (–2, 4) | 2 • 4 | (–2, 8) | $\frac{1}{2} \cdot 4$ | (–2, 2) |
| –1 | 1 | (–1, 1) | 2 • 1 | (–1, 2) | $\frac{1}{2} \cdot 1$ | $\left(-1, \frac{1}{2}\right)$ |
| 0 | 0 | (0, 0) | 2 • 0 | (0, 0) | $\frac{1}{2} \cdot 0$ | (0, 0) |
| 1 | 1 | (1, 1) | 2 • 1 | (1, 2) | $\frac{1}{2} \cdot 1$ | $\left(1, \frac{1}{2}\right)$ |
| 2 | 4 | (2, 4) | 2 • 4 | (2, 8) | $\frac{1}{2} \cdot 4$ | (2, 2) |

If we graph these functions, we can see the effect of the constant *a*, assuming $a > 0$.




The graph of the function $g(x) = 2x^2$ is "skinnier" than the graph of $f(x) = x^2$.

The graph of the function $h(x) = \frac{1}{2}x^2$ is "wider" than the graph of $f(x) = x^2$.

To graph a function with constant *a* it is easiest to choose a few points on $f(x) = x^2$ and multiply the *y*-values by *a*.

### Graph of a Quadratic Function of the form $f(x) = ax^2$

The coefficient *a* in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.

- If $0 < |a| < 1$, the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
- If $|a| > 1$, the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.

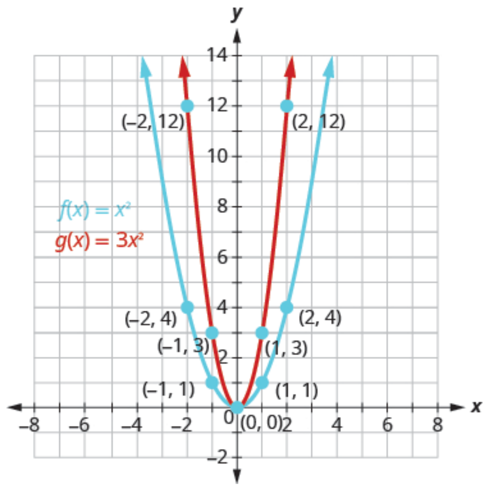
**EXAMPLE 9.58**

Graph $f(x) = 3x^2$.

### Solution

We will graph the functions $f(x) = x^2$ and $g(x) = 3x^2$ on the same grid. We will choose a few points on $f(x) = x^2$ and then multiply the *y*-values by 3 to get the points for $g(x) = 3x^2$.

| | $f(x) = x^2$ | $g(x) = 3x^2$ | |
|---|---|---|---|
| $x$ | $(x, f(x))$ | $(x, g(x))$ | |
| –2 | (–2, 4) | (–2, 12) | $3 \cdot 4 = 12$ |
| –1 | (–1, 1) | (–1, 3) | $3 \cdot 1 = 3$ |
| 0 | (0, 0) | (0, 0) | $3 \cdot 0 = 0$ |
| 1 | (1, 1) | (1, 3) | $3 \cdot 1 = 3$ |
| 2 | (2, 4) | (2, 12) | $3 \cdot 4 = 12$ |




**TRY IT : :** 9.115 Graph $f(x) = -3x^2$.

**TRY IT : :** 9.116 Graph $f(x) = 2x^2$.

## Graph Quadratic Functions Using Transformations

We have learned how the constants *a*, *h*, and *k* in the functions, $f(x) = x^2 + k$, $f(x) = (x - h)^2$, and $f(x) = ax^2$ affect their graphs. We can now put this together and graph quadratic functions $f(x) = ax^2 + bx + c$ by first putting them into the form $f(x) = a(x - h)^2 + k$ by completing the square. This form is sometimes known as the vertex form or standard form.

We must be careful to both add and subtract the number to the SAME side of the function to complete the square. We cannot add the number to both sides as we did when we completed the square with quadratic equations.

| Quadratic Equation | Quadratic Function |
|---|---|
| $x^2 + 8x + 6 = 0$ | $f(x) = x^2 + 8x + 6$ |
| $x^2 + 8x = -6$ | $f(x) = x^2 + 8x + 6$ |
| $x^2 + 8x + 16 = -6 + 16$ | $f(x) = x^2 + 8x + 16 + 6 - 16$ |
| $(x + 4)^2 = 10$ | $f(x) = (x + 4)^2 - 10$ |
| Add 16 to both sides | Add and subtract 16 from the same side |

When we complete the square in a function with a coefficient of $x^2$ that is not one, we have to factor that coefficient from just the *x*-terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to focus only on the *x*-terms.

Once we get the constant we want to complete the square, we must remember to multiply it by that coefficient before we then subtract it.

**EXAMPLE 9.59**

Rewrite $f(x) = -3x^2 - 6x - 1$ in the $f(x) = a(x - h)^2 + k$ form by completing the square.

## Solution

| | |
|---|---|
| | $f(x) = -3x^2 - 6x - 1$ |
| Separate the *x* terms from the constant. | $f(x) = -3x^2 - 6x \quad - 1$ |
| Factor the coefficient of $x^2$, $-3$. | $f(x) = -3(x^2 + 2x) \quad - 1$ |
| Prepare to complete the square. | $f(x) = -3(x^2 + 2x \quad\quad) - 1$ |
| Take half of 2 and then square it to complete the square. $\left(\frac{1}{2} \cdot 2\right)^2 = 1$ | |
| The constant 1 completes the square in the parentheses, but the parentheses is multiplied by $-3$. So we are really adding $-3$ We must then add 3 to not change the value of the function. | $f(x) = -3(x^2 + 2x + 1) - 1 + 3$ <br> $-3 \cdot 1 = -3$ so add $3$ |
| Rewrite the trinomial as a square and subtract the constants. | $f(x) = -3(x + 1) + 2$ |
| The function is now in the $f(x) = a(x - h)^2 + k$ form. | $f(x) = a(x - h)^2 + k$ <br> $f(x) = -3(x + 1)^2 + 2$ |

**TRY IT : :** 9.117

Rewrite $f(x) = -4x^2 - 8x + 1$ in the $f(x) = a(x - h)^2 + k$ form by completing the square.

**TRY IT : :** 9.118

Rewrite $f(x) = 2x^2 - 8x + 3$ in the $f(x) = a(x - h)^2 + k$ form by completing the square.

Once we put the function into the $f(x) = (x - h)^2 + k$ form, we can then use the transformations as we did in the last few problems. The next example will show us how to do this.

**EXAMPLE 9.60**

Graph $f(x) = x^2 + 6x + 5$ by using transformations.

## Solution

**Step 1.** Rewrite the function in $f(x) = a(x - h)^2 + k$ vertex form by completing the square.

| | |
|---|---|
| | $f(x) = x^2 + 6x + 5$ |
| Separate the *x* terms from the constant. | $f(x) = x^2 + 6x + 5$ |
| Take half of 6 and then square it to complete the square. $\left(\frac{1}{2} \cdot 6\right)^2 = 9$ | |
| We both add 9 and subtract 9 to not change the value of the function. | $f(x) = x^2 + 6x + 9 + 5 - 9$ |
| Rewrite the trinomial as a square and subtract the constants. | $f(x) = (x + 3)^2 - 4$ |
| The function is now in the $f(x) = (x - h)^2 + k$ form. | $f(x) = (x - h)^2 + k$<br>$f(x) = (x + 3)^2 - 4$ |

**Step 2:** Graph the function using transformations.

Looking at the *h*, *k* values, we see the graph will take the graph of $f(x) = x^2$ and shift it to the left 3 units and down 4 units.

$f(x) = x^2 \longrightarrow f(x) = (x + 3)^2 \longrightarrow f(x) = (x + 3)^2 - 4$

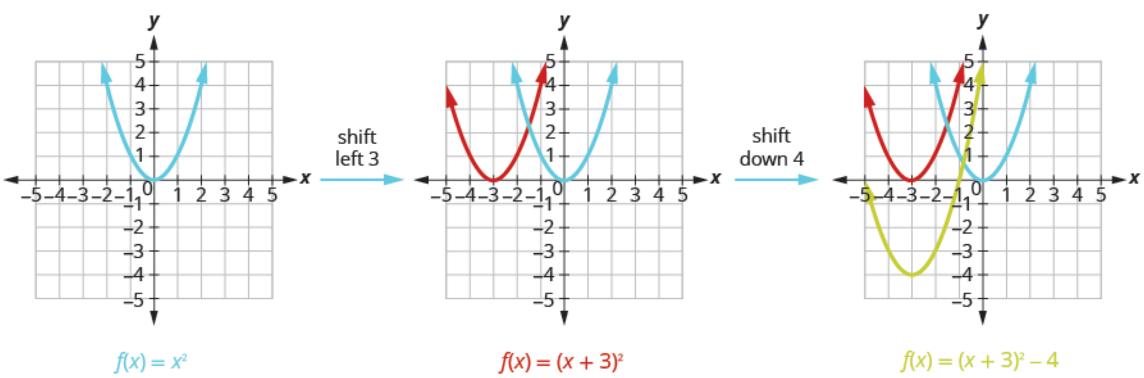
$h = -3$ — Shift left 3 units

$k = -4$ — Shift down 4 units

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = (x + 3)^2$, shift the graph $f(x) = x^2$ to the left 3 units.

To graph $f(x) = (x + 3)^2 - 4$, shift the graph $f(x) = (x + 3)^2$ down 4 units.




**TRY IT : :** 9.119 Graph $f(x) = x^2 + 2x - 3$ by using transformations.

**TRY IT : :** 9.120 Graph $f(x) = x^2 - 8x + 12$ by using transformations.

We list the steps to take to graph a quadratic function using transformations here.

**HOW TO : : GRAPH A QUADRATIC FUNCTION USING TRANSFORMATIONS.**

Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form by completing the square.

Step 2. Graph the function using transformations.

**EXAMPLE 9.61**

Graph $f(x) = -2x^2 - 4x + 2$ by using transformations.

## Solution

**Step 1.** Rewrite the function in $f(x) = a(x - h)^2 + k$ vertex form by completing the square.

| | |
|---|---|
| | $f(x) = -2x^2 - 4x + 2$ |
| Separate the *x* terms from the constant. | $f(x) = -2x^2 - 4x + 2$ |
| We need the coefficient of $x^2$ to be one.<br>We factor $-2$ from the *x*-terms. | $f(x) = -2(x^2 + 2x) + 2$ |
| Take half of 2 and then square it to complete the square.<br>$\left(\frac{1}{2} \cdot 2\right)^2 = 1$ | |
| We add 1 to complete the square in the parentheses, but the parentheses is multiplied by $-2$. Se we are really adding $-2$. To not change the value of the function we add 2. | $f(x) = -2(x^2 + 2x + 1) + 2 + 2$ |
| Rewrite the trinomial as a square and subtract the constants. | $f(x) = -2(x + 1)^2 + 4$ |
| The function is now in the $f(x) = a(x - h)^2 + k$ form. | $f(x) = a(x - h)^2 + k$<br>$f(x) = -2(x + 1)^2 + 4$ |

**Step 2.** Graph the function using transformations.

$f(x) = x^2 \rightarrow f(x) = -2x^2 \rightarrow f(x) = -2(x + 1)^2 \rightarrow f(x) = -2(x + 1)^2 + 4$

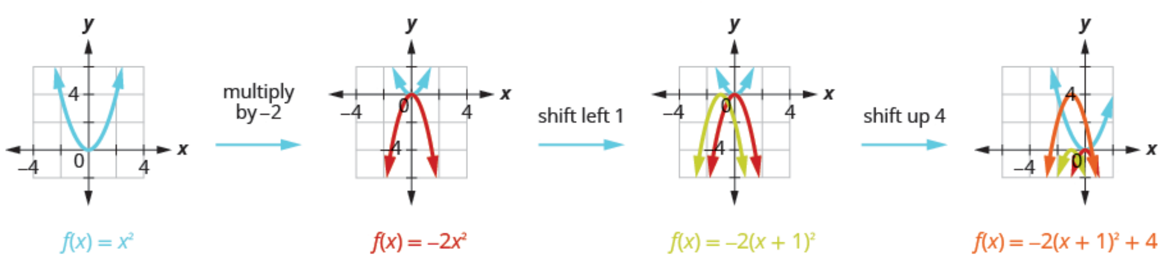
| $a = -2$ | $h = -1$ | $k = 4$ |
|---|---|---|
| Multiply *y*-values by $-2$ | Shift left 1 unit | Shift up 4 units |

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = -2x^2$, multiply the *y*-values in parabola of $f(x) = x^2$ by $-2$.

To graph $f(x) = -2(x + 1)^2$, shift the graph $f(x) = -2x^2$ to the left 1 unit.

To graph $f(x) = -2(x + 1)^2 + 4$, shift the graph $f(x) = (x + 1)^2$ up 4 units.

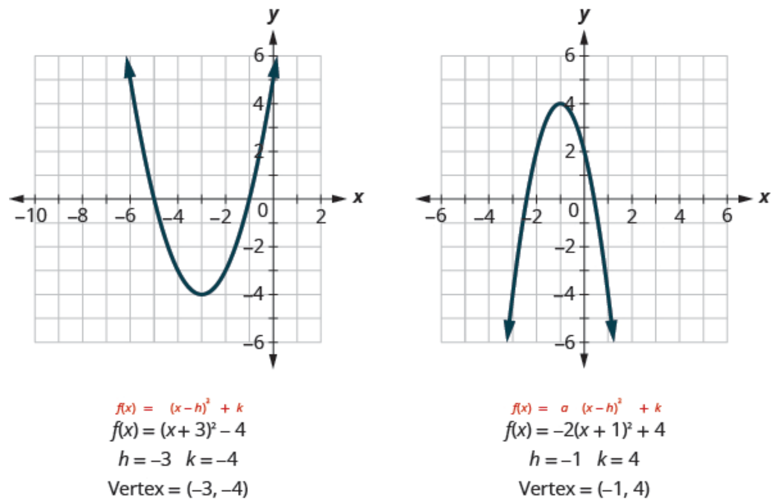



> **TRY IT : :** 9.121 Graph $f(x) = -3x^2 + 12x - 4$ by using transformations.

> **TRY IT : :** 9.122 Graph $f(x) = -2x^2 + 12x - 9$ by using transformations.

Now that we have completed the square to put a quadratic function into $f(x) = a(x - h)^2 + k$ form, we can also use this technique to graph the function using its properties as in the previous section.

If we look back at the last few examples, we see that the vertex is related to the constants *h* and *k*.




In each case, the vertex is (*h*, *k*). Also the axis of symmetry is the line *x* = *h*.

We rewrite our steps for graphing a quadratic function using properties for when the function is in $f(x) = a(x-h)^2 + k$ form.

**HOW TO : :** GRAPH A QUADRATIC FUNCTION IN THE FORM $f(x) = a(x-h)^2 + k$ USING PROPERTIES.

Step 1. Rewrite the function in $f(x) = a(x-h)^2 + k$ form.

Step 2. Determine whether the parabola opens upward, *a* > 0, or downward, *a* < 0.

Step 3. Find the axis of symmetry, *x* = *h*.

Step 4. Find the vertex, (*h*, *k*).

Step 5. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.

Step 6. Find the *x*-intercepts.

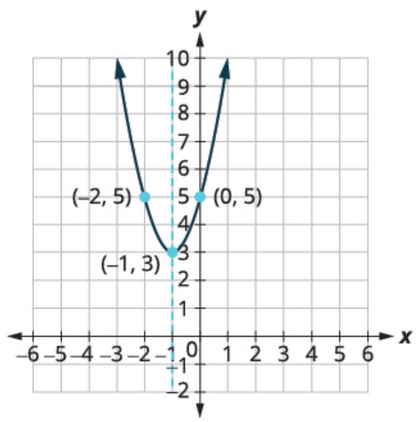
Step 7. Graph the parabola.

**EXAMPLE 9.62**

ⓐ Rewrite $f(x) = 2x^2 + 4x + 5$ in $f(x) = a(x-h)^2 + k$ form and ⓑ graph the function using properties.

## Solution

| | |
|---|---|
| Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square. | $f(x) = 2x^2 + 4x + 5$ |
| | $f(x) = 2(x^2 + 2x) + 5$ |
| | $f(x) = 2(x^2 + 2x + 1) + 5 - 2$ |
| | $f(x) = 2(x+1)^2 + 3$ |
| Identify the constants $a,\ h,\ k$. | $a = 2 \quad h = -1 \quad k = 3$ |

| | |
|---|---|
| Since $a = 2$, the parabola opens upward. |  |
| The axis of symmetry is $x = h$. | The axis of symmetry is $x = -1$. |
| The vertex is $(h, k)$. | The vertex is $(-1, 3)$. |
| Find the *y*-intercept by finding $f(0)$. | $f(0) = 2 \cdot 0^2 + 4 \cdot 0 + 5$ |
| | $f(0) = 5$ |
| | *y*-intercept $(0, 5)$ |
| Find the point symmetric to $(0, 5)$ across the axis of symmetry. | $(-2, 5)$ |
| Find the *x*-intercepts. | The discriminant negative, so there are no *x*-intercepts. Graph the parabola. |
| | <br> |

**TRY IT : :** 9.123

ⓐ Rewrite $f(x) = 3x^2 - 6x + 5$ in $f(x) = a(x - h)^2 + k$ form and ⓑ graph the function using properties.

**TRY IT : :** 9.124

ⓐ Rewrite $f(x) = -2x^2 + 8x - 7$ in $f(x) = a(x - h)^2 + k$ form and ⓑ graph the function using properties.

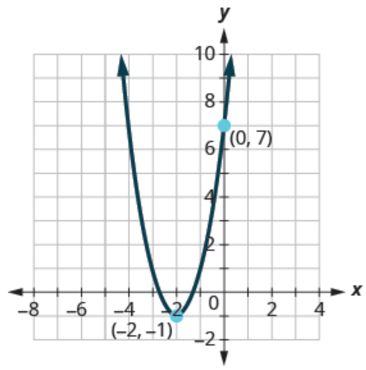
## Find a Quadratic Function from its Graph

So far we have started with a function and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the function.

**EXAMPLE 9.63**

Determine the quadratic function whose graph is shown.

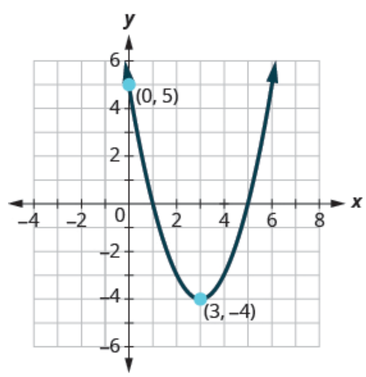



## Solution

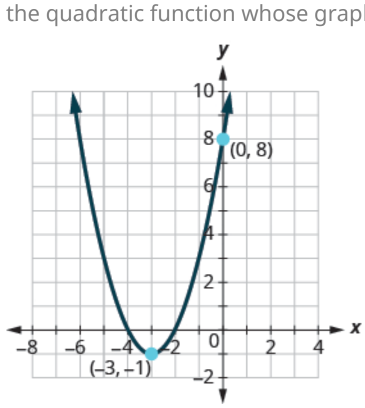
Since it is quadratic, we start with the $f(x) = a(x - h)^2 + k$ form.

| | |
|---|---|
| The vertex, $(h, k)$, is $(-2, -1)$ so $h = -2$ and $k = -1$. | $f(x) = a(x - (-2))^2 - 1$ |
| To fin $a$, we use the $y$-intercept, $(0, 7)$. | |
| So $f(0) = 7$. | $7 = a(0 + 2)^2 - 1$ |
| Solve for $a$. | $7 = 4a - 1$ |
| | $8 = 4a$ |
| | $2 = a$ |
| Write the function. | $f(x) = a(x - h)^2 + k$ |
| Substitute in $h = -2$, $k = -1$ and $a = 2$. | $f(x) = 2(x + 2)^2 - 1$ |

**TRY IT : :** 9.125 Write the quadratic function in $f(x) = a(x - h)^2 + k$ form whose graph is shown.




**TRY IT : :** 9.126 Determine the quadratic function whose graph is shown.



**MEDIA : :**

Access these online resources for additional instruction and practice with graphing quadratic functions using transformations.

- **Function Shift Rules Applied to Quadratic Functions (https://openstax.org/l/37QuadFuncTran1)**
- **Changing a Quadratic from Standard Form to Vertex Form (https://openstax.org/l/37QuadFuncTran2)**
- **Using Transformations to Graph Quadratic Functions (https://openstax.org/l/37QuadFuncTran3)**
- **Finding Quadratic Equation in Vertex Form from Graph (https://openstax.org/l/37QuadFuncTran4)**

# 9.7 EXERCISES

## Practice Makes Perfect

### Graph Quadratic Functions of the form $f(x) = x^2 + k$

*In the following exercises, ⓐ graph the quadratic functions on the same rectangular coordinate system and ⓑ describe what effect adding a constant, k, to the function has on the basic parabola.*

**293.** $f(x) = x^2,\ g(x) = x^2 + 4,$ and $h(x) = x^2 - 4.$

**294.** $f(x) = x^2,\ g(x) = x^2 + 7,$ and $h(x) = x^2 - 7.$

*In the following exercises, graph each function using a vertical shift.*

**295.** $f(x) = x^2 + 3$

**296.** $f(x) = x^2 - 7$

**297.** $g(x) = x^2 + 2$

**298.** $g(x) = x^2 + 5$

**299.** $h(x) = x^2 - 4$

**300.** $h(x) = x^2 - 5$

### Graph Quadratic Functions of the form $f(x) = (x - h)^2$

*In the following exercises, ⓐ graph the quadratic functions on the same rectangular coordinate system and ⓑ describe what effect adding a constant, $h$, to the function has on the basic parabola.*

**301.** $f(x) = x^2,\ g(x) = (x - 3)^2,$ and $h(x) = (x + 3)^2.$

**302.** $f(x) = x^2,\ g(x) = (x + 4)^2,$ and $h(x) = (x - 4)^2.$

*In the following exercises, graph each function using a horizontal shift.*

**303.** $f(x) = (x - 2)^2$

**304.** $f(x) = (x - 1)^2$

**305.** $f(x) = (x + 5)^2$

**306.** $f(x) = (x + 3)^2$

**307.** $f(x) = (x - 5)^2$

**308.** $f(x) = (x + 2)^2$

*In the following exercises, graph each function using transformations.*

**309.** $f(x) = (x + 2)^2 + 1$

**310.** $f(x) = (x + 4)^2 + 2$

**311.** $f(x) = (x - 1)^2 + 5$

**312.** $f(x) = (x - 3)^2 + 4$

**313.** $f(x) = (x + 3)^2 - 1$

**314.** $f(x) = (x + 5)^2 - 2$

**315.** $f(x) = (x - 4)^2 - 3$

**316.** $f(x) = (x - 6)^2 - 2$

### Graph Quadratic Functions of the form $f(x) = ax^2$

*In the following exercises, graph each function.*

**317.** $f(x) = -2x^2$

**318.** $f(x) = 4x^2$

**319.** $f(x) = -4x^2$

**320.** $f(x) = -x^2$

**321.** $f(x) = \frac{1}{2}x^2$

**322.** $f(x) = \frac{1}{3}x^2$

**323.** $f(x) = \frac{1}{4}x^2$

**324.** $f(x) = -\frac{1}{2}x^2$

### Graph Quadratic Functions Using Transformations

*In the following exercises, rewrite each function in the $f(x) = a(x-h)^2 + k$ form by completing the square.*

**325.** $f(x) = -3x^2 - 12x - 5$

**326.** $f(x) = 2x^2 - 12x + 7$

**327.** $f(x) = 3x^2 + 6x - 1$

**328.** $f(x) = -4x^2 - 16x - 9$

*In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)^2 + k$ form and ⓑ graph it by using transformations.*

**329.** $f(x) = x^2 + 6x + 5$

**330.** $f(x) = x^2 + 4x - 12$

**331.** $f(x) = x^2 + 4x - 12$

**332.** $f(x) = x^2 - 6x + 8$

**333.** $f(x) = x^2 - 6x + 15$

**334.** $f(x) = x^2 + 8x + 10$

**335.** $f(x) = -x^2 + 8x - 16$

**336.** $f(x) = -x^2 + 2x - 7$

**337.** $f(x) = -x^2 - 4x + 2$

**338.** $f(x) = -x^2 + 4x - 5$

**339.** $f(x) = 5x^2 - 10x + 8$

**340.** $f(x) = 3x^2 + 18x + 20$

**341.** $f(x) = 2x^2 - 4x + 1$

**342.** $f(x) = 3x^2 - 6x - 1$

**343.** $f(x) = -2x^2 + 8x - 10$

**344.** $f(x) = -3x^2 + 6x + 1$

*In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)^2 + k$ form and ⓑ graph it using properties.*

**345.** $f(x) = 2x^2 + 4x + 6$

**346.** $f(x) = 3x^2 - 12x + 7$

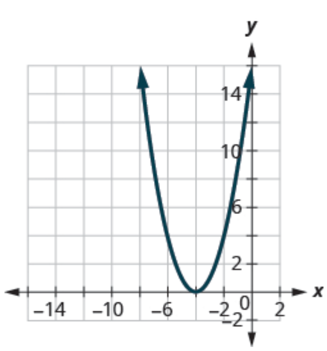
**347.** $f(x) = -x^2 + 2x - 4$

**348.** $f(x) = -2x^2 - 4x - 5$

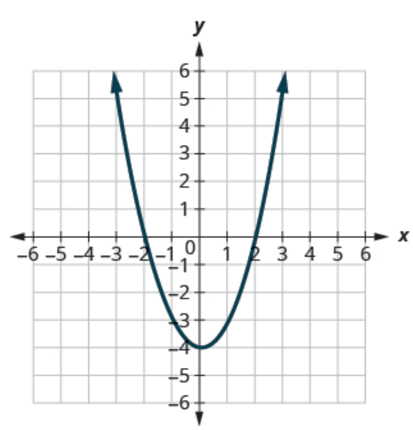
### Matching

*In the following exercises, match the graphs to one of the following functions: ⓐ $f(x) = x^2 + 4$ ⓑ $f(x) = x^2 - 4$ ⓒ $f(x) = (x+4)^2$ ⓓ $f(x) = (x-4)^2$ ⓔ $f(x) = (x+4)^2 - 4$ ⓕ $f(x) = (x+4)^2 + 4$ ⓖ $f(x) = (x-4)^2 - 4$ ⓗ $f(x) = (x-4)^2 + 4$*

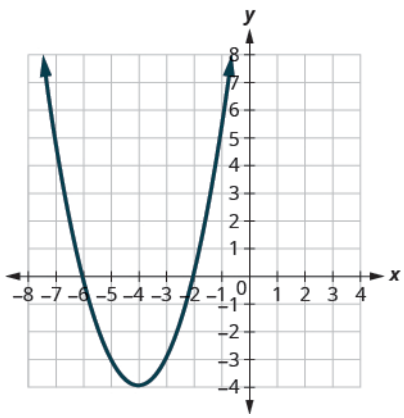
**349.**



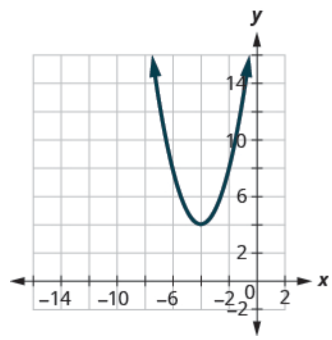

**350.**



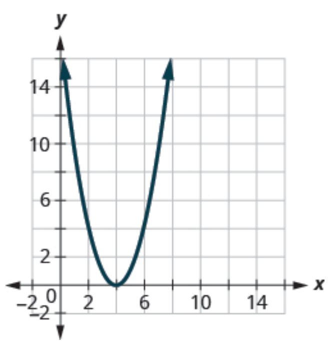
**351.**



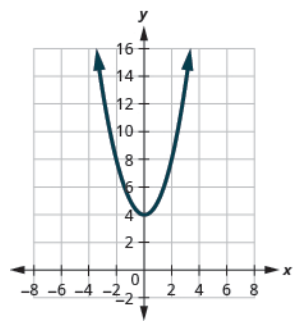

**352.**



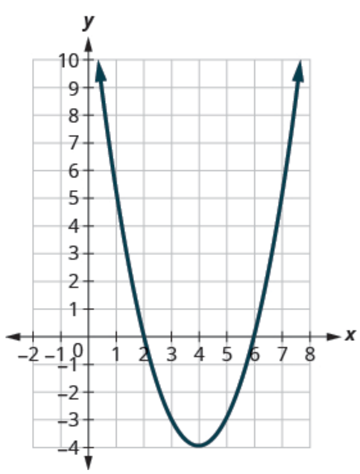

**353.**



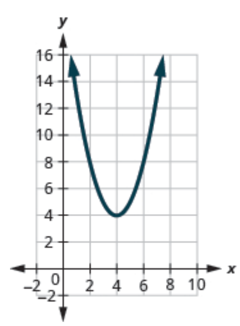

**354.**




**355.**



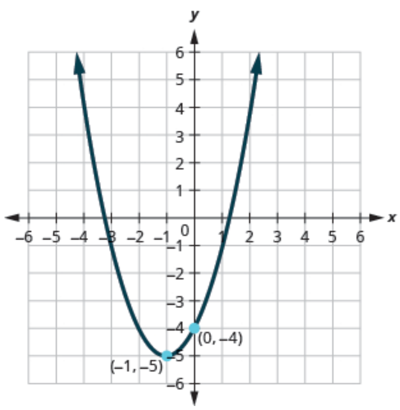

**356.**



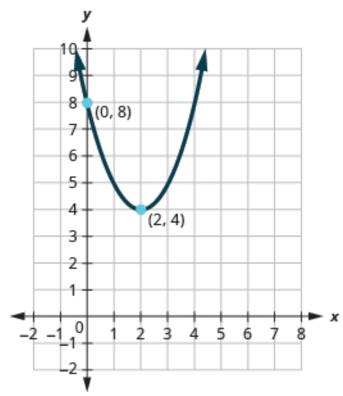
### Find a Quadratic Function from its Graph

*In the following exercises, write the quadratic function in* $f(x) = a(x - h)^2 + k$ *form whose graph is shown.*

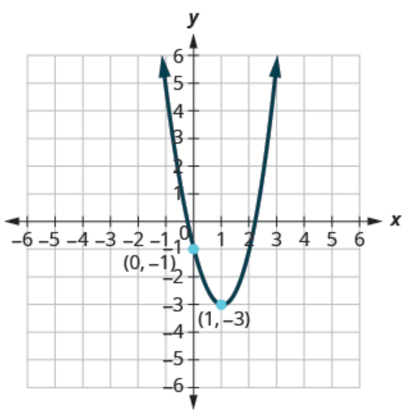
**357.**



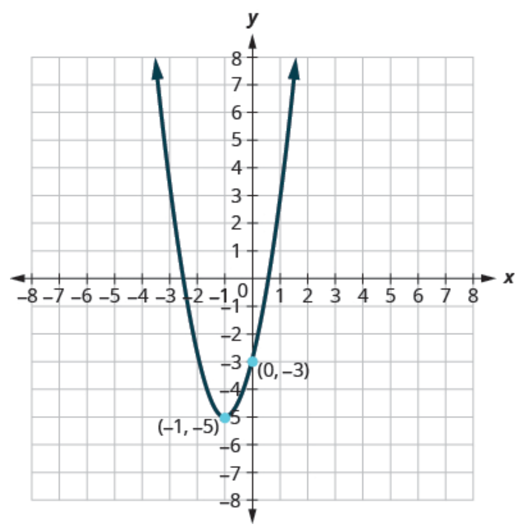

**358.**




**359.**




**360.**




## Writing Exercise

**361.** Graph the quadratic function $f(x) = x^2 + 4x + 5$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

**362.** Graph the quadratic function $f(x) = 2x^2 - 4x - 3$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| graph Quadratic Functions of the form $f(x) = x^2 + k$. | | | |
| graph Quadratic Functions of the form $f(x) = (x - h)^2$. | | | |
| graph Quadratic Functions of the form $f(x) = ax^2$. | | | |
| graph Quadratic Functions Using Transformations. | | | |
| find a Quadratic Function from its Graph. | | | |

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

# 9.8 Solve Quadratic Inequalities

## Learning Objectives

**By the end of this section, you will be able to:**

- Solve quadratic inequalities graphically
- Solve quadratic inequalities algebraically

### Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $2x - 3 = 0.$
   If you missed this problem, review **Example 2.2**.
2. Solve: $2y^2 + y = 15$.
   If you missed this problem, review **Example 6.45**.
3. Solve $\frac{1}{x^2 + 2x - 8} > 0$
   If you missed this problem, review **Example 7.56**.

We have learned how to solve linear inequalities and rational inequalities previously. Some of the techniques we used to solve them were the same and some were different.

We will now learn to solve inequalities that have a quadratic expression. We will use some of the techniques from solving linear and rational inequalities as well as quadratic equations.

We will solve quadratic inequalities two ways—both graphically and algebraically.

## Solve Quadratic Inequalities Graphically

A quadratic equation is in standard form when written as $ax^2 + bx + c = 0$. If we replace the equal sign with an inequality sign, we have a **quadratic inequality** in standard form.

### Quadratic Inequality

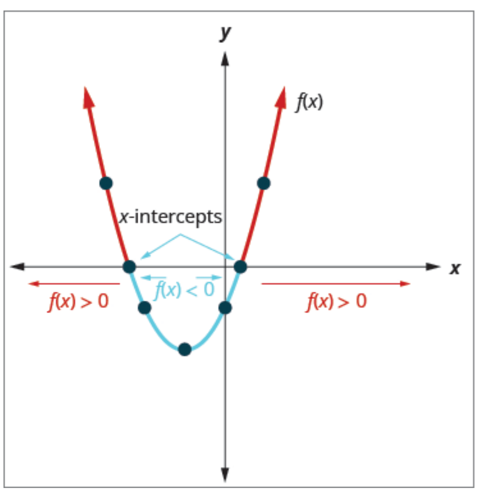
A **quadratic inequality** is an inequality that contains a quadratic expression.

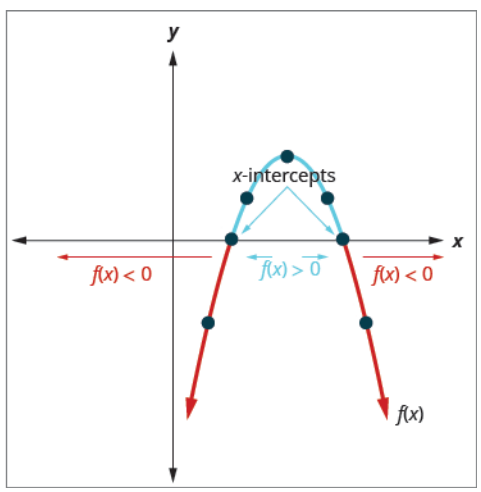
The standard form of a quadratic inequality is written:

$$ax^2 + bx + c < 0 \qquad ax^2 + bx + c \le 0$$
$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c \ge 0$$

The graph of a quadratic function $f(x) = ax^2 + bx + c = 0$ is a parabola. When we ask when is $ax^2 + bx + c < 0$, we are asking when is $f(x) < 0$. We want to know when the parabola is below the $x$-axis.

When we ask when is $ax^2 + bx + c > 0$, we are asking when is $f(x) > 0$. We want to know when the parabola is above the $y$-axis.



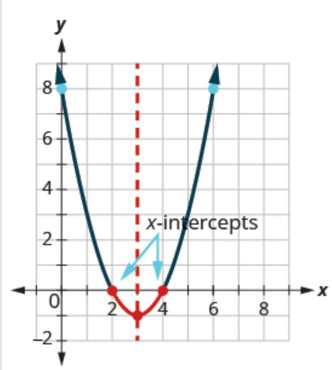



**EXAMPLE 9.64** HOW TO SOLVE A QUADRATIC INEQUALITY GRAPHICALLY

Solve $x^2 - 6x + 8 < 0$ graphically. Write the solution in interval notation.

## Solution

| | | |
|---|---|---|
| **Step 1.** Write the quadratic inequality in standard form. | The inequality is in standard form | $x^2 - 6x + 8 < 0$ |

| | | |
|---|---|---|
| **Step 2.** Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations. | We will graph using the properties.<br>Look at $a$ in the equation.<br>$f(x) = x^2 - 6x + 8$<br>Since $a$ is positive, the parabola opens upward.<br> | $f(x) = x^2 - 6x + 8$<br>$a = 1, b = -6, c = 8$<br>**The parabola opens upward.** |
| | $f(x) = x^2 - 6x + 8$<br>The axis of symmetry is the line $x = -\frac{b}{2a}$. | Axis of Symmetry<br>$x = -\frac{b}{2a}$<br>$x = -\frac{(-6)}{2 \cdot 1}$<br>$x = 3$<br>**The axis of symmetry is the line $x = 3$.** |
| | The vertex is on the axis of symmetry. Substitute $x = 3$ into the function. | Vertex<br>$f(x) = x^2 - 6x + 8$<br>$f(3) = (3)^2 - 6(3) + 8$<br>$f(3) = -1$<br>**The vertex is (3, –1).** |
| | We find $f(0)$ | $y$-intercept<br>$f(x) = x^2 - 6x + 8$<br>$f(0) = (0)^2 - 6(0) + 8$<br>$f(0) = 8$<br>**The $y$-intercept is (0, 8).** |
| | We use the axis of symmetry to find a point symmetric to the $y$-intercept. The $y$-intercept is 3 units left of the axis of symmetry, $x = 3$.<br>A point 3 units to the right of the axis of symmetry has $x = 6$. | Point symmetric to $y$-intercept<br>**The point is (6, 8).** |
| | We solve $f(x) = 0$.<br>We can solve this quadratic equation by factoring. | $x$-intercepts<br>$f(x) = x^2 - 6x + 8$<br>$0 = x^2 - 6x + 8$<br>$0 = (x - 2)(x - 4)$<br>$x = 2$ or $x = 4$<br>**The $x$-intercepts are (2, 0) and (4, 0).** |
| | We graph the vertex, intercepts, and the point symmetric to the $y$-intercept. We connect these 5 points to sketch the parabola. |  |

| | | |
|---|---|---|
| **Step 3.** Determine the solution from the graph. | $x^2 - 6x + 8 < 0$<br>The inequality asks for the values of *x* which make the function less than 0. Which values of *x* make the parabola below the *x*-axis.<br>We do not include the values 2, 4 as the inequality is less than only. | The solution, in interval notation, is (2, 4). |

**TRY IT : :** 9.127 ⓐ Solve $x^2 + 2x - 8 < 0$ graphically and ⓑ write the solution in interval notation.

**TRY IT : :** 9.128 ⓐ Solve $x^2 - 8x + 12 \geq 0$ graphically and ⓑ write the solution in interval notation.

We list the steps to take to solve a quadratic inequality graphically.

**HOW TO : :** SOLVE A QUADRATIC INEQUALITY GRAPHICALLY.

Step 1. Write the quadratic inequality in standard form.

Step 2. Graph the function $f(x) = ax^2 + bx + c$.

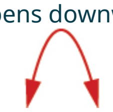
Step 3. Determine the solution from the graph.

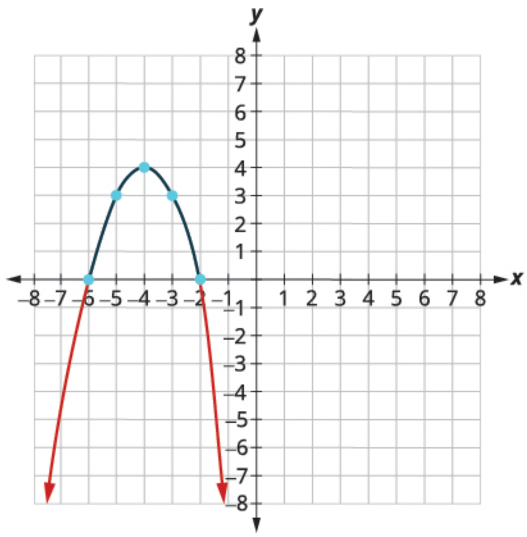
In the last example, the parabola opened upward and in the next example, it opens downward. In both cases, we are looking for the part of the parabola that is below the *x*-axis but note how the position of the parabola affects the solution.

**EXAMPLE 9.65**

Solve $-x^2 - 8x - 12 \leq 0$ graphically. Write the solution in interval notation.

## Solution

| | |
|---|---|
| The quadratic inequality in standard form. | $-x^2 - 8x - 12 \leq 0$ |
| Graph the function $f(x) = -x^2 - 8x - 12$. | The parabola opens downward.  |
| Find the line of symmetry. | $x = -\frac{b}{2a}$<br>$x = -\frac{-8}{2(-1)}$<br>$x = -4$ |
| Find the vertex. | $f(x) = -x^2 - 8x - 12$<br>$f(-4) = -(-4)^2 - 8(-4) - 12$<br>$f(-4) = -16 + 32 - 12$<br>$f(-4) = 4$<br>Vertex $(-4, 4)$ |

| | |
|---|---|
| Find the *x*-intercepts. Let $f(x) = 0$. | $f(x) = -x^2 - 8x - 12$<br>$0 = -x^2 - 8x - 12$ |
| Factor.<br>Use the Zero Product Property. | $0 = -1(x + 6)(x + 2)$<br>$x = -6 \quad x = -2$ |
| Graph the parabola. | *x*-intercepts $(-6, 0), (-2, 0)$<br> |
| Determine the solution from the graph.<br>We include the *x*-intercepts as the inequality is "less than or equal to." | $(-\infty, -6] \cup [-2, \infty)$ |

**TRY IT : :** 9.129 ⓐ Solve $-x^2 - 6x - 5 > 0$ graphically and ⓑ write the solution in interval notation.

**TRY IT : :** 9.130 ⓐ Solve $-x^2 + 10x - 16 \leq 0$ graphically and ⓑ write the solution in interval notation.

## Solve Quadratic Inequalities Algebraically

The algebraic method we will use is very similar to the method we used to solve rational inequalities. We will find the critical points for the inequality, which will be the solutions to the related quadratic equation. Remember a polynomial expression can change signs only where the expression is zero.

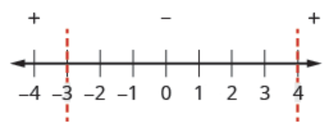
We will use the critical points to divide the number line into intervals and then determine whether the quadratic expression willl be postive or negative in the interval. We then determine the solution for the inequality.

**EXAMPLE 9.66** HOW TO SOLVE QUADRATIC INEQUALITIES ALGEBRAICALLY

Solve $x^2 - x - 12 \geq 0$ algebraically. Write the solution in interval notation.

### Solution

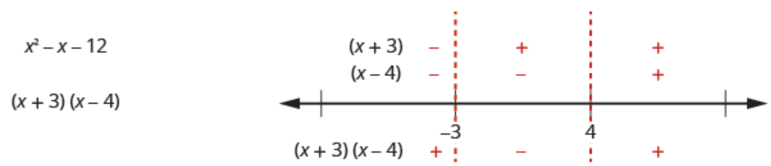
| | | |
|---|---|---|
| **Step 1.** Write the quadratic inequality in standard form. | The inequality is in standard form | $x^2 - x - 12 \geq 0$ |

| | | |
|---|---|---|
| **Step 2.** Determine the critical points—the solutions to the related quadratic equation. | Change the inequality sign to an equal sign and then solve the equation. | $x^2 - x - 12 = 0$<br>$(x + 3)(x - 4) = 0$<br>$x + 3 = 0 \quad x - 4 = 0$<br>$x = -3 \quad x = 4$ |
| **Step 3.** Use the critical points to divide the number line into intervals. | Use –3 and 4 to divide the number line into intervals |  |
| **Step 4.** Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality. | Test:<br>$x = -5$<br>$x = 0$<br>$x = 5$ | $x^2 - x - 12$   $x^2 - x - 12$   $x^2 - x - 12$<br>$(-5)^2 - (-5) - 12$   $0^2 - 0 - 12$   $5^2 - 5 - 12$<br>18   –12   8<br> |
| **Step 5.** Determine the intervals where the inequality is correct. Write the solution in interval notation. | $x^2 - x - 12 \geq 0$<br>The inequality is positive in the first and last intervals and equals 0 at the points –4, 3. | The solution, in interval notation, is $(-\infty, -3] \cup [4, \infty)$. |

**TRY IT : :** 9.131 Solve $x^2 + 2x - 8 \geq 0$ algebraically. Write the solution in interval notation.

**TRY IT : :** 9.132 Solve $x^2 - 2x - 15 \leq 0$ algebraically. Write the solution in interval notation.

In this example, since the expression $x^2 - x - 12$ factors nicely, we can also find the sign in each interval much like we did when we solved rational inequalities. We find the sign of each of the factors, and then the sign of the product. Our number line would like this:




The result is the same as we found using the other method.

We summarize the steps here.

**HOW TO : :** SOLVE A QUADRATIC INEQUALITY ALGEBRAICALLY.

Step 1. Write the quadratic inequality in standard form.

Step 2. Determine the critical points—the solutions to the related quadratic equation.

Step 3. Use the critical points to divide the number line into intervals.

Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.

Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

**EXAMPLE 9.67**

Solve $x^2 + 6x - 7 \geq 0$ algebraically. Write the solution in interval notation.

### Solution

| | |
|---|---|
| Write the quadratic inequality in standard form. | $-x^2 + 6x - 7 \geq 0$ |
| Multiply both sides of the inequality by $-1$. Remember to reverse the inequality sign. | $x^2 - 6x + 7 \leq 0$ |
| Determine the critical points by solving the related quadratic equation. | $x^2 - 6x + 7 = 0$ |
| Write the Quadratic Formula. | $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of $a, b, c$. | $x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (7)}}{2 \cdot 1}$ |
| Simplify. | $x = \dfrac{6 \pm \sqrt{8}}{2}$ |
| Simplify the radical. | $x = \dfrac{6 \pm 2\sqrt{2}}{2}$ |
| Remove the common factor, 2. | $x = \dfrac{2(3 \pm \sqrt{2})}{2}$<br>$x = 3 \pm \sqrt{2}$<br>$x = 3 + \sqrt{2} \qquad x = 3 - \sqrt{2}$<br>$x \approx 1.6 \qquad x \approx 4.4$ |
| Use the critical points to divide the number line into intervals. Test numbers from each interval in the original inequality. |  |
| Determine the intervals where the inequality is correct. Write the solution in interval notation. | $-x^2 + 6x - 7 \geq 0$ in the middle interval<br>$\left[3 - \sqrt{2},\ \ 3 + \sqrt{2}\right]$ |



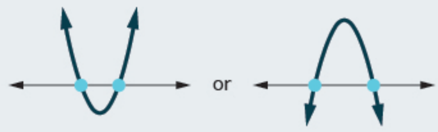
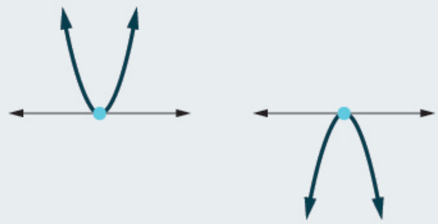
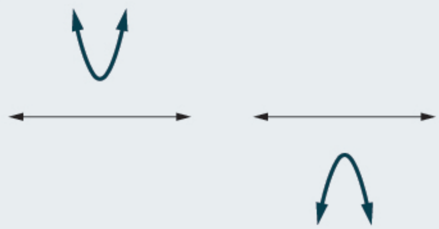
**TRY IT : :** 9.133 Solve $-x^2 + 2x + 1 \geq 0$ algebraically. Write the solution in interval notation.

**TRY IT : :** 9.134 Solve $-x^2 + 8x - 14 < 0$ algebraically. Write the solution in interval notation.

The solutions of the quadratic inequalities in each of the previous examples, were either an interval or the union of two intervals. This resulted from the fact that, in each case we found two solutions to the corresponding quadratic equation $ax^2 + bx + c = 0$. These two solutions then gave us either the two $x$-intercepts for the graph or the two critical points to divide the number line into intervals.

This correlates to our previous discussion of the number and type of solutions to a quadratic equation using the discriminant.

For a quadratic equation of the form $ax^2 + bx + c = 0,\ a \neq 0$.

| Discriminant | Number/Type of solution | Typical Graph |
|---|---|---|
| $b^2 - 4ac > 0$ | 2 real solutions<br>2 $x$-intercepts on graph |  |
| $b^2 - 4ac = 0$ | 1 real solution<br>1 $x$-intercept on graph |  |
| $b^2 - 4ac < 0$ | 2 complex solutions<br>No $x$-intercept |  |

The last row of the table shows us when the parabolas never intersect the $x$-axis. Using the Quadratic Formula to solve the quadratic equation, the radicand is a negative. We get two complex solutions.

In the next example, the quadratic inequality solutions will result from the solution of the quadratic equation being complex.

**EXAMPLE 9.68**

Solve, writing any solution in interval notation:

ⓐ $x^2 - 3x + 4 > 0$ ⓑ $x^2 - 3x + 4 \leq 0$

## Solution

ⓐ

| | |
|---|---|
| Write the quadratic inequality in standard form. | $-x^2 - 3x + 4 > 0$ |
| Determine the critical points by solving the related quadratic equation. | $x^2 - 3x + 4 = 0$ |
| Write the Quadratic Formula. | $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of $a, b, c$. | $x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$ |
| Simplify. | $x = \dfrac{3 \pm \sqrt{-7}}{2}$ |
| Simplify the radicand. | $x = \dfrac{3 \pm \sqrt{7}i}{2}$ |

| | |
|---|---|
| The complex solutions tell us the parabola does not intercept the *x*-axis. Also, the parabola opens upward. This tells us that the parabola is completely above the *x*-axis. | Complex solutions  |

We are to find the solution to $x^2 - 3x + 4 > 0$. Since for all values of $x$ the graph is above the *x*-axis, all values of *x* make the inequality true. In interval notation we write $(-\infty, \infty)$.

ⓑ

| | |
|---|---|
| Write the quadratic inequality in standard form. | $x^2 - 3x + 4 \leq 0$ |
| Determine the critical points by solving the related quadratic equation | $x^2 - 3x + 4 = 0$ |

Since the corresponding quadratic equation is the same as in part (a), the parabola will be the same. The parabola opens upward and is completely above the *x*-axis—no part of it is below the *x*-axis.

We are to find the solution to $x^2 - 3x + 4 \leq 0$. Since for all values of *x* the graph is never below the *x*-axis, no values of *x* make the inequality true. There is no solution to the inequality.

**TRY IT : :** 9.135 Solve and write any solution in interval notation:
ⓐ $-x^2 + 2x - 4 \leq 0$ ⓑ $-x^2 + 2x - 4 \geq 0$

**TRY IT : :** 9.136 Solve and write any solution in interval notation:
ⓐ $x^2 + 3x + 3 < 0$ ⓑ $x^2 + 3x + 3 > 0$

## 9.8 EXERCISES

### Practice Makes Perfect

#### Solve Quadratic Inequalities Graphically

*In the following exercises, ⓐ solve graphically and ⓑ write the solution in interval notation.*

**363.** $x^2 + 6x + 5 > 0$

**364.** $x^2 + 4x - 12 < 0$

**365.** $x^2 + 4x + 3 \leq 0$

**366.** $x^2 - 6x + 8 \geq 0$

**367.** $-x^2 - 3x + 18 \leq 0$

**368.** $-x^2 + 2x + 24 < 0$

**369.** $-x^2 + x + 12 \geq 0$

**370.** $-x^2 + 2x + 15 > 0$

*In the following exercises, solve each inequality algebraically and write any solution in interval notation.*

**371.** $x^2 + 3x - 4 \geq 0$

**372.** $x^2 + x - 6 \leq 0$

**373.** $x^2 - 7x + 10 < 0$

**374.** $x^2 - 4x + 3 > 0$

**375.** $x^2 + 8x > -15$

**376.** $x^2 + 8x < -12$

**377.** $x^2 - 4x + 2 \leq 0$

**378.** $-x^2 + 8x - 11 < 0$

**379.** $x^2 - 10x > -19$

**380.** $x^2 + 6x < -3$

**381.** $-6x^2 + 19x - 10 \geq 0$

**382.** $-3x^2 - 4x + 4 \leq 0$

**383.** $-2x^2 + 7x + 4 \geq 0$

**384.** $2x^2 + 5x - 12 > 0$

**385.** $x^2 + 3x + 5 > 0$

**386.** $x^2 - 3x + 6 \leq 0$

**387.** $-x^2 + x - 7 > 0$

**388.** $-x^2 - 4x - 5 < 0$

**389.** $-2x^2 + 8x - 10 < 0$

**390.** $-x^2 + 2x - 7 \geq 0$

### Writing Exercises

**391.** Explain critical points and how they are used to solve quadratic inequalities algebraically.

**392.** Solve $x^2 + 2x \geq 8$ both graphically and algebraically. Which method do you prefer, and why?

**393.** Describe the steps needed to solve a quadratic inequality graphically.

**394.** Describe the steps needed to solve a quadratic inequality algebraically.

### Self Check

*ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| solve quadratic inequalities graphically. | | | |
| solve quadratic inequalities algebraically. | | | |

*ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

# CHAPTER 9 REVIEW

## KEY TERMS

**discriminant** In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

**quadratic function** A quadratic function, where $a$, $b$, and $c$ are real numbers and $a \neq 0,$ is a function of the form $f(x) = ax^2 + bx + c.$

**quadratic inequality** A quadratic inequality is an inequality that contains a quadratic expression.

## KEY CONCEPTS

### 9.1 Solve Quadratic Equations Using the Square Root Property

- Square Root Property
  - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$ or $x = \pm\sqrt{k}$

How to solve a quadratic equation using the square root property.

Step 1. Isolate the quadratic term and make its coefficient one.

Step 2. Use Square Root Property.

Step 3. Simplify the radical.

Step 4. Check the solutions.

### 9.2 Solve Quadratic Equations by Completing the Square

- Binomial Squares Pattern
  If $a$ and $b$ are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- How to Complete a Square

  Step 1. Identify $b$, the coefficient of $x$.

  Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

  Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$

  Step 4. Rewrite the trinomial as a binomial square

- How to solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square.

  Step 1. Divide by $a$ to make the coefficient of $x^2$ term 1.

  Step 2. Isolate the variable terms on one side and the constant terms on the other.

  Step 3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

  Step 4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.

  Step 5. Use the Square Root Property.

  Step 6. Simplify the radical and then solve the two resulting equations.

  Step 7.

Check the solutions.

## 9.3 Solve Quadratic Equations Using the Quadratic Formula

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.

  Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of $a$, $b$, $c$.

  Step 2. Write the Quadratic Formula. Then substitute in the values of $a$, $b$, $c$.

  Step 3. Simplify.

  Step 4. Check the solutions.

- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,
    - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
    - if $b^2 - 4ac = 0$, the equation has 1 real solution.
    - if $b^2 - 4ac < 0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.

  Step 1. Try Factoring first. If the quadratic factors easily, this method is very quick.

  Step 2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.

  Step 3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

## 9.4 Solve Quadratic Equations in Quadratic Form

- How to solve equations in quadratic form.

  Step 1. Identify a substitution that will put the equation in quadratic form.

  Step 2. Rewrite the equation with the substitution to put it in quadratic form.

  Step 3. Solve the quadratic equation for $u$.

  Step 4. Substitute the original variable back into the results, using the substitution.

  Step 5. Solve for the original variable.

  Step 6. Check the solutions.

## 9.5 Solve Applications of Quadratic Equations

- Methods to Solve Quadratic Equations
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to use a Problem-Solving Strategy.

  Step 1. **Read** the problem. Make sure all the words and ideas are understood.

  Step 2. **Identify** what we are looking for.

  Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

  Step 4.

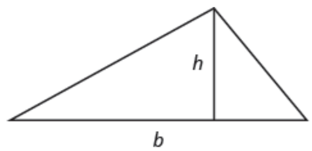
**Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5. **Solve** the equation using good algebra techniques.

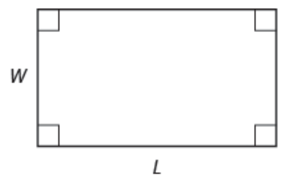
Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

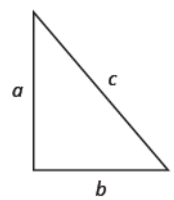
- Area of a Triangle
  - For a triangle with base, *b*, and height, *h*, the area, *A*, is given by the formula $A = \frac{1}{2}bh.$




- Area of a Rectangle
  - For a rectangle with length, *L*, and width, *W*, the area, *A*, is given by the formula $A = LW$.




- Pythagorean Theorem
  - In any right triangle, where *a* and *b* are the lengths of the legs, and *c* is the length of the hypotenuse, $a^2 + b^2 = c^2$.




- Projectile motion
  - The height in feet, *h*, of an object shot upwards into the air with initial velocity, $v_0$, after *t* seconds is given by the formula $h = -16t^2 + v_0t$.

## 9.6 Graph Quadratic Functions Using Properties

- Parabola Orientation
  - For the graph of the quadratic function $f(x) = ax^2 + bx + c,$ if
    - $a > 0$, the parabola opens upward.
    - $a < 0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:
  - the axis of symmetry is the vertical line $x = -\frac{b}{2a}.$
  - the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}.$
  - the *y*-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$ :

| **$y$-intercept** | **$x$-intercepts** |
|---|---|
| Let $x = 0$ and solve for $f(x)$. | Let $f(x) = 0$ and solve for $x$. |

- How to graph a quadratic function using properties.
  Step 1. Determine whether the parabola opens upward or downward.
  Step 2. Find the equation of the axis of symmetry.
  Step 3. Find the vertex.
  Step 4. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  Step 5. Find the *x*-intercepts. Find additional points if needed.
  Step 6. Graph the parabola.
- Minimum or Maximum Values of a Quadratic Equation
  - The *y*-coordinate of the vertex of the graph of a quadratic equation is the
  - *minimum* value of the quadratic equation if the parabola opens *upward*.
  - *maximum* value of the quadratic equation if the parabola opens *downward*.

## 9.7 Graph Quadratic Functions Using Transformations

- Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift
  - The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.
    - If $k > 0$, shift the parabola vertically up *k* units.
    - If $k < 0$, shift the parabola vertically down $|k|$ units.
- Graph a Quadratic Function of the form $f(x) = (x - h)^2$ Using a Horizontal Shift
  - The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.
    - If $h > 0$, shift the parabola horizontally left *h* units.
    - If $h < 0$, shift the parabola horizontally right $|h|$ units.
- Graph of a Quadratic Function of the form $f(x) = ax^2$
  - The coefficient *a* in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.
    If $0 < |a| < 1,$ then the graph of $f(x) = ax^2$ will be “wider” than the graph of $f(x) = x^2$.
    If $|a| > 1,$ then the graph of $f(x) = ax^2$ will be “skinnier” than the graph of $f(x) = x^2$.
- How to graph a quadratic function using transformations
  Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form by completing the square.
  Step 2. Graph the function using transformations.
- Graph a quadratic function in the vertex form $f(x) = a(x - h)^2 + k$ using properties
  Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form.
  Step 2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
  Step 3. Find the axis of symmetry, $x = h$.
  Step 4. Find the vertex, $(h, k)$.
  Step 5. Find the*y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  Step 6. Find the *x*-intercepts, if possible.
  Step 7. Graph the parabola.

## 9.8 Solve Quadratic Inequalities

- Solve a Quadratic Inequality Graphically

  Step 1. Write the quadratic inequality in standard form.

  Step 2. Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations.

  Step 3. Determine the solution from the graph.

- How to Solve a Quadratic Inequality Algebraically

  Step 1. Write the quadratic inequality in standard form.

  Step 2. Determine the critical points -- the solutions to the related quadratic equation.

  Step 3. Use the critical points to divide the number line into intervals.

  Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.

  Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

# REVIEW EXERCISES

## 9.1 Section 9.1 Solve Quadratic Equations Using the Square Root Property

### Solve Quadratic Equations of the form $ax^2 = k$ Using the Square Root Property

*In the following exercises, solve using the Square Root Property.*

**395.** $y^2 = 144$

**396.** $n^2 - 80 = 0$

**397.** $4a^2 = 100$

**398.** $2b^2 = 72$

**399.** $r^2 + 32 = 0$

**400.** $t^2 + 18 = 0$

**401.** $\frac{2}{3}w^2 - 20 = 30$

**402.** 11. $5c^2 + 3 = 19$

### Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

*In the following exercises, solve using the Square Root Property.*

**403.** $(p-5)^2 + 3 = 19$

**404.** $(u+1)^2 = 45$

**405.** $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$

**406.** $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$

**407.** $(n-4)^2 - 50 = 150$

**408.** $(4c-1)^2 = -18$

**409.** $n^2 + 10n + 25 = 12$

**410.** $64a^2 + 48a + 9 = 81$

## 9.2 Section 9.2 Solve Quadratic Equations by Completing the Square

### Solve Quadratic Equations Using Completing the Square

*In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.*

**411.** $x^2 + 22x$

**412.** $m^2 - 8m$

**413.** $a^2 - 3a$

**414.** $b^2 + 13b$

*In the following exercises, solve by completing the square.*

**415.** $d^2 + 14d = -13$

**416.** $y^2 - 6y = 36$

**417.** $m^2 + 6m = -109$

**418.** $t^2 - 12t = -40$

**419.** $v^2 - 14v = -31$

**420.** $w^2 - 20w = 100$

**421.** $m^2 + 10m - 4 = -13$

**422.** $n^2 - 6n + 11 = 34$

**423.** $a^2 = 3a + 8$

**424.** $b^2 = 11b - 5$

**425.** $(u + 8)(u + 4) = 14$

**426.** $(z - 10)(z + 2) = 28$

### Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

*In the following exercises, solve by completing the square.*

**427.** $3p^2 - 18p + 15 = 15$

**428.** $5q^2 + 70q + 20 = 0$

**429.** $4y^2 - 6y = 4$

**430.** $2x^2 + 2x = 4$

**431.** $3c^2 + 2c = 9$

**432.** $4d^2 - 2d = 8$

**433.** $2x^2 + 6x = -5$

**434.** $2x^2 + 4x = -5$

## 9.3 Section 9.3 Solve Quadratic Equations Using the Quadratic Formula

*In the following exercises, solve by using the Quadratic Formula.*

**435.** $4x^2 - 5x + 1 = 0$

**436.** $7y^2 + 4y - 3 = 0$

**437.** $r^2 - r - 42 = 0$

**438.** $t^2 + 13t + 22 = 0$

**439.** $4v^2 + v - 5 = 0$

**440.** $2w^2 + 9w + 2 = 0$

**441.** $3m^2 + 8m + 2 = 0$

**442.** $5n^2 + 2n - 1 = 0$

**443.** $6a^2 - 5a + 2 = 0$

**444.** $4b^2 - b + 8 = 0$

**445.** $u(u - 10) + 3 = 0$

**446.** $5z(z - 2) = 3$

**447.** $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$

**448.** $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$

**449.** $4c^2 + 4c + 1 = 0$

**450.** $9d^2 - 12d = -4$

### Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

*In the following exercises, determine the number of solutions for each quadratic equation.*

**451.**

ⓐ $9x^2 - 6x + 1 = 0$
ⓑ $3y^2 - 8y + 1 = 0$
ⓒ $7m^2 + 12m + 4 = 0$
ⓓ $5n^2 - n + 1 = 0$

**452.**

ⓐ $5x^2 - 7x - 8 = 0$
ⓑ $7x^2 - 10x + 5 = 0$
ⓒ $25x^2 - 90x + 81 = 0$
ⓓ $15x^2 - 8x + 4 = 0$

### Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

*In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.*

**453.**

ⓐ $16r^2 - 8r + 1 = 0$
ⓑ $5t^2 - 8t + 3 = 9$
ⓒ $3(c + 2)^2 = 15$

**454.**

ⓐ $4d^2 + 10d - 5 = 21$
ⓑ $25x^2 - 60x + 36 = 0$
ⓒ $6(5v - 7)^2 = 150$

## 9.4 Section 9.4 Solve Equations in Quadratic Form

### Solve Equations in Quadratic Form

*In the following exercises, solve.*

**455.** $x^4 - 14x^2 + 24 = 0$

**456.** $x^4 + 4x^2 - 32 = 0$

**457.** $4x^4 - 5x^2 + 1 = 0$

**458.** $(2y + 3)^2 + 3(2y + 3) - 28 = 0$

**459.** $x + 3\sqrt{x} - 28 = 0$

**460.** $6x + 5\sqrt{x} - 6 = 0$

**461.** $x^{\frac{2}{3}} - 10x^{\frac{1}{3}} + 24 = 0$

**462.** $x + 7x^{\frac{1}{2}} + 6 = 0$

**463.** $8x^{-2} - 2x^{-1} - 3 = 0$

## 9.5 Section 9.5 Solve Applications Modeled by Quadratic Equations

### Solve Applications Modeled by Quadratic Equations

*In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.*

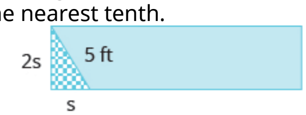
**464.** Find two consecutive odd numbers whose product is 323.

**465.** Find two consecutive even numbers whose product is 624.

**466.** A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.

**467.** Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.

**468.** A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.




**469.** A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

**470.** The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

**471.** For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

**472.** A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

**473.** The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

**474.** Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

**475.** Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

## 9.6 Section 9.6 Graphing Quadratic Functions Using Properties

### Recognize the Graph of a Quadratic Function

*In the following exercises, graph by plotting point.*

**476.** Graph $y = x^2 - 2$

**477.** Graph $y = -x^2 + 3$

*In the following exercises, determine if the following parabolas open up or down.*

**478.**

ⓐ $y = -3x^2 + 3x - 1$

ⓑ $y = 5x^2 + 6x + 3$

**479.**

ⓐ $y = x^2 + 8x - 1$

ⓑ $y = -4x^2 - 7x + 1$

### Find the Axis of Symmetry and Vertex of a Parabola

*In the following exercises, find ⓐ the equation of the axis of symmetry and ⓑ the vertex.*

**480.** $y = -x^2 + 6x + 8$

**481.** $y = 2x^2 - 8x + 1$

### Find the Intercepts of a Parabola

*In the following exercises, find the x- and y-intercepts.*

**482.** $y = x^2 - 4x + 5$

**483.** $y = x^2 - 8x + 15$

**484.** $y = x^2 - 4x + 10$

**485.** $y = -5x^2 - 30x - 46$

**486.** $y = 16x^2 - 8x + 1$

**487.** $y = x^2 + 16x + 64$

### Graph Quadratic Functions Using Properties

*In the following exercises, graph by using its properties.*

**488.** $y = x^2 + 8x + 15$

**489.** $y = x^2 - 2x - 3$

**490.** $y = -x^2 + 8x - 16$

**491.** $y = 4x^2 - 4x + 1$

**492.** $y = x^2 + 6x + 13$

**493.** $y = -2x^2 - 8x - 12$

### Solve Maximum and Minimum Applications

*In the following exercises, find the minimum or maximum value.*

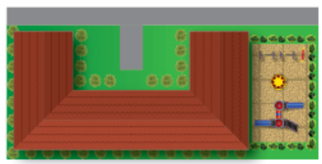
**494.** $y = 7x^2 + 14x + 6$

**495.** $y = -3x^2 + 12x - 10$

*In the following exercises, solve. Rounding answers to the nearest tenth.*

**496.** A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.

**497.** A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, $A$, of the yard for the length, $x$, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.



## 9.7 Section 9.7 Graphing Quadratic Functions Using Transformations

### Graph Quadratic Functions of the form $f(x) = x^2 + k$

*In the following exercises, graph each function using a vertical shift.*

**498.** $g(x) = x^2 + 4$

**499.** $h(x) = x^2 - 3$

*In the following exercises, graph each function using a horizontal shift.*

**500.** $f(x) = (x + 1)^2$

**501.** $g(x) = (x - 3)^2$

*In the following exercises, graph each function using transformations.*

**502.** $f(x) = (x + 2)^2 + 3$

**503.** $f(x) = (x + 3)^2 - 2$

**504.** $f(x) = (x - 1)^2 + 4$

**505.** $f(x) = (x - 4)^2 - 3$

### Graph Quadratic Functions of the form $f(x) = ax^2$

*In the following exercises, graph each function.*

**506.** $f(x) = 2x^2$

**507.** $f(x) = -x^2$

**508.** $f(x) = \frac{1}{2}x^2$

### Graph Quadratic Functions Using Transformations

*In the following exercises, rewrite each function in the $f(x) = a(x - h)^2 + k$ form by completing the square.*

**509.** $f(x) = 2x^2 - 4x - 4$

**510.** $f(x) = 3x^2 + 12x + 8$

*In the following exercises, ⓐ rewrite each function in $f(x) = a(x - h)^2 + k$ form and ⓑ graph it by using transformations.*

**511.** $f(x) = 3x^2 - 6x - 1$

**512.** $f(x) = -2x^2 - 12x - 5$

**513.** $f(x) = 2x^2 + 4x + 6$

**514.** $f(x) = 3x^2 - 12x + 7$

*In the following exercises, ⓐ rewrite each function in $f(x) = a(x - h)^2 + k$ form and ⓑ graph it using properties.*

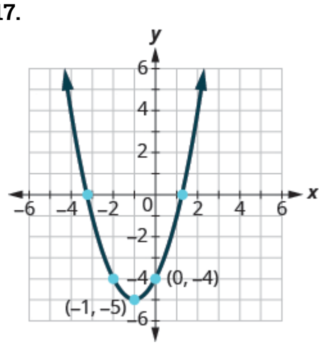
**515.** $f(x) = -3x^2 - 12x - 5$

**516.** $f(x) = 2x^2 - 12x + 7$

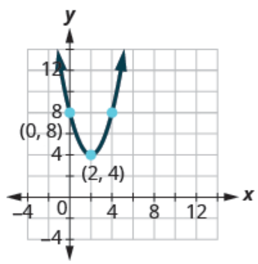
### Find a Quadratic Function from its Graph

*In the following exercises, write the quadratic function in $f(x) = a(x - h)^2 + k$ form.*

**517.**




**518.**




## 9.8 Section 9.8 Solve Quadratic Inequalities

### Solve Quadratic Inequalities Graphically

*In the following exercises, solve graphically and write the solution in interval notation.*

**519.** $x^2 - x - 6 > 0$

**520.** $x^2 + 4x + 3 \leq 0$

**521.** $-x^2 - x + 2 \geq 0$

**522.** $-x^2 + 2x + 3 < 0$

*In the following exercises, solve each inequality algebraically and write any solution in interval notation.*

**523.** $x^2 - 6x + 8 < 0$

**524.** $x^2 + x > 12$

**525.** $x^2 - 6x + 4 \leq 0$

**526.** $2x^2 + 7x - 4 > 0$

**527.** $-x^2 + x - 6 > 0$

**528.** $x^2 - 2x + 4 \geq 0$

## PRACTICE TEST

**529.** Use the Square Root Property to solve the quadratic equation $3(w+5)^2 = 27$.

**530.** Use Completing the Square to solve the quadratic equation $a^2 - 8a + 7 = 23$.

**531.** Use the Quadratic Formula to solve the quadratic equation $2m^2 - 5m + 3 = 0$.

*Solve the following quadratic equations. Use any method.*

**532.** $2x(3x-2) - 1 = 0$

**533.** $\frac{9}{4}y^2 - 3y + 1 = 0$

*Use the discriminant to determine the number and type of solutions of each quadratic equation.*

**534.** $6p^2 - 13p + 7 = 0$

**535.** $3q^2 - 10q + 12 = 0$

*Solve each equation.*

**536.** $4x^4 - 17x^2 + 4 = 0$

**537.** $y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - 3 = 0$

*For each parabola, find ⓐ which direction it opens, ⓑ the equation of the axis of symmetry, ⓒ the vertex, ⓓ the x- and y-intercepts, and e) the maximum or minimum value.*

**538.** $y = 3x^2 + 6x + 8$

**539.** $y = -x^2 - 8x + 16$

*Graph each quadratic function using intercepts, the vertex, and the equation of the axis of symmetry.*

**540.** $f(x) = x^2 + 6x + 9$

**541.** $f(x) = -2x^2 + 8x + 4$

*In the following exercises, graph each function using transformations.*

**542.** $f(x) = (x+3)^2 + 2$

**543.** $f(x) = x^2 - 4x - 1$

*In the following exercises, solve each inequality algebraically and write any solution in interval notation.*

**544.** $x^2 - 6x - 8 \leq 0$

**545.** $2x^2 + x - 10 > 0$

*Model the situation with a quadratic equation and solve by any method.*

**546.** Find two consecutive even numbers whose product is 360.

**547.** The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

**548.** A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.

# 10 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

**Figure 10.1** Hydroponic systems allow botanists to grow crops without land. (credit: “Izhamwong”/Wikimedia Commons)

## Chapter Outline



## Introduction

As the world population continues to grow, food supplies are becoming less able to meet the increasing demand. At the same time, available resources of fertile soil for growing plants is dwindling. One possible solution—grow plants without soil. Botanists around the world are expanding the potential of hydroponics, which is the process of growing plants without soil. To provide the plants with the nutrients they need, the botanists keep careful growth records. Some growth is described by the types of functions you will explore in this chapter—exponential and logarithmic. You will evaluate and graph these functions, and solve equations using them.

## 10.1 Finding Composite and Inverse Functions

### Learning Objectives

**By the end of this section, you will be able to:**

- Find and evaluate composite functions
- Determine whether a function is one-to-one
- Find the inverse of a function

#### Be Prepared!

Before you get started, take this readiness quiz.

1. If $f(x) = 2x - 3$ and $g(x) = x^2 + 2x - 3$, find $f(4)$.
   If you missed this problem, review **Example 3.48**.
2. Solve for $x$, $3x + 2y = 12$.
   If you missed this problem, review **Example 2.31**.
3. Simplify: $5\frac{(x+4)}{5} - 4$.
   If you missed this problem, review **Example 1.25**.

In this chapter, we will introduce two new types of functions, exponential functions and logarithmic functions. These functions are used extensively in business and the sciences as we will see.

## Find and Evaluate Composite Functions

Before we introduce the functions, we need to look at another operation on functions called composition. In composition, the output of one function is the input of a second function. For functions $f$ and $g,$ the composition is written $f \circ g$ and is defined by $(f \circ g)(x) = f(g(x)).$

We read $f(g(x))$ as "$f$ of $g$ of $x.$"


To do a composition, the output of the first function, $g(x),$ becomes the input of the second function, *f*, and so we must be sure that it is part of the domain of *f*.

### Composition of Functions

The composition of functions *f* and *g* is written $f \cdot g$ and is defined by

$$(f \circ g)(x) = f(g(x))$$

We read $f(g(x))$ as $f$ of $g$ of *x*.

We have actually used composition without using the notation many times before. When we graphed quadratic functions using translations, we were composing functions. For example, if we first graphed $g(x) = x^2$ as a parabola and then shifted it down vertically four units, we were using the composition defined by $(f \circ g)(x) = f(g(x))$ where $f(x) = x - 4.$


The next example will demonstrate that $(f \circ g)(x),$ $(g \circ f)(x)$ and $(f \cdot g)(x)$ usually result in different outputs.

**EXAMPLE 10.1**

For functions $f(x) = 4x - 5$ and $g(x) = 2x + 3,$ find: ⓐ $(f \circ g)(x),$ ⓑ $(g \circ f)(x),$ and ⓒ $(f \cdot g)(x).$

### Solution

ⓐ

| | |
|---|---|
| Use the definition of $(f \circ g)(x).$ | $(f \circ g)(x) = f(g(x))$ |
| Substitute $2x + 3$ for $g(x)$. | $(f \circ g)(x) = f(2x + 3)$ |
| Find $f(2x + 3)$ where $f(x) = 4x - 5$. | $(f \circ g)(x) = 4(2x + 3) - 5$ |
| Distribute. | $(f \circ g)(x) = 8x + 12 - 5$ |
| Simplify. | $(f \circ g)(x) = 8x + 7$ |

ⓑ

| | |
|---|---|
| Use the definition of $(f \circ g)(x)$. | $(g \circ f)(x) = g(f(x))$ |
| Substitute $4x - 5$ for $f(x)$. | $(g \circ f)(x) = g(4x - 5)$ |
| Find $g(4x - 5)$ where $g(x) = 2x + 3$. | $(g \circ f)(x) = 2(4x - 5) + 3$ |
| Distribute. | $(g \circ f)(x) = 8x - 10 + 3$ |
| Simplify. | $(g \circ f)(x) = 8x - 7$ |

Notice the difference in the result in part ⓐ and part ⓑ.

ⓒ Notice that $(f \cdot g)(x)$ is different than $(f \circ g)(x)$. In part ⓐ we did the composition of the functions. Now in part ⓒ we are not composing them, we are multiplying them.

| | |
|---|---|
| Use the definition o $(f \cdot g)(x)$. | $(f \cdot g)(x) = f(x) \cdot g(x)$ |
| Substitute $f(x) = 4x - 5$ and $g(x) = 2x + 3$. | $(f \cdot g)(x) = (4x - 5) \cdot (2x + 3)$ |
| Multiply. | $(f \cdot g)(x) = 8x^2 + 2x - 15$ |

**TRY IT : :** 10.1 For functions $f(x) = 3x - 2$ and $g(x) = 5x + 1$, find ⓐ $(f \circ g)(x)$ ⓑ $(g \circ f)(x)$ ⓒ $(f \cdot g)(x)$ .

**TRY IT : :** 10.2

For functions $f(x) = 4x - 3$, and $g(x) = 6x - 5$, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

In the next example we will evaluate a composition for a specific value.

**EXAMPLE 10.2**

For functions $f(x) = x^2 - 4$, and $g(x) = 3x + 2$, find: ⓐ $(f \circ g)(-3)$, ⓑ $(g \circ f)(-1)$, and ⓒ $(f \circ f)(2)$.

## Solution

ⓐ

| | |
|---|---|
| Use the definition of $(f \circ g)(-3)$. | $(f \circ g)(-3) = f(g(-3))$ |
| Find $g(-3)$ where $g(x) = 3x + 2$. | $(f \circ g)(-3) = f(3 \cdot (-3) + 2)$ |
| Simplify. | $(f \circ g)(-3) = f(-7)$ |
| Find $f(-7)$ where $f(x) = x^2 - 4$. | $(f \circ g)(-3) = (-7)^2 - 4$ |
| Simplify. | $(f \circ g)(-3) = 45$ |

ⓑ

| | |
|---|---|
| Use the definition of $(g \circ f)(-1)$. | $(g \circ f)(-1) = g(f(-1))$ |
| Find $f(-1)$ where $f(x) = x^2 - 4$. | $(g \circ f)(-1) = g((-1)^2 - 4)$ |
| Simplify. | $(g \circ f)(-1) = g(-3)$ |
| Find $g(-3)$ where $g(x) = 3x + 2$. | $(g \circ f)(-1) = 3(-3) + 2$ |
| Simplify. | $(g \circ f)(-1) = -7$ |

ⓒ

| | |
|---|---|
| Use the definition of $(f \circ f)(2)$. | $(f \circ f)(2) = f(f(2))$ |
| Find $f(2)$ where $f(x) = x^2 - 4$. | $(f \circ f)(2) = f(2^2 - 4)$ |
| Simplify. | $(f \circ f)(2) = f(0)$ |
| Find $f(0)$ where $f(x) = x^2 - 4$. | $(f \circ f)(2) = 0^2 - 4$ |
| Simplify. | $(f \circ f)(2) = -4$ |

**TRY IT : :** 10.3

For functions $f(x) = x^2 - 9$, and $g(x) = 2x + 5$, find ⓐ $(f \circ g)(-2)$, ⓑ $(g \circ f)(-3)$, and ⓒ $(f \circ f)(4)$.

**TRY IT : :** 10.4

For functions $f(x) = x^2 + 1$, and $g(x) = 3x - 5$, find ⓐ $(f \circ g)(-1)$, ⓑ $(g \circ f)(2)$, and ⓒ $(f \circ f)(-1)$.

## Determine Whether a Function is One-to-One

When we first introduced functions, we said a function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

We used the birthday example to help us understand the definition. Every person has a birthday, but no one has two birthdays and it is okay for two people to share a birthday. Since each person has exactly one birthday, that relation is a function.


A function is **one-to-one** if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

Our example of the birthday relation is not a one-to-one function. Two people can share the same birthday. The range value August 2 is the birthday of Liz and June, and so one range value has two domain values. Therefore, the function is not one-to-one.

### One-to-One Function

A function is **one-to-one** if each value in the range corresponds to one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value. There are no repeated *y*-values.

## EXAMPLE 10.3

For each set of ordered pairs, determine if it represents a function and, if so, if the function is one-to-one.

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$ and ⓑ $\{(0, 0), (1, 1), (4, 2), (9, 3), (16, 4)\}$.

### Solution

ⓐ

$$\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

But each *y*-value is not paired with only one *x*-value, $(-3, 27)$ and $(3, 27)$, for example. So this function is not one-to-one.

ⓑ

$$\{(0, 0), (1, 1), (4, 2), (9, 3), (16, 4)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

Since each *y*-value is paired with only one *x*-value, this function is one-to-one.

**TRY IT : :** 10.5

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$

ⓑ $\{(-4, 8), (-2, 4), (-1, 2), (0, 0), (1, 2), (2, 4), (4, 8)\}$

**TRY IT : :** 10.6

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(27, -3), (8, -2), (1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$

ⓑ $\{(7, -3), (-5, -4), (8, 0), (0, 0), (-6, 4), (-2, 2), (-1, 3)\}$

To help us determine whether a relation is a function, we use the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. Also, if any vertical line intersects the graph in more than one point, the graph does not represent a function.

The vertical line is representing an *x*-value and we check that it intersects the graph in only one *y*-value. Then it is a function.

To check if a function is one-to-one, we use a similar process. We use a horizontal line and check that each horizontal line intersects the graph in only one point. The horizontal line is representing a *y*-value and we check that it intersects the graph in only one *x*-value. If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function. This is the **horizontal line test**.

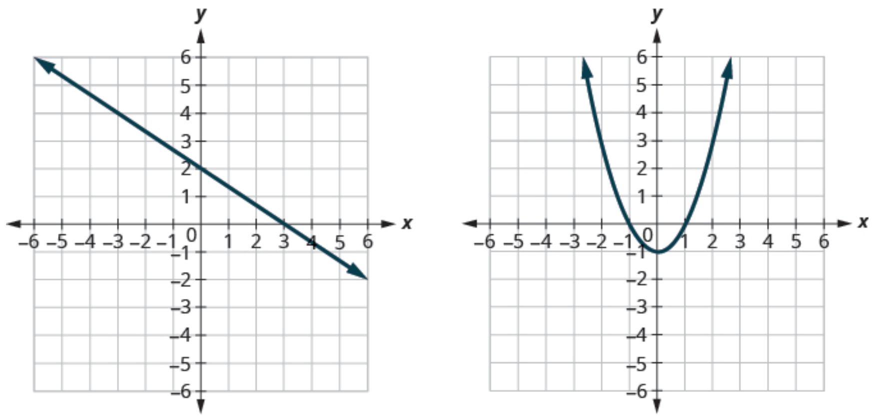
### Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function.

We can test whether a graph of a relation is a function by using the vertical line test. We can then tell if the function is one-to-one by applying the horizontal line test.

## EXAMPLE 10.4

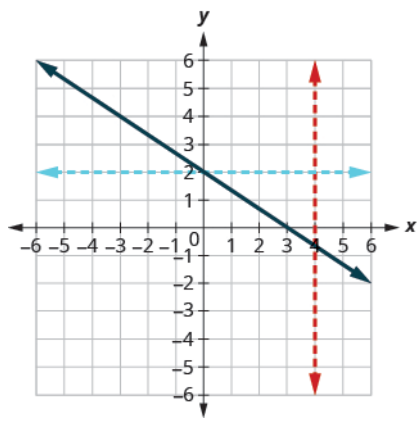
Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



(a) (b)

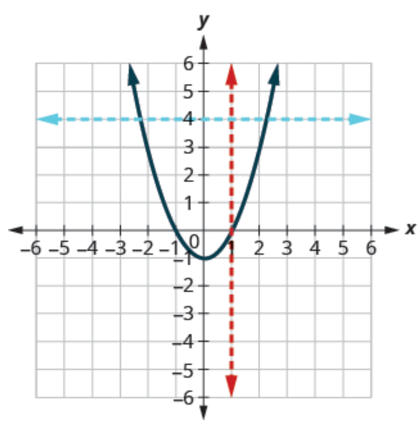
## Solution

ⓐ



Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.

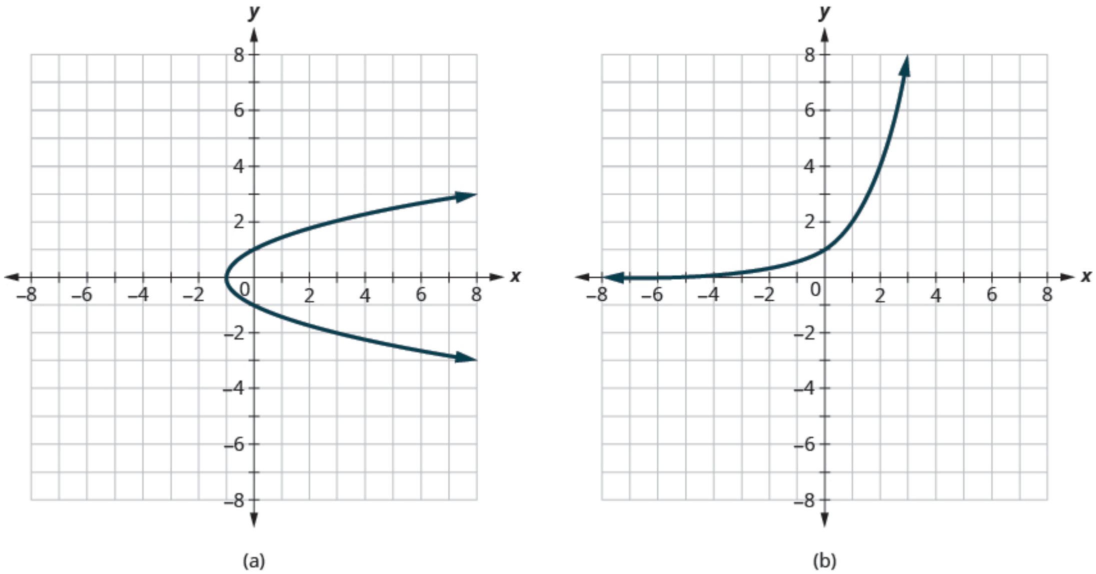
ⓑ



Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. The horizontal line shown on the graph intersects it in two points. This graph does not represent a one-to-one function.

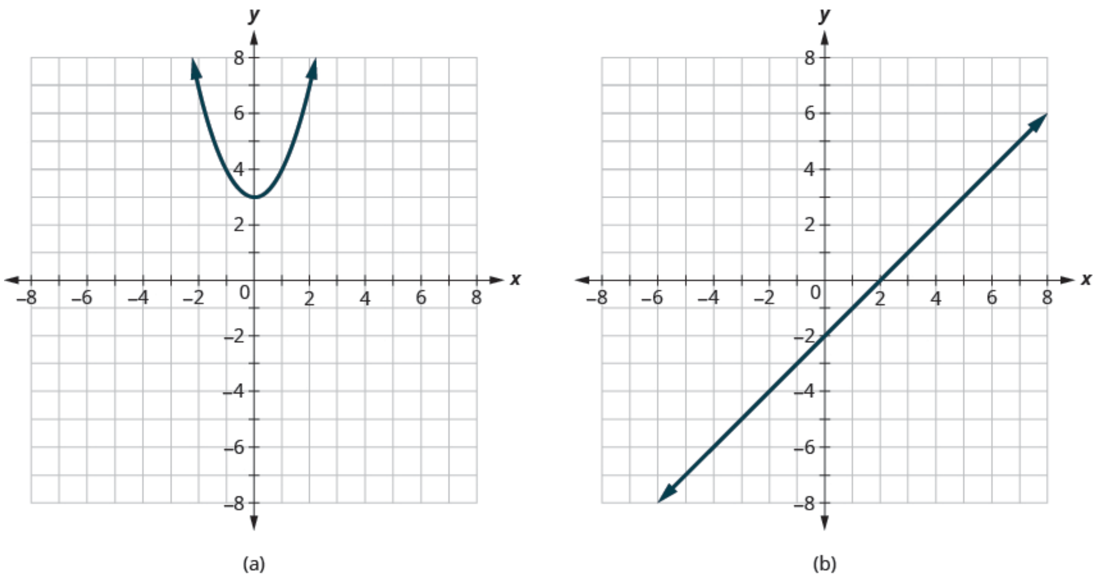
> **TRY IT : :** 10.7

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



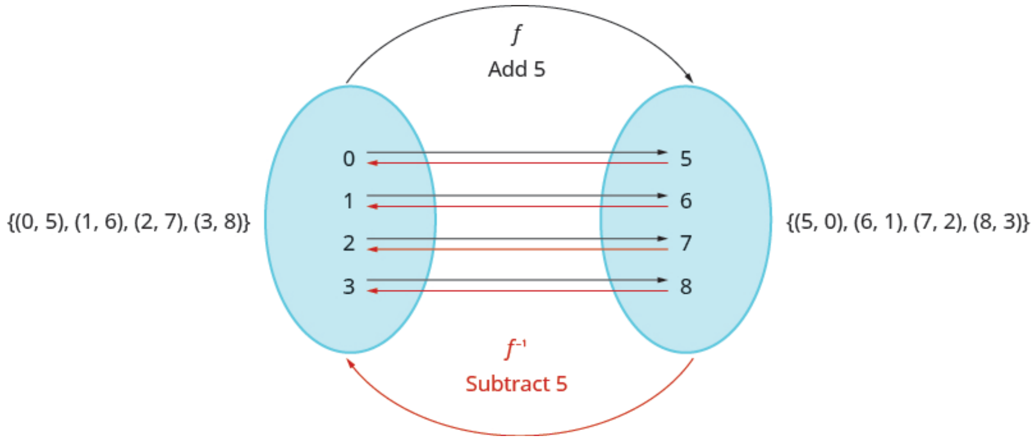

> **TRY IT : :** 10.8

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.




## Find the Inverse of a Function

Let's look at a one-to one function, $f$, represented by the ordered pairs $\{(0,\ 5),\ (1,\ 6),\ (2,\ 7),\ (3,\ 8)\}$. For each $x$-value, $f$ adds 5 to get the $y$-value. To 'undo' the addition of 5, we subtract 5 from each $y$-value and get back to the original $x$-value. We can call this "taking the inverse of $f$" and name the function $f^{-1}$.




Notice that that the ordered pairs of $f$ and $f^{-1}$ have their $x$-values and $y$-values reversed. The domain of $f$ is the range of $f^{-1}$ and the domain of $f^{-1}$ is the range of $f$.

**Inverse of a Function Defined by Ordered Pairs**

If $f(x)$ is a one-to-one function whose ordered pairs are of the form $(x, y)$, then its inverse function $f^{-1}(x)$ is the set of ordered pairs $(y, x)$.

In the next example we will find the inverse of a function defined by ordered pairs.

**EXAMPLE 10.5**

Find the inverse of the function $\{(0, 3), (1, 5), (2, 7), (3, 9)\}$. Determine the domain and range of the inverse function.

## Solution

This function is one-to-one since every $x$-value is paired with exactly one $y$-value.

To find the inverse we reverse the $x$-values and $y$-values in the ordered pairs of the function.

| | |
|---|---|
| Function | $\{(0, 3), (1, 5), (2, 7), (3, 9)\}$ |
| Inverse Function | $\{(3, 0), (5, 1), (7, 2), (9, 3)\}$ |
| Domain of Inverse Function | $\{3, 5, 7, 9\}$ |
| Range of Inverse Function | $\{0, 1, 2, 3\}$ |

**TRY IT : :** 10.9

Find the inverse of $\{(0, 4), (1, 7), (2, 10), (3, 13)\}$. Determine the domain and range of the inverse function.

**TRY IT : :** 10.10

Find the inverse of $\{(-1, 4), (-2, 1), (-3, 0), (-4, 2)\}$. Determine the domain and range of the inverse function.

We just noted that if $f(x)$ is a one-to-one function whose ordered pairs are of the form $(x, y)$, then its inverse function $f^{-1}(x)$ is the set of ordered pairs $(y, x)$.

So if a point $(a, b)$ is on the graph of a function $f(x)$, then the ordered pair $(b, a)$ is on the graph of $f^{-1}(x)$. See Figure 10.2.

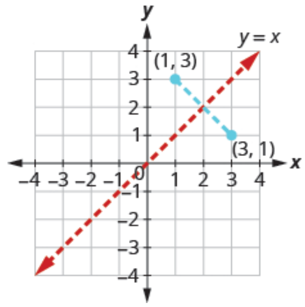


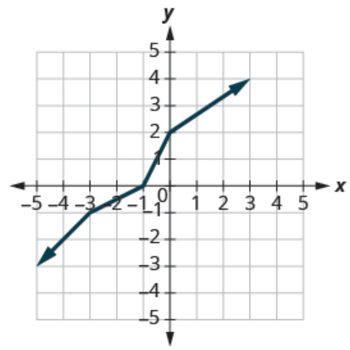

**Figure 10.2**

The distance between any two pairs $(a, b)$ and $(b, a)$ is cut in half by the line $y = x.$ So we say the points are mirror images of each other through the line $y = x.$

Since every point on the graph of a function $f(x)$ is a mirror image of a point on the graph of $f^{-1}(x),$ we say the graphs are mirror images of each other through the line $y = x.$ We will use this concept to graph the inverse of a function in the next example.

**EXAMPLE 10.6**

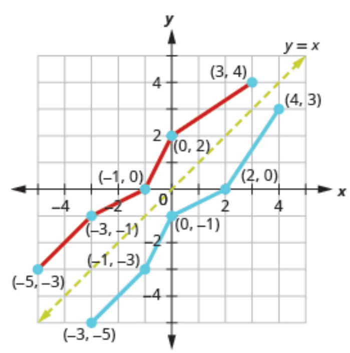
Graph, on the same coordinate system, the inverse of the one-to one function shown.




## Solution

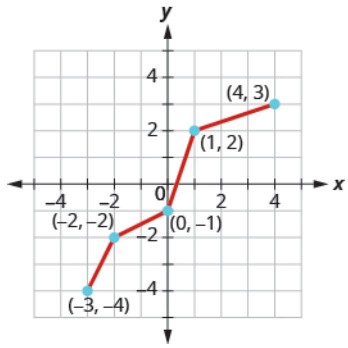
We can use points on the graph to find points on the inverse graph. Some points on the graph are: $(-5, -3), (-3, -1), (-1, 0), (0, 2), (3, 4)$.

So, the inverse function will contain the points: $(-3, -5), (-1, -3), (0, -1), (2, 0), (4, 3)$.

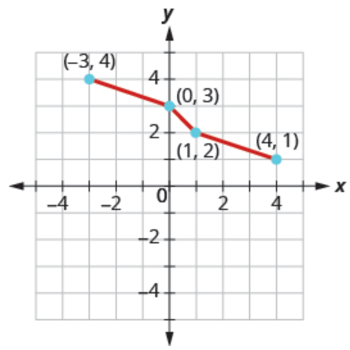



Notice how the graph of the original function and the graph of the inverse functions are mirror images through the line $y = x.$

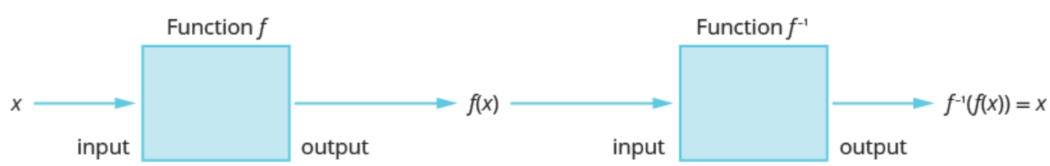
**TRY IT : :** 10.11 Graph, on the same coordinate system, the inverse of the one-to one function.




**TRY IT : :** 10.12 Graph, on the same coordinate system, the inverse of the one-to one function.




When we began our discussion of an inverse function, we talked about how the inverse function 'undoes' what the original function did to a value in its domain in order to get back to the original *x*-value.




**Inverse Functions**

$$\begin{aligned} f^{-1}(f(x)) &= x, \text{ for all } x \text{ in the domain of } f \\ f\left(f^{-1}(x)\right) &= x, \text{ for all } x \text{ in the domain of } f^{-1} \end{aligned}$$

We can use this property to verify that two functions are inverses of each other.

**EXAMPLE 10.7**

Verify that $f(x) = 5x - 1$ and $g(x) = \dfrac{x+1}{5}$ are inverse functions.

## Solution

The functions are inverses of each other if $g(f(x)) = x$ and $f(g(x)) = x$.

| | |
|---|---|
| | $g(f(x)) \stackrel{?}{=} x$ |
| Substitute $5x-1$ for $f(x)$. | $g(5x-1) \stackrel{?}{=} x$ |
| Find $g(5x-1)$ where $g(x)=\frac{x+1}{5}$. | $\frac{(5x-1)+1}{5} \stackrel{?}{=} x$ |
| Simplify. | $\frac{5x}{5} \stackrel{?}{=} x$ |
| Simplify. | $x = x$ ✓ |

| | |
|---|---|
| | $f(g(x)) \stackrel{?}{=} x$ |
| Substitute $\frac{x+1}{5}$ for $g(x)$. | $f\left(\frac{x+1}{5}\right) \stackrel{?}{=} x$ |
| Find $f\left(\frac{x+1}{5}\right)$ where $f(x)=5x-1$. | $5\left(\frac{x+1}{5}\right)-1 \stackrel{?}{=} x$ |
| Simplify. | $x+1-1 \stackrel{?}{=} x$ |
| Simplify. | $x = x$ ✓ |

Since both $g(f(x)) = x$ and $f(g(x)) = x$ are true, the functions $f(x) = 5x - 1$ and $g(x) = \frac{x+1}{5}$ are inverse functions. That is, they are inverses of each other.

**TRY IT : :** 10.13 Verify that the functions are inverse functions.

$f(x) = 4x - 3$ and $g(x) = \frac{x+3}{4}$.

**TRY IT : :** 10.14 Verify that the functions are inverse functions.

$f(x) = 2x + 6$ and $g(x) = \frac{x-6}{2}$.

We have found inverses of function defined by ordered pairs and from a graph. We will now look at how to find an inverse using an algebraic equation. The method uses the idea that if $f(x)$ is a one-to-one function with ordered pairs $(x, y)$, then its inverse function $f^{-1}(x)$ is the set of ordered pairs $(y, x)$.

If we reverse the *x* and *y* in the function and then solve for *y*, we get our inverse function.

**EXAMPLE 10.8** HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

Find the inverse of $f(x) = 4x + 7$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Substitute *y* for *f*(*x*). | Replace *f*(*x*) with *y*. | $f(x) = 4x + 7$<br>$y = 4x + 7$ |
| **Step 2.** Interchange the variables *x* and *y*. | Replace *x* with *y* and then *y* with *x*. | $x = 4y + 7$ |

| | | |
|---|---|---|
| **Step 3.** Solve for *y*. | Subtract 7 from each side.<br>Divide by 4. | $x-7=4y$<br>$\frac{x-7}{4}=y$ |
| **Step 4.** Substitute $f^{-1}(x)$ for *y*. | Replace *y* with $f^{-1}(x)$. | $\frac{x-7}{4}=f^{-1}(x)$ |
| **Step 5.** Verify that the functions are inverses. | Show $f^{-1}(f(x))=x$<br>and $f(f^{-1}(x))=x$ | $f^{-1}(f(x))\stackrel{?}{=}x$<br>$f^{-1}(4x+7)\stackrel{?}{=}x$<br>$\frac{(4x+7)-7}{4}\stackrel{?}{=}x$<br>$\frac{4x}{4}\stackrel{?}{=}x$<br>$x=x$ ✓<br>$f(f^{-1}(x))\stackrel{?}{=}x$<br>$f\left(\frac{x-7}{4}\right)\stackrel{?}{=}x$<br>$4\left(\frac{x-7}{4}\right)+7\stackrel{?}{=}x$<br>$x-7+7\stackrel{?}{=}x$<br>$x=x$ ✓ |

**TRY IT : :** 10.15 Find the inverse of the function $f(x)=5x-3$.

**TRY IT : :** 10.16 Find the inverse of the function $f(x)=8x+5$.

We summarize the steps below.

**HOW TO : : HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION**

Step 1. Substitute *y* for $f(x)$.

Step 2. Interchange the variables *x* and *y*.

Step 3. Solve for *y*.

Step 4. Substitute $f^{-1}(x)$ for *y*.

Step 5. Verify that the functions are inverses.

**EXAMPLE 10.9** HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

Find the inverse of $f(x)=\sqrt[5]{2x-3}$.

## Solution

| | |
|---|---|
| | $f(x) = \sqrt[5]{2x-3}$ |
| Substitute $y$ for $f(x)$. | $y = \sqrt[5]{2x-3}$ |
| Interchange the variables $x$ and $y$. | $x = \sqrt[5]{2y-3}$ |
| Solve for $y$. | $(x)^5 = \left(\sqrt[5]{2y-3}\right)^5$ |
| | $x^5 = 2y-3$ |
| | $x^5+3 = 2y$ |
| | $\frac{x^5+3}{2} = y$ |
| Substitute $f^{-1}(x)$ for $y$. | $f^{-1}(x) = \frac{x^5+3}{2}$ |

Verify that the functions are inverses.

$$\begin{aligned} f^{-1}(f(x)) &\stackrel{?}{=} x \\ f^{-1}\left(\sqrt[5]{2x-3}\right) &\stackrel{?}{=} x \\ \frac{\left(\sqrt[5]{2x-3}\right)^5+3}{2} &\stackrel{?}{=} x \\ \frac{2x-3+3}{2} &\stackrel{?}{=} x \\ \frac{2x}{2} &\stackrel{?}{=} x \\ x &= x \checkmark \end{aligned}$$

$$\begin{aligned} f\left(f^{-1}(x)\right) &\stackrel{?}{=} x \\ f\left(\frac{x^5+3}{2}\right) &\stackrel{?}{=} x \\ \sqrt[5]{2\left(\frac{x^5+3}{2}\right)-3} &\stackrel{?}{=} x \\ \sqrt[5]{x^5+3-3} &\stackrel{?}{=} x \\ \sqrt[5]{x^5} &\stackrel{?}{=} x \\ x &= x \checkmark \end{aligned}$$

**TRY IT : :** 10.17 Find the inverse of the function $f(x) = \sqrt[5]{3x-2}$.

**TRY IT : :** 10.18 Find the inverse of the function $f(x) = \sqrt[4]{6x-7}$.

## 10.1 EXERCISES

### Practice Makes Perfect

#### Find and Evaluate Composite Functions

*In the following exercises, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.*

**1.** $f(x) = 4x + 3$ and $g(x) = 2x + 5$

**2.** $f(x) = 3x - 1$ and $g(x) = 5x - 3$

**3.** $f(x) = 6x - 5$ and $g(x) = 4x + 1$

**4.** $f(x) = 2x + 7$ and $g(x) = 3x - 4$

**5.** $f(x) = 3x$ and $g(x) = 2x^2 - 3x$

**6.** $f(x) = 2x$ and $g(x) = 3x^2 - 1$

**7.** $f(x) = 2x - 1$ and $g(x) = x^2 + 2$

**8.** $f(x) = 4x + 3$ and $g(x) = x^2 - 4$

*In the following exercises, find the values described.*

**9.** For functions $f(x) = 2x^2 + 3$ and $g(x) = 5x - 1,$ find
ⓐ $(f \circ g)(-2)$
ⓑ $(g \circ f)(-3)$
ⓒ $(f \circ f)(-1)$

**10.** For functions $f(x) = 5x^2 - 1$ and $g(x) = 4x - 1,$ find
ⓐ $(f \circ g)(1)$
ⓑ $(g \circ f)(-1)$
ⓒ $(f \circ f)(2)$

**11.** For functions $f(x) = 2x^3$ and $g(x) = 3x^2 + 2,$ find
ⓐ $(f \circ g)(-1)$
ⓑ $(g \circ f)(1)$
ⓒ $(g \circ g)(1)$

**12.** For functions $f(x) = 3x^3 + 1$ and $g(x) = 2x^2 - 3,$ find
ⓐ $(f \circ g)(-2)$
ⓑ $(g \circ f)(-1)$
ⓒ $(g \circ g)(1)$

#### Determine Whether a Function is One-to-One

*In the following exercises, determine if the set of ordered pairs represents a function and if so, is the function one-to-one.*

**13.** $\{(-3, 9), (-2, 4), (-1, 1), (0, 0),$ $(1, 1), (2, 4), (3, 9)\}$

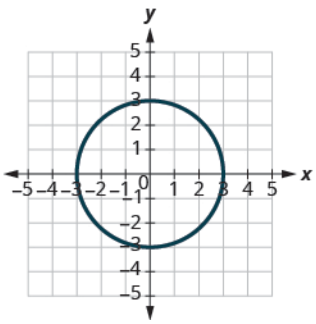
**14.** $\{(9, -3), (4, -2), (1, -1), (0, 0),$ $(1, 1), (4, 2), (9, 3)\}$

**15.** $\{(-3, -5), (-2, -3), (-1, -1),$ $(0, 1), (1, 3), (2, 5), (3, 7)\}$

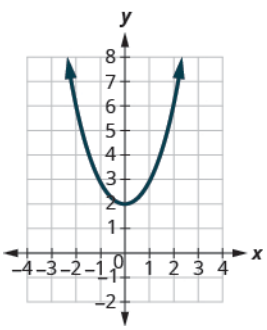
**16.** $\{(5, 3), (4, 2), (3, 1), (2, 0),$ $(1, -1), (0, -2), (-1, -3)\}$

*In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.*

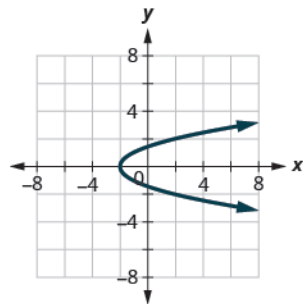
**17.** ⓐ



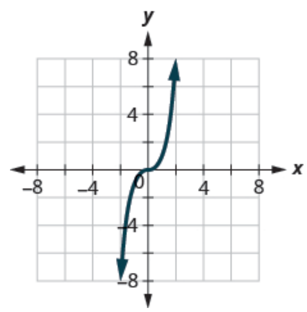

ⓑ



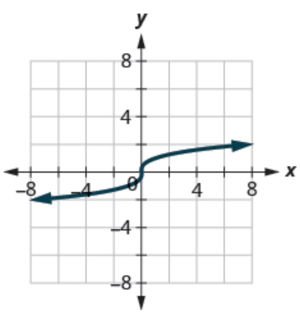

**18.** ⓐ



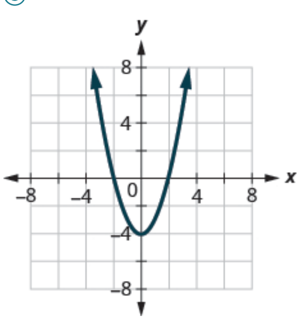

ⓑ



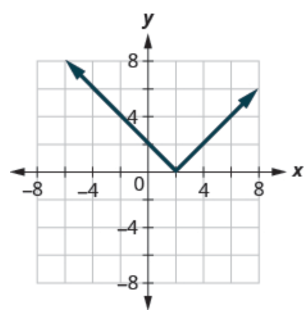

**19.** ⓐ



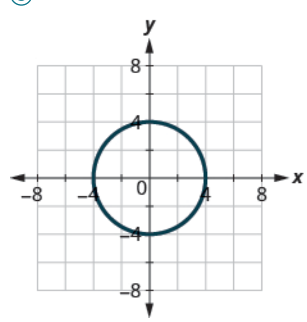

ⓑ




**20.** ⓐ




ⓑ




*In the following exercises, find the inverse of each function. Determine the domain and range of the inverse function.*

**21.** $\{(2, 1), (4, 2), (6, 3), (8, 4)\}$

**22.** $\{(6, 2), (9, 5), (12, 8), (15, 11)\}$

**23.** $\{(0, -2), (1, 3), (2, 7), (3, 12)\}$

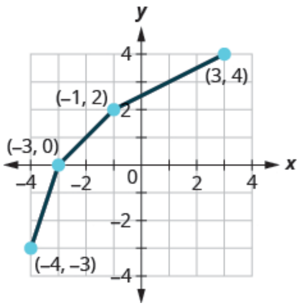
**24.** $\{(0, 0), (1, 1), (2, 4), (3, 9)\}$

**25.** $\{(-2, -3), (-1, -1), (0, 1), (1, 3)\}$

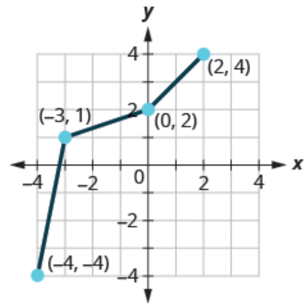
**26.** $\{(5, 3), (4, 2), (3, 1), (2, 0)\}$

*In the following exercises, graph, on the same coordinate system, the inverse of the one-to-one function shown.*

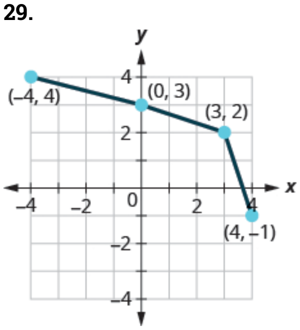
**27.**



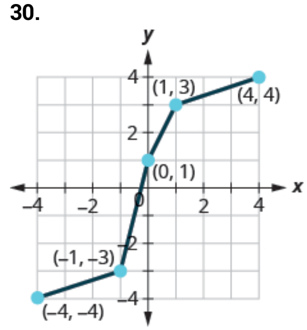

**28.**




**29.**




**30.**




*In the following exercises, determine whether or not the given functions are inverses.*

**31.** $f(x) = x + 8$ and $g(x) = x - 8$

**32.** $f(x) = x - 9$ and $g(x) = x + 9$

**33.** $f(x) = 7x$ and $g(x) = \frac{x}{7}$

**34.** $f(x) = \frac{x}{11}$ and $g(x) = 11x$

**35.** $f(x) = 7x + 3$ and $g(x) = \frac{x - 3}{7}$

**36.** $f(x) = 5x - 4$ and $g(x) = \frac{x - 4}{5}$

**37.** $f(x) = \sqrt{x + 2}$ and $g(x) = x^2 - 2$

**38.** $f(x) = \sqrt[3]{x - 4}$ and $g(x) = x^3 + 4$

*In the following exercises, find the inverse of each function.*

**39.** $f(x) = x - 12$

**40.** $f(x) = x + 17$

**41.** $f(x) = 9x$

**42.** $f(x) = 8x$

**43.** $f(x) = \frac{x}{6}$

**44.** $f(x) = \frac{x}{4}$

**45.** $f(x) = 6x - 7$

**46.** $f(x) = 7x - 1$

**47.** $f(x) = -2x + 5$

**48.** $f(x) = -5x - 4$

**49.** $f(x) = x^2 + 6, \quad x \geq 0$

**50.** $f(x) = x^2 - 9, \quad x \geq 0$

**51.** $f(x) = x^3 - 4$

**52.** $f(x) = x^3 + 6$

**53.** $f(x) = \frac{1}{x+2}$

**54.** $f(x) = \frac{1}{x-6}$

**55.** $f(x) = \sqrt{x-2}, \quad x \geq 2$

**56.** $f(x) = \sqrt{x+8}, \quad x \geq -8$

**57.** $f(x) = \sqrt[3]{x-3}$

**58.** $f(x) = \sqrt[3]{x+5}$

**59.** $f(x) = \sqrt[4]{9x-5}, \quad x \geq \frac{5}{9}$

**60.** $f(x) = \sqrt[4]{8x-3}, \quad x \geq \frac{3}{8}$

**61.** $f(x) = \sqrt[5]{-3x+5}$

**62.** $f(x) = \sqrt[5]{-4x-3}$

## Writing Exercises

**63.** Explain how the graph of the inverse of a function is related to the graph of the function.

**64.** Explain how to find the inverse of a function from its equation. Use an example to demonstrate the steps.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| find and evaluate composite functions. | | | |
| determine whether a function is one-to-one. | | | |
| find the inverse of a function. | | | |

ⓑ *If most of your checks were:*

*...confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

*...with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

*...no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

# 10.2 Evaluate and Graph Exponential Functions

## Learning Objectives

**By the end of this section, you will be able to:**

- Graph exponential functions
- Solve Exponential equations
- Use exponential models in applications

### Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\left(\frac{x^3}{x^2}\right)$.

   If you missed this problem, review **Example 5.13**.

2. Evaluate: ⓐ $2^0$ ⓑ $\left(\frac{1}{3}\right)^0$.

   If you missed this problem, review **Example 5.14**.

3. Evaluate: ⓐ $2^{-1}$ ⓑ $\left(\frac{1}{3}\right)^{-1}$.

   If you missed this problem, review **Example 5.15**.

## Graph Exponential Functions

The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

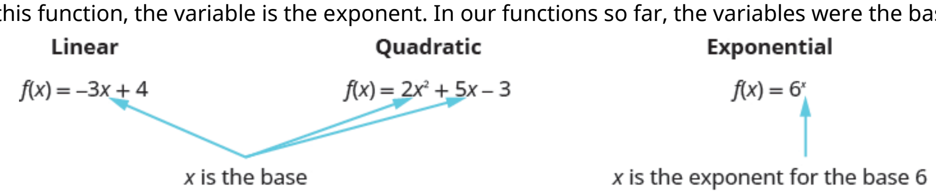
An **exponential function** is a function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$.

### Exponential Function

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form

$$f(x) = a^x$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.




Our definition says $a \neq 1$. If we let $a = 1$, then $f(x) = a^x$ becomes $f(x) = 1^x$. Since $1^x = 1$ for all real numbers, $f(x) = 1$. This is the constant function.

Our definition also says $a > 0$. If we let a base be negative, say $-4$, then $f(x) = (-4)^x$ is not a real number when $x = \frac{1}{2}$.

$$\begin{aligned} f(x) &= (-4)^x \\ f\left(\frac{1}{2}\right) &= (-4)^{\frac{1}{2}} \\ f\left(\frac{1}{2}\right) &= \sqrt{-4} \quad \text{not a real number} \end{aligned}$$

In fact, $f(x) = (-4)^x$ would not be a real number any time $x$ is a fraction with an even denominator. So our definition requires $a > 0$.

By graphing a few exponential functions, we will be able to see their unique properties.

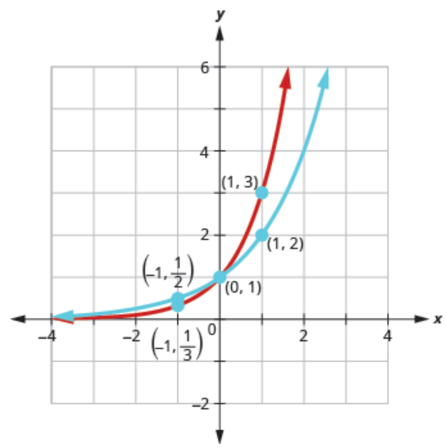
**EXAMPLE 10.10**

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

## Solution

We will use point plotting to graph the functions.

| $x$ | $f(x) = 2^x$ | $(x, f(x))$ | $g(x) = 3^x$ | $(x, g(x))$ |
|---|---|---|---|---|
| −2 | $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$ | $\left(-2, \frac{1}{4}\right)$ | $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ | $\left(-2, \frac{1}{9}\right)$ |
| −1 | $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$ | $\left(-1, \frac{1}{2}\right)$ | $3^{-1} = \frac{1}{3^1} = \frac{1}{3}$ | $\left(-1, \frac{1}{3}\right)$ |
| 0 | $2^0 = 1$ | (0, 1) | $3^0 = 1$ | (0, 1) |
| 1 | $2^1 = 2$ | (1, 2) | $3^1 = 3$ | (1, 3) |
| 2 | $2^2 = 4$ | (2, 4) | $3^2 = 9$ | (2, 9) |
| 3 | $2^3 = 8$ | (3, 8) | $3^3 = 27$ | (3, 27) |




**TRY IT : :** 10.19 Graph: $f(x) = 4^x$.

**TRY IT : :** 10.20 Graph: $g(x) = 5^x$.

If we look at the graphs from the previous Example and Try Its, we can identify some of the properties of exponential functions.

The graphs of $f(x) = 2^x$ and $g(x) = 3^x$, as well as the graphs of $f(x) = 4^x$ and $g(x) = 5^x$, all have the same basic shape. This is the shape we expect from an exponential function where $a > 1$.

We notice, that for each function, the graph contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any *a*.

The graph of each function, $f(x) = a^x$ also contains the point $(1, a)$. The graph of $f(x) = 2^x$ contained $(1, 2)$ and the graph of $g(x) = 3^x$ contained $(1, 3)$. This makes sense as $a^1 = a$.

Notice too, the graph of each function $f(x) = a^x$ also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x) = 2^x$ contained $\left(-1, \frac{1}{2}\right)$ and the graph of $g(x) = 3^x$ contained $\left(-1, \frac{1}{3}\right)$. This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty, \infty)$.

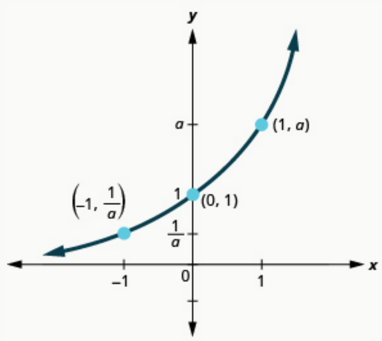
Look at each graph. What is the range of the function? The graph never hits the $x$-axis. The range is all positive numbers.

We write the range in interval notation as $(0,\ \infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the $x$-axis very closely but will never cross it, we call the line $y = 0$, the *x*-axis, a horizontal asymptote.

### Properties of the Graph of $f(x) = a^x$ when $a > 1$

| | |
|---|---|
| Domain | $(-\infty,\ \infty)$ |
| Range | $(0,\ \infty)$ |
| *x*-intercept | None |
| *y*-intercept | $(0,\ 1)$ |
| Contains | $(1,\ a),\ \left(-1,\ \frac{1}{a}\right)$ |
| Asymptote | $x$-axis, the line $y = 0$ |




Our definition of an exponential function $f(x) = a^x$ says $a > 0,$ but the examples and discussion so far has been about functions where $a > 1$. What happens when $0 < a < 1$? The next example will explore this possibility.

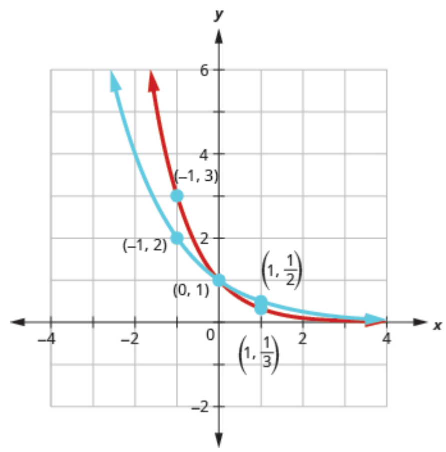
**EXAMPLE 10.11**

On the same coordinate system, graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

### Solution

We will use point plotting to graph the functions.

| $x$ | $f(x)=\left(\frac{1}{2}\right)^x$ | $(x, f(x))$ | | $g(x)=\left(\frac{1}{3}\right)^x$ | $(x, g(x))$ |
|---|---|---|---|---|---|
| −2 | $\left(\frac{1}{2}\right)^{-2}=2^2=4$ | (−2, 4) | | $\left(\frac{1}{3}\right)^{-2}=3^2=9$ | (−2, 9) |
| −1 | $\left(\frac{1}{2}\right)^{-1}=2^1=2$ | (−1, 2) | | $\left(\frac{1}{3}\right)^{-1}=3^1=3$ | (−1, 3) |
| 0 | $\left(\frac{1}{2}\right)^{0}=1$ | (0, 1) | | $\left(\frac{1}{3}\right)^{0}=1$ | (0, 1) |
| 1 | $\left(\frac{1}{2}\right)^{1}=\frac{1}{2}$ | $\left(1, \frac{1}{2}\right)$ | | $\left(\frac{1}{3}\right)^{1}=\frac{1}{3}$ | $\left(1, \frac{1}{3}\right)$ |
| 2 | $\left(\frac{1}{2}\right)^{2}=\frac{1}{4}$ | $\left(2, \frac{1}{4}\right)$ | | $\left(\frac{1}{3}\right)^{2}=\frac{1}{9}$ | $\left(2, \frac{1}{9}\right)$ |
| 3 | $\left(\frac{1}{2}\right)^{3}=\frac{1}{8}$ | $\left(3, \frac{1}{8}\right)$ | | $\left(\frac{1}{3}\right)^{3}=\frac{1}{27}$ | $\left(3, \frac{1}{27}\right)$ |




**TRY IT : :** 10.21

Graph: $f(x)=\left(\frac{1}{4}\right)^x$.

**TRY IT : :** 10.22

Graph: $g(x)=\left(\frac{1}{5}\right)^x$.

Now let's look at the graphs from the previous Example and Try Its so we can now identify some of the properties of exponential functions where $0<a<1$.

The graphs of $f(x)=\left(\frac{1}{2}\right)^x$ and $g(x)=\left(\frac{1}{3}\right)^x$ as well as the graphs of $f(x)=\left(\frac{1}{4}\right)^x$ and $g(x)=\left(\frac{1}{5}\right)^x$ all have the same basic shape. While this is the shape we expect from an exponential function where $0<a<1$, the graphs go down from left to right while the previous graphs, when $a>1$, went from up from left to right.

We notice that for each function, the graph still contains the point (0, 1). This make sense because $a^0=1$ for any *a*.

As before, the graph of each function, $f(x)=a^x$, also contains the point $(1, a)$. The graph of $f(x)=\left(\frac{1}{2}\right)^x$ contained $\left(1, \frac{1}{2}\right)$ and the graph of $g(x)=\left(\frac{1}{3}\right)^x$ contained $\left(1, \frac{1}{3}\right)$. This makes sense as $a^1=a$.

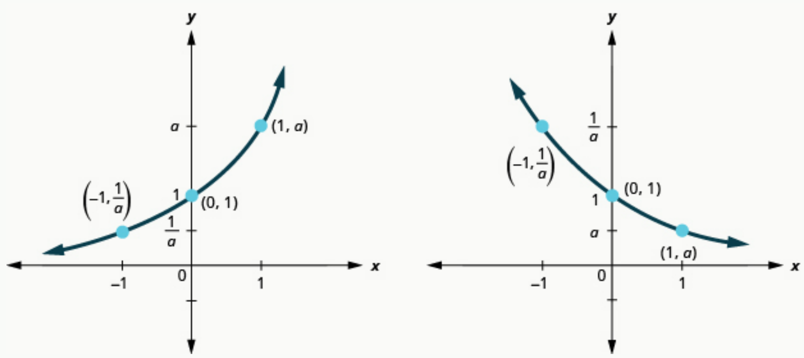
Notice too that the graph of each function, $f(x)=a^x$, also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x)=\left(\frac{1}{2}\right)^x$ contained $(-1, 2)$ and the graph of $g(x)=\left(\frac{1}{3}\right)^x$ contained $(-1, 3)$. This makes sense as $a^{-1}=\frac{1}{a}$.

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $(-\infty, \infty)$. Again, the graph never hits the $x$-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

We will summarize these properties in the chart below. Which also include when $a > 1$.

**Properties of the Graph of $f(x) = a^x$**

| when $a > 1$ | | when $0 < a < 1$ | |
|---|---|---|---|
| Domain | $(-\infty, \infty)$ | Domain | $(-\infty, \infty)$ |
| Range | $(0, \infty)$ | Range | $(0, \infty)$ |
| $x$-intercept | none | $x$-intercept | none |
| $y$-intercept | $(0, 1)$ | $y$-intercept | $(0, 1)$ |
| Contains | $(1, a)$, $\left(-1, \frac{1}{a}\right)$ | Contains | $(1, a)$, $\left(-1, \frac{1}{a}\right)$ |
| Asymptote | $x$-axis, the line $y = 0$ | Asymptote | $x$-axis, the line $y = 0$ |
| Basic shape | increasing | Basic shape | decreasing |




It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

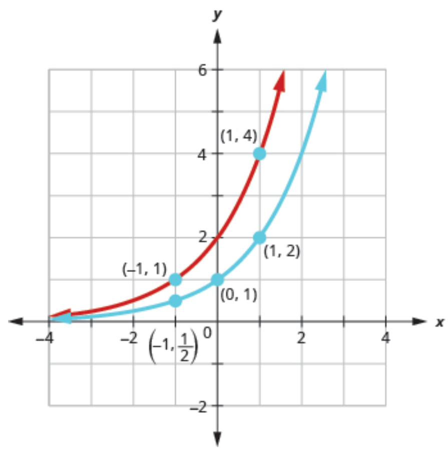
**EXAMPLE 10.12**

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 2^{x+1}$.

### Solution

We will use point plotting to graph the functions.

| $x$ | $f(x)=2^x$ | $(x, f(x))$ | $g(x)=2^{x+1}$ | $(x, g(x))$ |
|---|---|---|---|---|
| $-2$ | $2^{-2}=\frac{1}{2^2}=\frac{1}{4}$ | $\left(-2, \frac{1}{4}\right)$ | $2^{-2+1}=\frac{1}{2^1}=\frac{1}{2}$ | $\left(-2, \frac{1}{2}\right)$ |
| $-1$ | $2^{-1}=\frac{1}{2^1}=\frac{1}{2}$ | $\left(-1, \frac{1}{2}\right)$ | $2^{-1+1}=2^0=1$ | $(-1, 1)$ |
| $0$ | $2^0=1$ | $(0, 1)$ | $2^{0+1}=2^1=2$ | $(0, 2)$ |
| $1$ | $2^1=2$ | $(1, 2)$ | $2^{1+1}=2^2=4$ | $(1, 4)$ |
| $2$ | $2^2=4$ | $(2, 4)$ | $2^{2+1}=2^3=8$ | $(2, 8)$ |
| $3$ | $2^3=8$ | $(3, 8)$ | $2^{3+1}=2^4=16$ | $(3, 16)$ |




TRY IT : : 10.23 On the same coordinate system, graph: $f(x)=2^x$ and $g(x)=2^{x-1}$.

TRY IT : : 10.24 On the same coordinate system, graph: $f(x)=3^x$ and $g(x)=3^{x+1}$.

Looking at the graphs of the functions $f(x)=2^x$ and $g(x)=2^{x+1}$ in the last example, we see that adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

Let's now consider another situation that might be graphed more easily by translation, once we recognize the pattern.

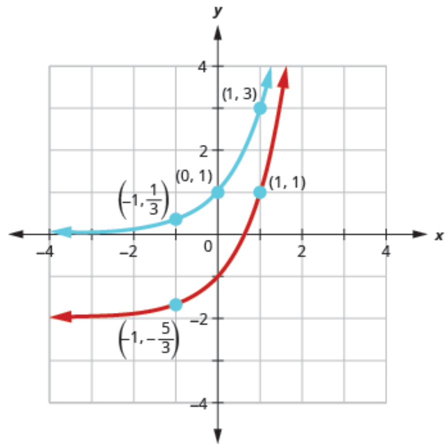
**EXAMPLE 10.13**

On the same coordinate system graph $f(x)=3^x$ and $g(x)=3^x-2$.

## Solution

We will use point plotting to graph the functions.

| $x$ | $f(x)=3^x$ | $(x, g(x))$ | $g(x)=3^x-2$ | $(x, g(x))$ |
|---|---|---|---|---|
| $-2$ | $3^{-2}=\frac{1}{9}$ | $\left(-2, \frac{1}{9}\right)$ | $3^{-2}-2=\frac{1}{9}-2=-\frac{17}{9}$ | $\left(-2, -\frac{17}{9}\right)$ |
| $-1$ | $3^{-1}=\frac{1}{3}$ | $\left(-1, \frac{1}{3}\right)$ | $3^{-1}-2=\frac{1}{3}-2=-\frac{5}{3}$ | $\left(-1, -\frac{5}{3}\right)$ |
| $0$ | $3^0=1$ | $(0, 1)$ | $3^0-2=1-2=-1$ | $(0, -1)$ |
| $1$ | $3^1=3$ | $(1, 3)$ | $3^1-2=3-2=1$ | $(1, 1)$ |
| $2$ | $3^2=9$ | $(2, 9)$ | $3^2-2=9-2=7$ | $(2, 8)$ |




**TRY IT : :** 10.25 On the same coordinate system, graph: $f(x) = 3^x$ and $g(x) = 3^x + 2$.

**TRY IT : :** 10.26 On the same coordinate system, graph: $f(x) = 4^x$ and $g(x) = 4^x - 2$.

Looking at the graphs of the functions $f(x) = 3^x$ and $g(x) = 3^x - 2$ in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, *e*. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $\left(1 + \frac{1}{n}\right)^n$ as *n* gets larger and larger. We say, as *n* approaches infinity, or increases without bound. The table shows the value of $\left(1 + \frac{1}{n}\right)^n$ for several values of $n$.

| $n$ | $\left(1 + \frac{1}{n}\right)^n$ |
|---|---|
| 1 | 2 |
| 2 | 2.25 |
| 5 | 2.48832 |
| 10 | 2.59374246 |
| 100 | 2.704813829... |
| 1,000 | 2.716923932... |
| 10,000 | 2.718145927... |
| 100,000 | 2.718268237... |
| 1,000,000 | 2.718280469... |
| 1,000,000,000 | 2.718281827... |

**Table 10.7**

$$e \approx 2.718281827$$

The number *e* is like the number $\pi$ in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number *e* is called the **natural base**.

**Natural Base $e$**

The number *e* is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as *n* increases without bound. We say, as *n* approaches infinity,

$$e \approx 2.718281827...$$

The exponential function whose base is $e$, $f(x)=e^x$ is called the **natural exponential function**.

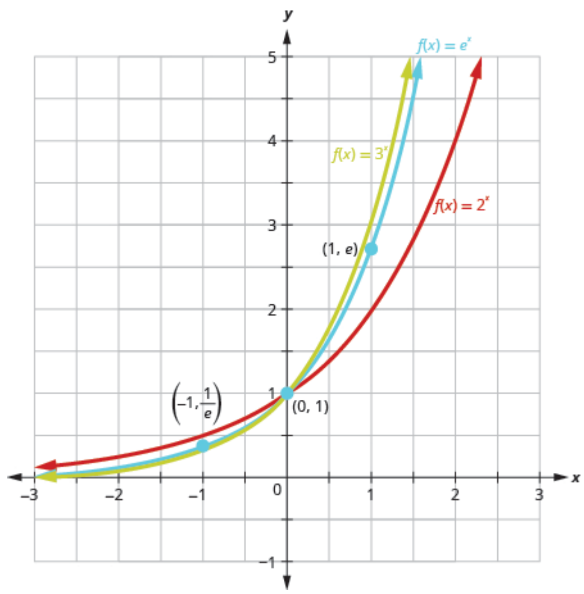
**Natural Exponential Function**

The natural exponential function is an exponential function whose base is $e$

$$f(x)=e^x$$

The domain is $(-\infty,\ \infty)$ and the range is $(0,\ \infty)$.

Let's graph the function $f(x)=e^x$ on the same coordinate system as $g(x)=2^x$ and $h(x)=3^x$.




Notice that the graph of $f(x)=e^x$ is "between" the graphs of $g(x)=2^x$ and $h(x)=3^x$. Does this make sense as $2<e<3$?

## Solve Exponential Equations

Equations that include an exponential expression $a^x$ are called exponential equations. To solve them we use a property that says as long as $a>0$ and $a \neq 1$, if $a^x=a^y$ then it is true that $x=y$. In other words, in an exponential equation, if the bases are equal then the exponents are equal.

**One-to-One Property of Exponential Equations**

For $a>0$ and $a \neq 1$,

$$\text{If } a^x=a^y, \text{ then } x=y.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

**EXAMPLE 10.14** HOW TO SOLVE AN EXPONENTIAL EQUATION

Solve: $3^{2x-5}=27$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Write both sides of the equation with the same base. | Since the left side has base 3, we write the right side with base 3. $27 = 3^3$ | $3^{2x-5} = 27$<br>$3^{2x-5} = 3^3$ |
| **Step 2.** Write a new equation by setting the exponents equal. | Since the bases are the same, the exponents must be equal. | $2x - 5 = 3$ |
| **Step 3.** Solve the equation. | Add 5 to each side.<br>Divide by 2. | $2x = 8$<br>$x = 4$ |
| **Step 4.** Check the solution. | Substitute $x = 4$ into the original equation. | $3^{2x-5} = 27$<br>$3^{2\cdot4-5} \stackrel{?}{=} 27$<br>$3^3 \stackrel{?}{=} 27$<br>$27 = 27$ ✓ |

**TRY IT : :** 10.27 Solve: $3^{3x-2} = 81$.

**TRY IT : :** 10.28 Solve: $7^{x-3} = 7$.

The steps are summarized below.

**HOW TO : :** HOW TO SOLVE AN EXPONENTIAL EQUATION

Step 1. Write both sides of the equation with the same base, if possible.
Step 2. Write a new equation by setting the exponents equal.
Step 3. Solve the equation.
Step 4. Check the solution.

In the next example, we will use our properties on exponents.

### EXAMPLE 10.15

Solve $\dfrac{e^{x^2}}{e^3} = e^{2x}$.

## Solution

| | |
|---|---|
| | $\dfrac{e^{x^2}}{e^3} = e^{2x}$ |
| Use the Property of Exponents: $\dfrac{a^m}{a^n} = a^{m-n}$. | $e^{x^2-3} = e^{2x}$ |
| Write a new equation by setting the exponents equal. | $x^2 - 3 = 2x$ |
| Solve the equation. | $x^2 - 2x - 3 = 0$ |

$$(x-3)(x+1)=0$$

$$x=3, \quad x=-1$$

Check the solutions.

| $x=3$ | $x=-1$ |
|---|---|
| $\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$ | $\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$ |
| $\frac{e^{3^2}}{e^3} \stackrel{?}{=} e^{2\cdot 3}$ | $\frac{e^{(-1)^2}}{e^3} \stackrel{?}{=} e^{2\cdot(-1)}$ |
| $\frac{e^9}{e^3} \stackrel{?}{=} e^6$ | $\frac{e^1}{e^3} \stackrel{?}{=} e^{-2}$ |
| $e^6 = e^6$ ✓ | $e^{-2} = e^{-2}$ ✓ |

**TRY IT : :** 10.29 Solve: $\frac{e^{x^2}}{e^x}=e^2$.

**TRY IT : :** 10.30 Solve: $\frac{e^{x^2}}{e^x}=e^6$.

## Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, *P*, is invested at an interest rate, *r*, for *t* years, the new balance, *A*, will depend on how often the interest is compounded. If the interest is compounded *n* times a year we use the formula $A=P\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A=Pe^{rt}$. These are the formulas for **compound interest**.

### Compound Interest

For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A*, is:

$$A=P\left(1+\frac{r}{n}\right)^{nt} \quad \text{when compounded } n \text{ times a year.}$$

$$A=Pe^{rt} \quad \text{when compounded continuously.}$$

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

**EXAMPLE 10.16**

A total of $\$10,000$ was invested in a college fund for a new grandchild. If the interest rate is $5\%$, how much will be in the account in 18 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

## Solution

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$\begin{aligned} A &= ? \\ P &= \$10,000 \\ r &= 0.05 \\ t &= 18 \text{ years} \end{aligned}$$

ⓐ

For quarterly compounding, $n = 4$. There are 4 quarters in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1 + \frac{0.05}{4}\right)^{4 \cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24,459.20$$

ⓑ

For monthly compounding, $n = 12$. There are 12 months in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 18}$$

Compute the amount.

$$A = \$24,550.08$$

ⓒ

For compounding continuously,

$$A = Pe^{rt}$$

Substitute the values in the formula.

$$A = 10,000e^{0.05 \cdot 18}$$

Compute the amount.

$$A = \$24,596.03$$

**TRY IT : :** 10.31

Angela invested $\$15,000$ in a savings account. If the interest rate is $4\%,$ how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

**TRY IT : :** 10.32

Allan invested \$10,000 in a mutual fund. If the interest rate is $5\%,$ how much will be in the account in 15 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use $A_0,$ as the original amount instead of calling it $P,$ the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

**Exponential Growth and Decay**

For an original amount, $A_0$, that grows or decays at a rate, *r*, for a certain time, *t*, the final amount, *A*, is:

$$A = A_0 e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

**EXAMPLE 10.17**

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

**Solution**

| | |
|---|---|
| Identify the values of each variable in the formulas. Be sure to put the percent in decimal form. Be sure the units match—the rate is per hour and the time is in hours. | $A = ?$<br>$A_0 = 100$<br>$r = 0.25/\text{hour}$<br>$t = 24 \text{ hours}$ |
| Substitute the values in the formula: $A = A_0 e^{rt}$. | $A = 100e^{0.25 \cdot 24}$ |
| Compute the amount. | $A = 40{,}342.88$ |
| Round to the nearest whole virus. | $A = 40{,}343$<br>The researcher will find 40,34 viruses. |

**TRY IT : :** 10.33

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of $15\%$ per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

**TRY IT : :** 10.34

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of $10\%$ per hour. She will check on the virus in 24 hours. How many viruses will she find?

**MEDIA : :**

Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

- Graphing Exponential Functions (https://openstax.org/l/37Graphexponent)
- Solving Exponential Equations (https://openstax.org/l/37Solvelikebase)
- Applications of Exponential Functions (https://openstax.org/l/37Exponentapp)
- Continuously Compound Interest (https://openstax.org/l/37Compoundint)
- Radioactive Decay and Exponential Growth (https://openstax.org/l/37Exponentdecay)

# 10.2 EXERCISES

## Practice Makes Perfect

### Graph Exponential Functions

*In the following exercises, graph each exponential function.*

**65.** $f(x) = 2^x$

**66.** $g(x) = 3^x$

**67.** $f(x) = 6^x$

**68.** $g(x) = 7^x$

**69.** $f(x) = (1.5)^x$

**70.** $g(x) = (2.5)^x$

**71.** $f(x) = \left(\frac{1}{2}\right)^x$

**72.** $g(x) = \left(\frac{1}{3}\right)^x$

**73.** $f(x) = \left(\frac{1}{6}\right)^x$

**74.** $g(x) = \left(\frac{1}{7}\right)^x$

**75.** $f(x) = (0.4)^x$

**76.** $g(x) = (0.6)^x$

*In the following exercises, graph each function in the same coordinate system.*

**77.** $f(x) = 4^x,\ g(x) = 4^{x-1}$

**78.** $f(x) = 3^x,\ g(x) = 3^{x-1}$

**79.** $f(x) = 2^x,\ g(x) = 2^{x-2}$

**80.** $f(x) = 2^x,\ g(x) = 2^{x+2}$

**81.** $f(x) = 3^x,\ g(x) = 3^x + 2$

**82.** $f(x) = 4^x,\ g(x) = 4^x + 2$

**83.** $f(x) = 2^x,\ g(x) = 2^x + 1$

**84.** $f(x) = 2^x,\ g(x) = 2^x - 1$

*In the following exercises, graph each exponential function.*

**85.** $f(x) = 3^{x+2}$

**86.** $f(x) = 3^{x-2}$

**87.** $f(x) = 2^x + 3$

**88.** $f(x) = 2^x - 3$

**89.** $f(x) = \left(\frac{1}{2}\right)^{x-4}$

**90.** $f(x) = \left(\frac{1}{2}\right)^x - 3$

**91.** $f(x) = e^x + 1$

**92.** $f(x) = e^{x-2}$

**93.** $f(x) = -2^x$

**94.** $f(x) = 3^x$

### Solve Exponential Equations

*In the following exercises, solve each equation.*

**95.** $2^{3x-8} = 16$

**96.** $2^{2x-3} = 32$

**97.** $3^{x+3} = 9$

**98.** $3^{x^2} = 81$

**99.** $4^{x^2} = 4$

**100.** $4^x = 32$

**101.** $4^{x+2} = 64$

**102.** $4^{x+3} = 16$

**103.** $2^{x^2 + 2x} = \frac{1}{2}$

**104.** $3^{x^2 - 2x} = \frac{1}{3}$

**105.** $e^{3x} \cdot e^4 = e^{10}$

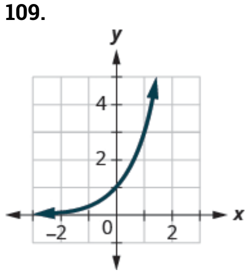
**106.** $e^{2x} \cdot e^3 = e^9$

**107.** $\frac{e^{x^2}}{e^2} = e^x$

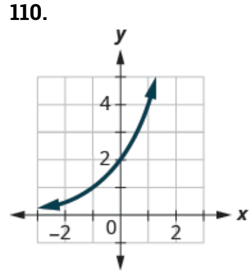
**108.** $\frac{e^{x^2}}{e^3} = e^{2x}$

*In the following exercises, match the graphs to one of the following functions: ⓐ $2^x$ ⓑ $2^{x+1}$ ⓒ $2^{x-1}$ ⓓ $2^x + 2$ ⓔ $2^x - 2$ ⓕ $3^x$*

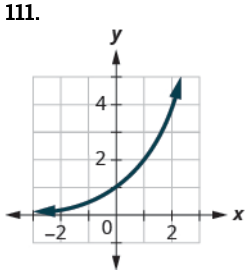
**109.**



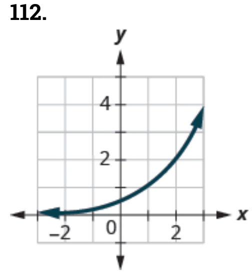

**110.**



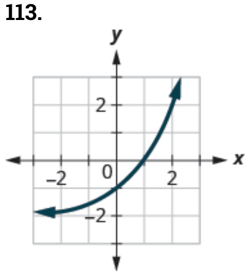

**111.**



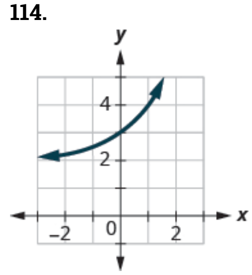

**112.**




**113.**




**114.**



### Use exponential models in applications

*In the following exercises, use an exponential model to solve.*

**115.** Edgar accumulated $\$5,000$ in credit card debt. If the interest rate is $20\%$ per year, and he does not make any payments for 2 years, how much will he owe on this debt in 2 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

**116.** Cynthia invested $\$12,000$ in a savings account. If the interest rate is $6\%,$ how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

**117.** Rochelle deposits $\$5,000$ in an IRA. What will be the value of her investment in 25 years if the investment is earning $8\%$ per year and is compounded continuously?

**118.** Nazerhy deposits $\$8,000$ in a certificate of deposit. The annual interest rate is $6\%$ and the interest will be compounded quarterly. How much will the certificate be worth in 10 years?

**119.** A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. He starts his experiment with 100 of the bacteria that grows at a rate of $6\%$ per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

**120.** A biologist is observing the growth pattern of a virus. She starts with 50 of the virus that grows at a rate of $20\%$ per hour. She will check on the virus in 24 hours. How many viruses will she find?

**121.** In the last ten years the population of Indonesia has grown at a rate of $1.12\%$ per year to 258,316,051. If this rate continues, what will be the population in 10 more years?

**122.** In the last ten years the population of Brazil has grown at a rate of $0.9\%$ per year to 205,823,665. If this rate continues, what will be the population in 10 more years?

## Writing Exercises

**123.** Explain how you can distinguish between exponential functions and polynomial functions.

**124.** Compare and contrast the graphs of $y = x^2$ and $y = 2^x$.

**125.** What happens to an exponential function as the values of $x$ decreases? Will the graph ever cross the $y$-axis? Explain.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| graph exponential eunctions. | | | |
| solve exponential equations. | | | |
| use exponential models in applications. | | | |

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

# 10.3 Evaluate and Graph Logarithmic Functions

## Learning Objectives

**By the end of this section, you will be able to:**

- Convert between exponential and logarithmic form
- Evaluate logarithmic functions
- Graph Logarithmic functions
- Solve logarithmic equations
- Use logarithmic models in applications

### Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 81$.
   If you missed this problem, review **Example 6.46**.
2. Evaluate: $3^{-2}$.
   If you missed this problem, review **Example 5.15**.
3. Solve: $2^4 = 3x - 5$.
   If you missed this problem, review **Example 2.2**.

We have spent some time finding the inverse of many functions. It works well to 'undo' an operation with another operation. Subtracting 'undoes' addition, multiplication 'undoes' division, taking the square root 'undoes' squaring.

As we studied the exponential function, we saw that it is one-to-one as its graphs pass the horizontal line test. This means an exponential function does have an inverse. If we try our algebraic method for finding an inverse, we run into a problem.

| | |
|---|---|
| | $f(x) = a^x$ |
| Rewrite with $y = f(x)$. | $y = a^x$ |
| Interchange the variables $x$ and $y$. | $x = a^y$ |
| Solve for $y$. | Oops! We have no way to solve for $y$! |

To deal with this we define the logarithm function with base $a$ to be the inverse of the exponential function $f(x) = a^x$.

We use the notation $f^{-1}(x) = \log_a x$ and say the inverse function of the exponential function is the logarithmic function.

### Logarithmic Function

The function $f(x) = \log_a x$ is the **logarithmic function** with base $a$, where $a > 0$, $x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

## Convert Between Exponential and Logarithmic Form

Since the equations $y = \log_a x$ and $x = a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See **Figure 10.3**. Notice the positions of the exponent and base.

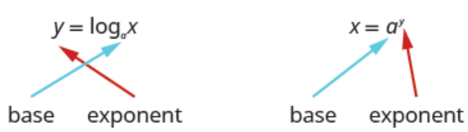



**Figure 10.3**

If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, "base to the exponent give us the number."

**EXAMPLE 10.18**

Convert to logarithmic form: ⓐ $2^3 = 8$, ⓑ $5^{\frac{1}{2}} = \sqrt{5}$, and ⓒ $\left(\frac{1}{2}\right)^x = \frac{1}{16}$.

**Solution**

Identify the base and the exponent.

| (a) | (b) | (c) |
|---|---|---|
| $2^3 = 8$ | $5^{\frac{1}{2}} = \sqrt{5}$ | $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ |
| $y = \log_a x$ | $y = \log_a x$ | $y = \log_a x$ |
| $3 = \log_2 8$ | $\frac{1}{2} = \log_5 \sqrt{5}$ | $4 = \log_{\frac{1}{2}} \frac{1}{16}$ |
| If $2^3 = 8$, then $3 = \log_2 8$. | If $5^{\frac{1}{2}} = \sqrt{5}$, then $\frac{1}{2} = \log_5 \sqrt{5}$. | If $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, then $4 = \log_{\frac{1}{2}} \frac{1}{16}$. |

**TRY IT : :** 10.35

Convert to logarithmic form: ⓐ $3^2 = 9$ ⓑ $7^{\frac{1}{2}} = \sqrt{7}$ ⓒ $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

**TRY IT : :** 10.36

Convert to logarithmic form: ⓐ $4^3 = 64$ ⓑ $4^{\frac{1}{3}} = \sqrt[3]{4}$ ⓒ $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

In the next example we do the reverse—convert logarithmic form to exponential form.

**EXAMPLE 10.19**

Convert to exponential form: ⓐ $2 = \log_8 64$, ⓑ $0 = \log_4 1$, and ⓒ $-3 = \log_{10} \frac{1}{1000}$.

**Solution**

Identify the base and the exponent.

| (a) | (b) | (c) |
|---|---|---|
| $2 = \log_8 64$ | $0 = \log_4 1$ | $-3 = \log_{10} \frac{1}{1000}$ |
| $x = a^y$ | $x = a^y$ | $x = a^y$ |
| $64 = 8^2$ | $1 = 4^0$ | $\frac{1}{1000} = \log^{-3}$ |
| If $2 = \log_8 64$, then $64 = 8^2$. | If $0 = \log_4 1$, then $1 = 4^0$. | If $-3 = \log_{10} \frac{1}{1000}$, then $\frac{1}{1000} = 10^{-3}$. |

**TRY IT : :** 10.37

Convert to exponential form: ⓐ $3 = \log_4 64$ ⓑ $0 = \log_x 1$ ⓒ $-2 = \log_{10} \frac{1}{100}$

**TRY IT : :** 10.38

Convert to exponential form: ⓐ $3 = \log_3 27$ ⓑ $0 = \log_x 1$ ⓒ $-1 = \log_{10} \frac{1}{10}$

## Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

**EXAMPLE 10.20**

Find the value of *x*: ⓐ $\log_x 36 = 2$, ⓑ $\log_4 x = 3$, and ⓒ $\log_{\frac{1}{2}} \frac{1}{8} = x$.

**Solution**

ⓐ

| | |
|---|---|
| | $\log_x 36 = 2$ |
| Convert to exponential form. | $x^2 = 36$ |
| Solve the quadratic. | $x = 6,\quad \cancel{x = -6}$ |
| The base of a logarithmic function must be positive, so we eliminate $x = -6$. | $x = 6$ Therefore, $\log_6 36 = 2$. |

ⓑ

| | |
|---|---|
| | $\log_4 x = 3$ |
| Convert to exponential form. | $4^3 = x$ |
| Simplify. | $x = 64$ Therefore, $\log_4 64 = 3$. |

ⓒ

| | |
|---|---|
| | $\log_{\frac{1}{2}} \frac{1}{8} = x$ |
| Convert to exponential form. | $\left(\frac{1}{2}\right)^x = \frac{1}{8}$ |
| Rewrite $\frac{1}{8}$ as $\left(\frac{1}{2}\right)^3$. | $\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$ |
| With the same base, the exponents must be equal. | $x = 3$ Therefore, $\log_{\frac{1}{2}} \frac{1}{8} = 3$ |

**TRY IT : :** 10.39 Find the value of $x$ : ⓐ $\log_x 64 = 2$ ⓑ $\log_5 x = 3$ ⓒ $\log_{\frac{1}{2}} \frac{1}{4} = x$

**TRY IT : :** 10.40 Find the value of $x$ : ⓐ $\log_x 81 = 2$ ⓑ $\log_3 x = 5$ ⓒ $\log_{\frac{1}{3}} \frac{1}{27} = x$

When see an expression such as $\log_3 27$, we can find its exact value two ways. By inspection we realize it means "3 to what power will be 27"? Since $3^3 = 27$, we know $\log_3 27 = 3$. An alternate way is to set the expression equal to $x$ and then convert it into an exponential equation.

**EXAMPLE 10.21**

Find the exact value of each logarithm without using a calculator:

ⓐ $\log_5 25$,

ⓑ $\log_9 3$, and ⓒ $\log_2 \frac{1}{16}$.

**Solution**

ⓐ

| | |
|---|---|
| | $\log_5 25$ |
| 5 to what power will be 25? | $\log_5 25 = 2$ |
| Or | |
| Set the expression equal to $x$. | $\log_5 25 = x$ |
| Change to exponential form. | $5^x = 25$ |
| Rewrite 25 as $5^2$. | $5^x = 5^2$ |
| With the same base the exponents must be equal. | $x = 2$ Therefore, $\log_5 25 = 2$. |

ⓑ

| | |
|---|---|
| | $\log_9 3$ |
| Set the expression equal to $x$. | $\log_9 3 = x$ |
| Change to exponential form. | $9^x = 3$ |
| Rewrite 9 as $3^2$. | $\left(3^2\right)^x = 3^1$ |
| Simplify the exponents. | $3^{2x} = 3^1$ |
| With the same base the exponents must be equal. | $2x = 1$ |
| Solve the equation. | $x = \frac{1}{2}$ Therefore, $\log_9 3 = \frac{1}{2}$. |

ⓒ

| | |
|---|---|
| | $\log_2 \frac{1}{16}$ |
| Set the expression equal to $x$. | $\log_2 \frac{1}{16} = x$ |
| Change to exponential form. | $2^x = \frac{1}{16}$ |
| Rewrite 16 as $2^4$. | $2^x = \frac{1}{2^4}$ |
| | $2^x = 2^{-4}$ |
| With the same base the exponents must be equal. | $x = -4$ Therefore, $\log_2 \frac{1}{16} = -4$. |

**TRY IT : :** 10.41 Find the exact value of each logarithm without using a calculator:

ⓐ $\log_{12} 144$

ⓑ $\log_4 2$

ⓒ $\log_2 \frac{1}{32}$

**TRY IT : :** 10.42 Find the exact value of each logarithm without using a calculator:

ⓐ $\log_9 81$

ⓑ $\log_8 2$

ⓒ $\log_3 \frac{1}{9}$

## Graph Logarithmic Functions

To graph a logarithmic function $y = \log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$. Generally, when we look for ordered pairs for the graph of a function, we usually choose an *x*-value and then determine its corresponding *y*-value. In this case you may find it easier to choose *y*-values and then determine its corresponding *x*-value.

**EXAMPLE 10.22**

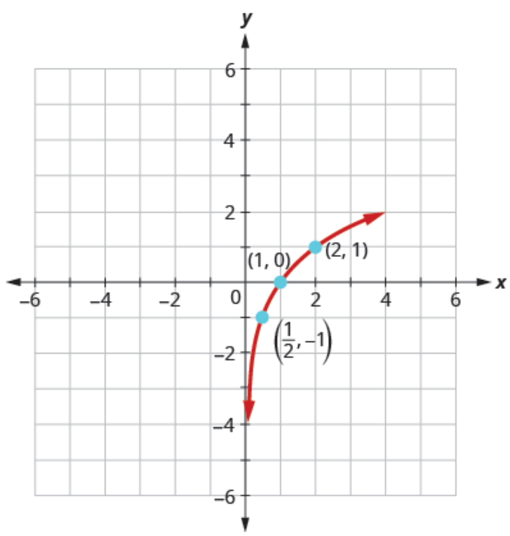
Graph $y = \log_2 x$.

## Solution

To graph the function, we will first rewrite the logarithmic equation, $y = \log_2 x$, in exponential form, $2^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

| $y$ | $2^y = x$ | $(x, y)$ |
|---|---|---|
| $-2$ | $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$ | $\left(\frac{1}{4}, 2\right)$ |
| $-1$ | $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$ | $\left(\frac{1}{2}, -1\right)$ |
| 0 | $2^0 = 1$ | $(1, 0)$ |
| 1 | $2^1 = 2$ | $(2, 1)$ |
| 2 | $2^2 = 4$ | $(4, 2)$ |
| 3 | $2^3 = 8$ | $(8, 3)$ |




TRY IT : : 10.43 Graph: $y = \log_3 x$.

TRY IT : : 10.44 Graph: $y = \log_5 x$.

The graphs of $y = \log_2 x$, $y = \log_3 x$, and $y = \log_5 x$ are the shape we expect from a logarithmic function where $a > 1$.

We notice that for each function the graph contains the point $(1, 0)$. This make sense because $0 = \log_a 1$ means $a^0 = 1$ which is true for any *a*.

The graph of each function, also contains the point $(a, 1)$. This makes sense as $1 = \log_a a$ means $a^1 = a$. which is true for any *a*.

Notice too, the graph of each function $y = \log_a x$ also contains the point $\left(\frac{1}{a}, -1\right)$. This makes sense as $-1 = \log_a \frac{1}{a}$ means $a^{-1} = \frac{1}{a}$, which is true for any *a*.

Look at each graph again. Now we will see that many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

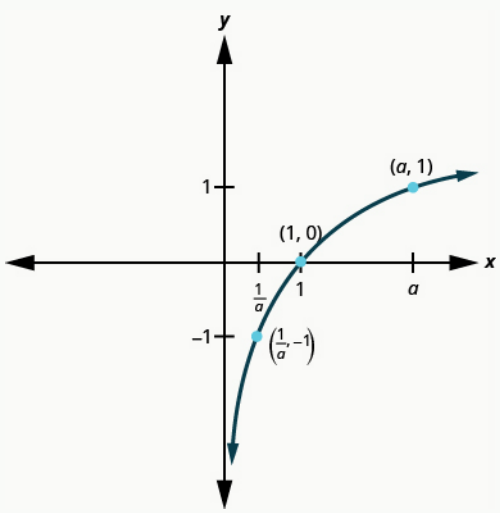
What is the domain of the function? The graph never hits the *y*-axis. The domain is all positive numbers. We write the domain in interval notation as $(0, \infty)$.

What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as $(-\infty, \infty)$.

When the graph approaches the *y*-axis so very closely but will never cross it, we call the line $x = 0$, the *y*-axis, a vertical asymptote.

### Properties of the Graph of $y = \log_a x$ when $a > 1$

| | |
|---|---|
| Domain | $(0, \infty)$ |
| Range | $(-\infty, \infty)$ |
| $x$-intercept | $(1, 0)$ |
| $y$-intercept | None |
| Contains | $(a, 1)$, $\left(\frac{1}{a}, -1\right)$ |
| Asymptote | $y$-axis |




Our next example looks at the graph of $y = \log_a x$ when $0 < a < 1$.

**EXAMPLE 10.23**

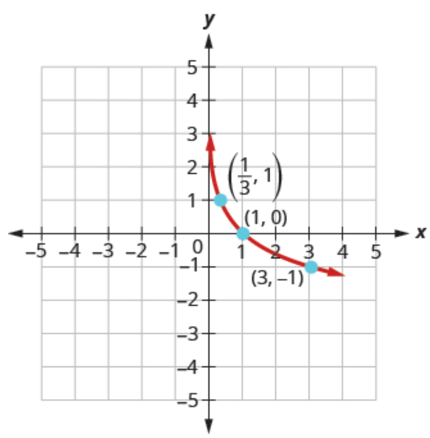
Graph $y = \log_{\frac{1}{3}} x$.

## Solution

To graph the function, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}} x,$ in exponential form, $\left(\frac{1}{3}\right)^y = x.$

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

| $y$ | $\left(\frac{1}{3}\right)^y = x$ | $(x, y)$ |
|---|---|---|
| $-2$ | $\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$ | $(9, -2)$ |
| $-1$ | $\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$ | $(3, -1)$ |
| 0 | $\left(\frac{1}{3}\right)^0 = 1$ | $(1, 0)$ |
| 1 | $\left(\frac{1}{3}\right)^1 = \frac{1}{3}$ | $\left(\frac{1}{3}, 1\right)$ |
| 2 | $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ | $\left(\frac{1}{9}, 2\right)$ |
| 3 | $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$ | $\left(\frac{1}{27}, 3\right)$ |




**TRY IT : :** 10.45 Graph: $y = \log_{\frac{1}{2}} x.$

**TRY IT : :** 10.46 Graph: $y = \log_{\frac{1}{4}} x.$

Now, let's look at the graphs $y = \log_{\frac{1}{2}} x,$ $y = \log_{\frac{1}{3}} x$ and $y = \log_{\frac{1}{4}} x$, so we can identify some of the properties of logarithmic functions where $0 < a < 1.$

The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where $0 < a < 1.$

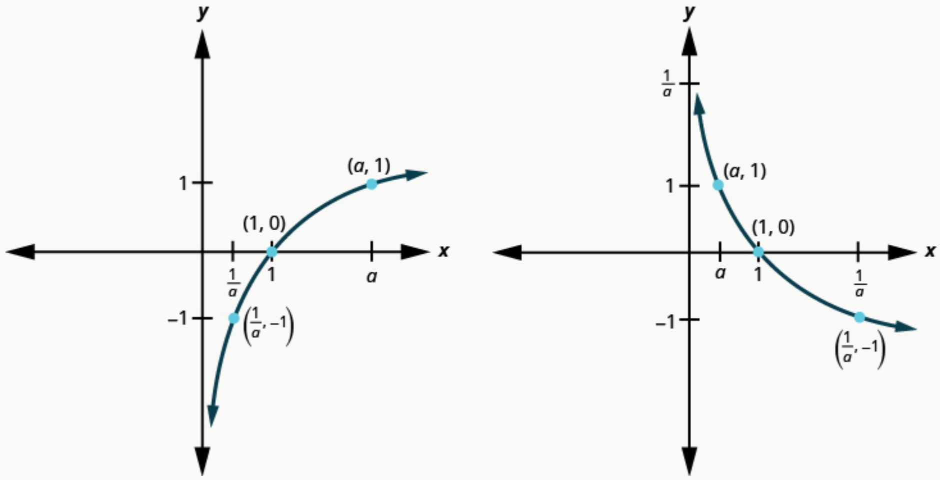
We notice, that for each function again, the graph contains the points, $(1, 0),$ $(a, 1),$ $\left(\frac{1}{a}, -1\right).$ This make sense for the same reasons we argued above.

We notice the domain and range are also the same—the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. The $y$-axis is again the vertical asymptote.

We will summarize these properties in the chart below. Which also include when $a > 1$.

**Properties of the Graph of $y = \log_a x$**

| when $a > 1$ | | when $0 < a < 1$ | |
|---|---|---|---|
| Domain | $(0, \infty)$ | Domain | $(0, \infty)$ |
| Range | $(-\infty, \infty)$ | Range | $(-\infty, \infty)$ |
| $x$-intercept | $(1, 0)$ | $x$-intercept | $(1, 0)$ |
| $y$-intercept | none | $y$-intercept | None |
| Contains | $(a, 1),\ \left(\frac{1}{a}, -1\right)$ | Contains | $(a, 1),\ \left(\frac{1}{a}, -1\right)$ |
| Asymptote | $y$-axis | Asymptote | $y$-axis |
| Basic shape | increasing | Basic shape | Decreasing |




We talked earlier about how the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$. The graphs in Figure 10.4 show both the exponential (blue) and logarithmic (red) functions on the same graph for both $a > 1$ and $0 < a < 1$.

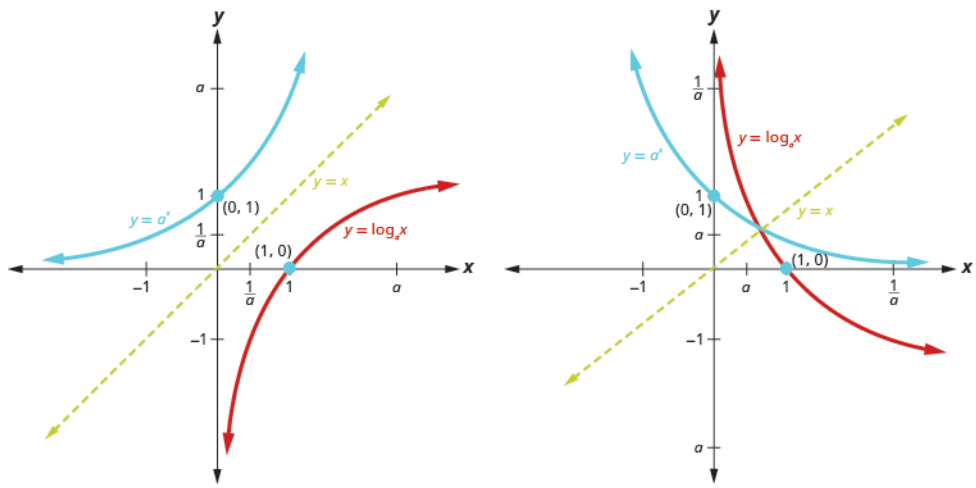



**Figure 10.4**

Notice how the graphs are reflections of each other through the line $y = x$. We know this is true of inverse functions. Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the *x*-axis is the horizontal asymptote for the exponential functions and the *y*-axis is the vertical asymptote for the logarithmic functions.

## Solve Logarithmic Equations

When we talked about exponential functions, we introduced the number *e*. Just as *e* was a base for an exponential function, it can be used a base for logarithmic functions too. The logarithmic function with base *e* is called the **natural logarithmic function**. The function $f(x) = \log_e x$ is generally written $f(x) = \ln x$ and we read it as "el en of $x$."

### Natural Logarithmic Function

The function $f(x) = \ln x$ is the **natural logarithmic function** with base $e$, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

When the base of the logarithm function is 10, we call it the **common logarithmic function** and the base is not shown. If the base *a* of a logarithm is not shown, we assume it is 10.

### Common Logarithmic Function

The function $f(x) = \log x$ is the **common logarithmic function** with base $10$, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

It will be important for you to use your calculator to evaluate both common and natural logarithms.

Look for the log and ln keys on your calculator.

To solve logarithmic equations, one strategy is to change the equation to exponential form and then solve the exponential equation as we did before. As we solve logarithmic equations, $y = \log_a x$, we need to remember that for the base *a*, $a > 0$ and $a \neq 1$. Also, the domain is $x > 0$. Just as with radical equations, we must check our solutions to eliminate any extraneous solutions.

**EXAMPLE 10.24**

Solve: ⓐ $\log_a 49 = 2$ and ⓑ $\ln x = 3$.

### Solution

ⓐ

| | |
|---|---|
| | $\log_a 49 = 2$ |
| Rewrite in exponential form. | $a^2 = 49$ |
| Solve the equation using the square root property. | $a = \pm 7$ |
| The base cannot be negative, so we eliminate $a = -7$. | $a = 7, \quad \cancel{a = -7}$ |

Check.

$a = 7$

$$\begin{aligned} \log_a 49 &= 2 \\ \log_7 49 &\stackrel{?}{=} 2 \\ 7^2 &\stackrel{?}{=} 49 \\ 49 &= 49 \checkmark \end{aligned}$$

ⓑ

| | |
|---|---|
| | $\ln x = 3$ |
| Rewrite in exponential form. | $e^3 = x$ |

Check.

$x = e^3$

$$\begin{aligned} \ln x &= 3 \\ \ln e^3 &\stackrel{?}{=} 3 \\ e^3 &= e^3 \checkmark \end{aligned}$$

**TRY IT : :** 10.47 Solve: ⓐ $\log_a 121 = 2$ ⓑ $\ln x = 7$

**TRY IT : :** 10.48 Solve: ⓐ $\log_a 64 = 3$ ⓑ $\ln x = 9$

**EXAMPLE 10.25**

Solve: ⓐ $\log_2(3x - 5) = 4$ and ⓑ $\ln e^{2x} = 4$.

## Solution

ⓐ

| | |
|---|---|
| | $\log_2(3x - 5) = 4$ |
| Rewrite in exponential form. | $2^4 = 3x - 5$ |
| Simplify. | $16 = 3x - 5$ |
| Solve the equation. | $21 = 3x$ |
| | $7 = x$ |

Check.

$x = 7$

$$\begin{aligned} \log_2(3x - 5) &= 4 \\ \log_2(3 \cdot 7 - 5) &\stackrel{?}{=} 4 \\ \log_2(16) &\stackrel{?}{=} 4 \\ 2^4 &\stackrel{?}{=} 16 \\ 16 &= 16 \checkmark \end{aligned}$$

ⓑ

| | |
|---|---|
| | $\ln e^{2x} = 4$ |
| Rewrite in exponential form. | $e^4 = e^{2x}$ |
| Since the bases are the same the exponents are equal. | $4 = 2x$ |
| Solve the equation. | $2 = x$ |

Check.

$x = 2$

$$\begin{aligned} \ln e^{2x} &= 4 \\ \ln e^{2\cdot 2} &\stackrel{?}{=} 4 \\ \ln e^{4} &\stackrel{?}{=} 4 \\ e^4 &= e^4 \checkmark \end{aligned}$$

> **TRY IT : :** 10.49 Solve: ⓐ $\log_2(5x-1)=6$ ⓑ $\ln e^{3x}=6$

> **TRY IT : :** 10.50 Solve: ⓐ $\log_3(4x+3)=3$ ⓑ $\ln e^{4x}=4$

## Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The $10^{-12}$ in the formula represents the intensity of sound that is barely audible.

**Decibel Level of Sound**

The loudness level, *D*, measured in decibels, of a sound of intensity, *I*, measured in watts per square inch is

$$D = 10\log\left(\frac{I}{10^{-12}}\right)$$

**EXAMPLE 10.26**

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity $10^{-2}$ watts per square inch?

### Solution

| | |
|---|---|
| | $D = 10\log\left(\frac{I}{10^{-12}}\right)$ |
| Substitute in the intensity level, *I*. | $D = 10\log\left(\frac{10^{-2}}{10^{-12}}\right)$ |
| Simplify. | $D = 10\log(10^{10})$ |
| Since $\log 10^{10} = 10$. | $D = 10 \cdot 10$ |
| Multiply. | $D = 100$ |
| | The decibel level of music coming through earphones is 100 dB. |

> **TRY IT : :** 10.51
>
> What is the decibel level of one of the new quiet dishwashers with intensity $10^{-7}$ watts per square inch?

> **TRY IT : :** 10.52 What is the decibel level heavy city traffic with intensity $10^{-3}$ watts per square inch?

The magnitude $R$ of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where $I$ is the intensity of the shock wave. This model provides a way to measure earthquake intensity.

**Earthquake Intensity**

The magnitude $R$ of an earthquake is measured by $R = \log I$, where $I$ is the intensity of its shock wave.

**EXAMPLE 10.27**

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $108 million dollars of damage. Compare the intensities of the two earthquakes.

**Solution**

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

| | |
|---|---|
| Convert the magnitudes to intensities. | $R = \log I$ |
| 1906 earthquake | $7.8 = \log I$ |
| Convert to exponential form. | $I = 10^{7.8}$ |
| 2014 earthquake | $5.1 = \log I$ |
| Convert to exponential form. | $I = 10^{5.1}$ |
| Form a ratio of the intensities. | $\dfrac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$ |
| Substitute in the values. | $\dfrac{10^{7.8}}{10^{5.1}}$ |
| Divide by subtracting the exponents. | $10^{2.7}$ |
| Evaluate. | $501$ |
| | The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake. |

**TRY IT : :** 10.53

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

**TRY IT : :** 10.54

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

**MEDIA : :**

Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

- **Re-writing logarithmic equations in exponential form (https://openstax.org/l/37logasexponent)**
- **Simplifying Logarithmic Expressions (https://openstax.org/l/37Simplifylog)**
- **Graphing logarithmic functions (https://openstax.org/l/37Graphlog)**
- **Using logarithms to calculate decibel levels (https://openstax.org/l/37Finddecibel)**

# 10.3 EXERCISES

## Practice Makes Perfect

### Convert Between Exponential and Logarithmic Form

*In the following exercises, convert from exponential to logarithmic form.*

**126.** $4^2 = 16$

**127.** $2^5 = 32$

**128.** $3^3 = 27$

**129.** $5^3 = 125$

**130.** $10^3 = 1000$

**131.** $10^{-2} = \frac{1}{100}$

**132.** $x^{\frac{1}{2}} = \sqrt{3}$

**133.** $x^{\frac{1}{3}} = \sqrt[3]{6}$

**134.** $32^x = \sqrt[4]{32}$

**135.** $17^x = \sqrt[5]{17}$

**136.** $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

**137.** $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

**138.** $3^{-2} = \frac{1}{9}$

**139.** $4^{-3} = \frac{1}{64}$

**140.** $e^x = 6$

**141.** $e^3 = x$

*In the following exercises, convert each logarithmic equation to exponential form.*

**142.** $3 = \log_4 64$

**143.** $6 = \log_2 64$

**144.** $4 = \log_x 81$

**145.** $5 = \log_x 32$

**146.** $0 = \log_{12} 1$

**147.** $0 = \log_7 1$

**148.** $1 = \log_3 3$

**149.** $1 = \log_9 9$

**150.** $-4 = \log_{10} \frac{1}{10{,}000}$

**151.** $3 = \log_{10} 1{,}000$

**152.** $5 = \log_e x$

**153.** $x = \log_e 43$

### Evaluate Logarithmic Functions

*In the following exercises, find the value of $x$ in each logarithmic equation.*

**154.** $\log_x 49 = 2$

**155.** $\log_x 121 = 2$

**156.** $\log_x 27 = 3$

**157.** $\log_x 64 = 3$

**158.** $\log_3 x = 4$

**159.** $\log_5 x = 3$

**160.** $\log_2 x = -6$

**161.** $\log_3 x = -5$

**162.** $\log_{\frac{1}{4}} \frac{1}{16} = x$

**163.** $\log_{\frac{1}{3}} \frac{1}{9} = x$

**164.** $\log_{\frac{1}{4}} 64 = x$

**165.** $\log_{\frac{1}{9}} 81 = x$

*In the following exercises, find the exact value of each logarithm without using a calculator.*

**166.** $\log_7 49$

**167.** $\log_6 36$

**168.** $\log_4 1$

**169.** $\log_5 1$

**170.** $\log_{16} 4$

**171.** $\log_{27} 3$

**172.** $\log_{\frac{1}{2}} 2$

**173.** $\log_{\frac{1}{2}} 4$

**174.** $\log_2 \frac{1}{16}$

**175.** $\log_3 \frac{1}{27}$

**176.** $\log_4 \frac{1}{16}$

**177.** $\log_9 \frac{1}{81}$

### Graph Logarithmic Functions

*In the following exercises, graph each logarithmic function.*

**178.** $y = \log_2 x$

**179.** $y = \log_4 x$

**180.** $y = \log_6 x$

**181.** $y = \log_7 x$

**182.** $y = \log_{1.5} x$

**183.** $y = \log_{2.5} x$

**184.** $y = \log_{\frac{1}{3}} x$

**185.** $y = \log_{\frac{1}{5}} x$

**186.** $y = \log_{0.4} x$

**187.** $y = \log_{0.6} x$

### Solve Logarithmic Equations

*In the following exercises, solve each logarithmic equation.*

**188.** $\log_a 16 = 2$

**189.** $\log_a 81 = 2$

**190.** $\log_a 8 = 3$

**191.** $\log_a 27 = 3$

**192.** $\log_a 32 = 2$

**193.** $\log_a 24 = 3$

**194.** $\ln x = 5$

**195.** $\ln x = 4$

**196.** $\log_2 (5x + 1) = 4$

**197.** $\log_2 (6x + 2) = 5$

**198.** $\log_3 (4x - 3) = 2$

**199.** $\log_3 (5x - 4) = 4$

**200.** $\log_4 (5x + 6) = 3$

**201.** $\log_4 (3x - 2) = 2$

**202.** $\ln e^{4x} = 8$

**203.** $\ln e^{2x} = 6$

**204.** $\log x^2 = 2$

**205.** $\log(x^2 - 25) = 2$

**206.** $\log_2\left(x^2 - 4\right) = 5$

**207.** $\log_3\left(x^2 + 2\right) = 3$

### Use Logarithmic Models in Applications

*In the following exercises, use a logarithmic model to solve.*

**208.** What is the decibel level of normal conversation with intensity $10^{-6}$ watts per square inch?

**209.** What is the decibel level of a whisper with intensity $10^{-10}$ watts per square inch?

**210.** What is the decibel level of the noise from a motorcycle with intensity $10^{-2}$ watts per square inch?

**211.** What is the decibel level of the sound of a garbage disposal with intensity $10^{-2}$ watts per square inch?

**212.** In 2014, Chile experienced an intense earthquake with a magnitude of $8.2$ on the Richter scale. In 2010, Haiti also experienced an intense earthquake which measured $7.0$ on the Richter scale. Compare the intensities of the two earthquakes.

**213.** The Los Angeles area experiences many earthquakes. In 1994, the Northridge earthquake measured magnitude of $6.7$ on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured $5.1$ on the Richter scale. Compare the intensities of the two earthquakes.

## Writing Exercises

**214.** Explain how to change an equation from logarithmic form to exponential form.

**215.** Explain the difference between common logarithms and natural logarithms.

**216.** Explain why $\log_a a^x = x$.

**217.** Explain how to find the $\log_7 32$ on your calculator.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| convert between exponential and logarithmic form. | | | |
| evaluate logarithmic functions. | | | |
| graph logarithmic functions. | | | |
| solve logarithmic functions. | | | |
| use logarithmic models in applications. | | | |

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

# 10.4 Use the Properties of Logarithms

## Learning Objectives

**By the end of this section, you will be able to:**

- Use the properties of logarithms
- Use the Change of Base Formula

### Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate: ⓐ $a^0$ ⓑ $a^1$.
   If you missed this problem, review **Example 5.14**.
2. Write with a rational exponent: $\sqrt[3]{x^2 y}$.
   If you missed this problem, review **Example 8.27**.
3. Round to three decimal places: 2.5646415.
   If you missed this problem, review **Example 1.34**.

## Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0 = 1,$ we can convert this to logarithmic form and get $\log_a 1 = 0.$ Also, since $a^1 = a,$ we get $\log_a a = 1.$

### Properties of Logarithms

$$\log_a 1 = 0 \qquad \log_a a = 1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

**EXAMPLE 10.28**

Evaluate using the properties of logarithms: ⓐ $\log_8 1$ and ⓑ $\log_6 6.$

**Solution**

ⓐ

| | | |
|---|---|---|
| | $\log_8 1$ | |
| Use the property, $\log_a 1 = 0.$ | $0$ | $\log_8 1 = 0$ |

ⓑ

| | | |
|---|---|---|
| | $\log_6 6$ | |
| Use the property, $\log_a a = 1.$ | $1$ | $\log_6 6 = 1$ |

**TRY IT : :** 10.55 Evaluate using the properties of logarithms: ⓐ $\log_{13} 1$ ⓑ $\log_9 9.$

**TRY IT : :** 10.56 Evaluate using the properties of logarithms: ⓐ $\log_5 1$ ⓑ $\log_7 7.$

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $a^{\log_a x} = x$ converts to the logarithmic equation $\log_a x = \log_a x,$ which is a true statement for positive values for *x* only.

The logarithmic equation $\log_a a^x = x$ converts to the exponential equation $a^x = a^x,$ which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power. These two properties show the composition of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

### Inverse Properties of Logarithms

For $a > 0,\quad x > 0$ and $a \neq 1,$

$$a^{\log_a x} = x \qquad \log_a a^x = x$$

In the next example, apply the inverse properties of logarithms.

### EXAMPLE 10.29

Evaluate using the properties of logarithms: ⓐ $4^{\log_4 9}$ and ⓑ $\log_3 3^5$.

#### Solution

ⓐ

| | | |
|---|---|---|
| | $4^{\log_4 9}$ | |
| Use the property, $a^{\log_a x} = x$. | $9$ | $4^{\log_4 9} = 9$ |

ⓑ

| | | |
|---|---|---|
| | $\log_3 3^5$ | |
| Use the property, $a^{\log_a x} = x$. | $5$ | $\log_3 3^5 = 5$ |

**TRY IT : :** 10.57 Evaluate using the properties of logarithms: ⓐ $5^{\log_5 15}$ ⓑ $\log_7 7^4$.

**TRY IT : :** 10.58 Evaluate using the properties of logarithms: ⓐ $2^{\log_2 8}$ ⓑ $\log_2 2^{15}$.

There are three more properties of logarithms that will be useful in our work. We know exponential functions and logarithmic function are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $a^m \cdot a^n = a^{m+n},$ we see that to multiply the same base, we add the exponents. The **Product Property of Logarithms**, $\log_a M \cdot N = \log_a M + \log_a N$ tells us to take the log of a product, we add the log of the factors.

### Product Property of Logarithms

If $M > 0,\ N > 0, \text{a} > 0$ and $\text{a} \neq 1,$ then,

$$\log_a (M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

### EXAMPLE 10.30

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible: ⓐ $\log_3 7x$ and ⓑ $\log_4 64xy$.

## Solution

ⓐ

| | |
|---|---|
| | $\log_3 7x$ |
| Use the Product Property, $\log_a (M \cdot N) = \log_a M + \log_a N$. | $\log_3 7 + \log_3 x$ |
| | $\log_3 7x = \log_3 7 + \log_3 x$ |

ⓑ

| | |
|---|---|
| | $\log_4 64xy$ |
| Use the Product Property, $\log_a (M \cdot N) = \log_a M + \log_a N$. | $\log_4 64 + \log_4 x + \log_4 y$ |
| Simplify by evaluating $\log_4 64$. | $3 + \log_4 x + \log_4 y$ |
| | $\log_4 64xy = 3 + \log_4 x + \log_4 y$ |

**TRY IT : :** 10.59

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

ⓐ $\log_3 3x$ ⓑ $\log_2 8xy$

**TRY IT : :** 10.60

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

ⓐ $\log_9 9x$ ⓑ $\log_3 27xy$

Similarly, in the Quotient Property of Exponents, $\frac{a^m}{a^n} = a^{m-n}$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $\log_a \frac{M}{N} = \log_a M - \log_a N$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

### Quotient Property of Logarithms

If $M > 0$, $N > 0$,a $> 0$ and a $\neq 1$, then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_a M - \log_a N \neq \log_a (M - N)$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

**EXAMPLE 10.31**

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_5 \frac{5}{7}$ and ⓑ $\log \frac{x}{100}$

## Solution

ⓐ

| | |
|---|---|
| | $\log_5 \frac{5}{7}$ |
| Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$. | $\log_5 5 - \log_5 7$ |
| Simplify. | $1 - \log_5 7$ |
| | $\log_5 \frac{5}{7} = 1 - \log_5 7$ |

ⓑ

| | |
|---|---|
| | $\log \frac{x}{100}$ |
| Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$. | $\log x - \log 100$ |
| Simplify. | $\log x - 2$ |
| | $\log \frac{x}{100} = \log x - 2$ |

**TRY IT : :** 10.61

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_4 \frac{3}{4}$ ⓑ $\log \frac{x}{1000}$

**TRY IT : :** 10.62

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_2 \frac{5}{4}$ ⓑ $\log \frac{10}{y}$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_a M^p = p \log_a M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

**Power Property of Logarithms**

If $M > 0$, $a > 0$, $a \neq 1$ and $p$ is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power as the product product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

**EXAMPLE 10.32**

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_5 4^3$ and ⓑ $\log x^{10}$

## Solution

ⓐ

| | |
|---|---|
| | $\log_5 4^3$ |
| Use the Power Property, $\log_a M^p = p \log_a M$. | $3\log_5 4$ |
| | $\log_5 4^3 = 3\log_5 4$ |

ⓑ

| | |
|---|---|
| | $\log x^{10}$ |
| Use the Power Property, $\log_a M^p = p \log_a M$. | $10 \log x$ |
| | $\log x^{10} = 10\log x$ |

**TRY IT : :** 10.63

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_7 5^4$ ⓑ $\log x^{100}$

**TRY IT : :** 10.64

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_2 3^7$ ⓑ $\log x^{20}$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

**Properties of Logarithms**

If $M > 0,\ a > 0,\ a \neq 1$ and $p$ is any real number then,

| Property | Base $a$ | Base $e$ |
|---|---|---|
| | $\log_a 1 = 0$ | $\ln 1 = 0$ |
| | $\log_a a = 1$ | $\ln e = 1$ |
| **Inverse Properties** | $a^{\log_a x} = x$<br>$\log_a a^x = x$ | $e^{\ln x} = x$<br>$\ln e^x = x$ |
| **Product Property of Logarithms** | $\log_a (M \cdot N) = \log_a M + \log_a N$ | $\ln(M \cdot N) = \ln M + \ln N$ |
| **Quotient Property of Logarithms** | $\log_a \frac{M}{N} = \log_a M - \log_a N$ | $\ln \frac{M}{N} = \ln M - \ln N$ |
| **Power Property of Logarithms** | $\log_a M^p = p \log_a M$ | $\ln M^p = p \ln M$ |

Now that we have the properties we can use them to "expand" a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

**EXAMPLE 10.33**

Use the Properties of Logarithms to expand the logarithm $\log_4\left(2x^3 y^2\right)$. Simplify, if possible.

## Solution

| | |
|---|---|
| | $\log_4(2x^3y^2)$ |
| Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$. | $\log_4 2 + \log_4 x^3 + \log_4 y^2$ |
| Use the Power Property, $\log_a M^p = p\log_a M$, on the last two terms. | $\log_4 2 + 3\log_4 x + 2\log_4 y$ |
| Simplify. | $\frac{1}{2} + 3\log_4 x + 2\log_4 y$ |
| | $\log_4(2x^3y^2) = \frac{1}{2} + 3\log_4 x + 2\log_4 y$ |

**TRY IT : :** 10.65

Use the Properties of Logarithms to expand the logarithm $\log_2(5x^4y^2)$. Simplify, if possible.

**TRY IT : :** 10.66

Use the Properties of Logarithms to expand the logarithm $\log_3(7x^5y^3)$. Simplify, if possible.

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

**EXAMPLE 10.34**

Use the Properties of Logarithms to expand the logarithm $\log_2\sqrt[4]{\frac{x^3}{3y^2z}}$. Simplify, if possible.

## Solution

| | |
|---|---|
| | $\log_2\sqrt[4]{\frac{x^3}{3y^2z}}$ |
| Rewrite the radical with a rational exponent. | $\log_2\left(\frac{x^3}{3y^2z}\right)^{\frac{1}{4}}$ |
| Use the Power Property, $\log_a M^p = p\log_a M$. | $\frac{1}{4}\log_2\left(\frac{x^3}{3y^2z}\right)$ |
| Use the Quotient Property, $\log_a M \cdot N = \log_a M - \log_a N$. | $\frac{1}{4}\left(\log_2(x^3) - \log_2(3y^2z)\right)$ |
| Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$, in the second term. | $\frac{1}{4}\left(\log_2(x^3) - \left(\log_2 3 + \log_2 y^2 + \log_2 z\right)\right)$ |
| Use the Power Property, $\log_a M^p = p\log_a M$, inside the parentheses. | $\frac{1}{4}(3\log_2 x - (\log_2 3 + 2\log_2 y + \log_2 z))$ |
| Simplify by distributing. | $\frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$ |
| | $\log_2\sqrt[4]{\frac{x^3}{3y^2z}} = \frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$ |

**TRY IT : :** 10.67

Use the Properties of Logarithms to expand the logarithm $\log_4 \sqrt[5]{\frac{x^4}{2y^3 z^2}}$. Simplify, if possible.

**TRY IT : :** 10.68

Use the Properties of Logarithms to expand the logarithm $\log_3 \sqrt[3]{\frac{x^2}{5y z}}$. Simplify, if possible.

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

**EXAMPLE 10.35**

Use the Properties of Logarithms to condense the logarithm $\log_4 3 + \log_4 x - \log_4 y$. Simplify, if possible.

### Solution

| | |
|---|---|
| The log expressions all have the same base, 4. | $\log_4 3 + \log_4 x - \log_4 y$ |
| The fir t two terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$. | $\log_4 3x - \log_4 y$ |
| Since the logs are subtracted, we use the Quotient Property, $\log_a M - \log_a N = \log_a \frac{M}{N}$. | $\log_4 \frac{3x}{y}$ |
| | $\log_4 3 + \log_4 x - \log_4 y = \log_4 \frac{3x}{y}$ |

**TRY IT : :** 10.69

Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

**TRY IT : :** 10.70

Use the Properties of Logarithms to condense the logarithm $\log_3 6 - \log_3 x - \log_3 y$. Simplify, if possible.

**EXAMPLE 10.36**

Use the Properties of Logarithms to condense the logarithm $2\log_3 x + 4\log_3 (x+1)$. Simplify, if possible.

### Solution

| | |
|---|---|
| The log expressions have the same base, 3. | $2\log_3 x + 4\log_3 (x+1)$ |
| Use the Power Property, $\log_a M + \log_a N = \log_a M \cdot N$. | $\log_3 x^2 + \log_3 (x+1)^4$ |
| The terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$. | $\log_3 x^2 (x+1)^4$ |
| | $2\log_3 x + 4\log_3 (x+1) = \log_3 x^2 (x+1)^4$ |

**TRY IT : :** 10.71

Use the Properties of Logarithms to condense the logarithm $3\log_2 x + 2\log_2 (x-1)$. Simplify, if possible.

**TRY IT : :** 10.72

Use the Properties of Logarithms to condense the logarithm $2\log x + 2\log(x+1)$. Simplify, if possible.

## Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the Change-of-Base Formula. We will show how this is derived.

| | |
|---|---|
| Suppose we want to evaluate $\log_a M$. | $\log_a M$ |
| Let $y = \log_a M$. | $y = \log_a M$ |
| Rewrite the expression in exponential form. | $a^y = M$ |
| Take the $\log_b$ of each side. | $\log_b a^y = \log_b M$ |
| Use the Power Property. | $y\log_b a = \log_b M$ |
| Solve for $y$. | $y = \dfrac{\log_b M}{\log_b a}$ |
| Substitute $y = \log_a M$. | $\log_a M = \dfrac{\log_b M}{\log_b a}$ |

The Change-of-Base Formula introduces a new base $b$. This can be any base *b* we want where $b > 0, b \neq 1$. Because our calculators have keys for logarithms base 10 and base *e*, we will rewrite the Change-of-Base Formula with the new base as 10 or *e*.

**Change-of-Base Formula**

For any logarithmic bases $a, b$ and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$

new base $b$     new base 10     new base $e$

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol $(\approx)$.

**EXAMPLE 10.37**

Rounding to three decimal places, approximate $\log_4 35$.

### Solution

| | $\log_4 35$ |
|---|---|
| Use the Change-of-Base Formula. | $\log_a M = \dfrac{\log_b M}{\log_b a}$ |
| Identify *a* and *M*. Choose 10 for *b*. | $\log_4 35 = \dfrac{\log 35}{\log 4}$ |
| Enter the expression $\dfrac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places. | $\log_4 35 \approx 2.565$ |

**TRY IT : :** 10.73 Rounding to three decimal places, approximate $\log_3 42$.

**TRY IT : :** 10.74 Rounding to three decimal places, approximate $\log_5 46$.

**MEDIA : :**

Access these online resources for additional instruction and practice with using the properties of logarithms.

- Using Properties of Logarithms to Expand Logs (https://openstax.org/l/37Logproperties)
- Using Properties of Logarithms to Condense Logs (https://openstax.org/l/37Condenselogs)
- Change of Base (https://openstax.org/l/37Changeofbase)

## 10.4 EXERCISES

### Practice Makes Perfect

#### Use the Properties of Logarithms

*In the following exercises, use the properties of logarithms to evaluate.*

**218.** ⓐ $\log_4 1$ ⓑ $\log_8 8$

**219.** ⓐ $\log_{12} 1$ ⓑ $\ln e$

**220.** ⓐ $3^{\log_3 6}$ ⓑ $\log_2 2^7$

**221.** ⓐ $5^{\log_5 10}$ ⓑ $\log_4 4^{10}$

**222.** ⓐ $8^{\log_8 7}$ ⓑ $\log_6 6^{-2}$

**223.** ⓐ $6^{\log_6 15}$ ⓑ $\log_8 8^{-4}$

**224.** ⓐ $10^{\log\sqrt{5}}$ ⓑ $\log 10^{-2}$

**225.** ⓐ $10^{\log\sqrt{3}}$ ⓑ $\log 10^{-1}$

**226.** ⓐ $e^{\ln 4}$ ⓑ $\ln e^2$

**227.** ⓐ $e^{\ln 3}$ ⓑ $\ln e^7$

*In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.*

**228.** $\log_4 6x$

**229.** $\log_5 8y$

**230.** $\log_2 32xy$

**231.** $\log_3 81xy$

**232.** $\log 100x$

**233.** $\log 1000y$

*In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.*

**234.** $\log_3 \frac{3}{8}$

**235.** $\log_6 \frac{5}{6}$

**236.** $\log_4 \frac{16}{y}$

**237.** $\log_5 \frac{125}{x}$

**238.** $\log \frac{x}{10}$

**239.** $\log \frac{10{,}000}{y}$

**240.** $\ln \frac{e^3}{3}$

**241.** $\ln \frac{e^4}{16}$

*In the following exercises, use the Power Property of Logarithms to expand each. Simplify if possible.*

**242.** $\log_3 x^2$

**243.** $\log_2 x^5$

**244.** $\log x^{-2}$

**245.** $\log x^{-3}$

**246.** $\log_4 \sqrt{x}$

**247.** $\log_5 \sqrt[3]{x}$

**248.** $\ln x^{\sqrt{3}}$

**249.** $\ln x^{\sqrt[3]{4}}$

*In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.*

**250.** $\log_5\left(4x^6y^4\right)$

**251.** $\log_2\left(3x^5y^3\right)$

**252.** $\log_3\left(\sqrt{2}x^2\right)$

**253.** $\log_5\left(\sqrt[4]{21}y^3\right)$

**254.** $\log_3\frac{xy^2}{z^2}$

**255.** $\log_5\frac{4ab^3c^4}{d^2}$

**256.** $\log_4\frac{\sqrt{x}}{16y^4}$

**257.** $\log_3\frac{\sqrt[3]{x^2}}{27y^4}$

**258.** $\log_2\frac{\sqrt{2x+y^2}}{z^2}$

**259.** $\log_3\frac{\sqrt{3x+2y^2}}{5z^2}$

**260.** $\log_2\sqrt[4]{\frac{5x^3}{2y^2z^4}}$

**261.** $\log_5\sqrt[3]{\frac{3x^2}{4y^3z}}$

*In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.*

**262.** $\log_6 4 + \log_6 9$

**263.** $\log 4 + \log 25$

**264.** $\log_2 80 - \log_2 5$

**265.** $\log_3 36 - \log_3 4$

**266.** $\log_3 4 + \log_3 (x+1)$

**267.** $\log_2 5 - \log_2 (x-1)$

**268.** $\log_7 3 + \log_7 x - \log_7 y$

**269.** $\log_5 2 - \log_5 x - \log_5 y$

**270.** $4\log_2 x + 6\log_2 y$

**271.** $6\log_3 x + 9\log_3 y$

**272.** $\log_3\left(x^2-1\right) - 2\log_3(x-1)$

**273.** $\log\left(x^2+2x+1\right) - 2\log(x+1)$

**274.** $4\log x - 2\log y - 3\log z$

**275.** $3\ln x + 4\ln y - 2\ln z$

**276.** $\frac{1}{3}\log x - 3\log(x+1)$

**277.** $2\log(2x+3) + \frac{1}{2}\log(x+1)$

### Use the Change-of-Base Formula

*In the following exercises, use the Change-of-Base Formula, rounding to three decimal places, to approximate each logarithm.*

**278.** $\log_3 42$

**279.** $\log_5 46$

**280.** $\log_{12} 87$

**281.** $\log_{15} 93$

**282.** $\log_{\sqrt{2}} 17$

**283.** $\log_{\sqrt{3}} 21$

## Writing Exercises

**284.** Write the Product Property in your own words. Does it apply to each of the following? $\log_a 5x$, $\log_a(5+x)$. Why or why not?

**285.** Write the Power Property in your own words. Does it apply to each of the following? $\log_a x^p$, $(\log_a x)^r$. Why or why not?

**286.** Use an example to show that $\log(a+b) \neq \log a + \log b$?

**287.** Explain how to find the value of $\log_7 15$ using your calculator.

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| use the properties of logarithms. | | | |
| use the Change of Base Formula. | | | |

ⓑ *On a scale of* $1-10$, *how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

# 10.5 Solve Exponential and Logarithmic Equations

## Learning Objectives

**By the end of this section, you will be able to:**

- Solve logarithmic equations using the properties of logarithms
- Solve exponential equations using logarithms
- Use exponential models in applications

### Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 16.$
   If you missed this problem, review **Example 6.46**.
2. Solve: $x^2 - 5x + 6 = 0.$
   If you missed this problem, review **Example 6.45**.
3. Solve: $x(x + 6) = 2x + 5.$
   If you missed this problem, review **Example 6.47**.

## Solve Logarithmic Equations Using the Properties of Logarithms

In the section on logarithmic functions, we solved some equations by rewriting the equation in exponential form. Now that we have the properties of logarithms, we have additional methods we can use to solve logarithmic equations.

If our equation has two logarithms we can use a property that says that if $\log_a M = \log_a N$ then it is true that $M = N.$ This is the **One-to-One Property of Logarithmic Equations**.

### One-to-One Property of Logarithmic Equations

For $M > 0,\ N > 0,\ \text{a>0},$ and $\text{a} \neq 1$ is any real number:

$$\text{If } \log_a M = \log_a N, \ \text{ then } M = N.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Remember that logarithms are defined only for positive real numbers. Check your results in the original equation. You may have obtained a result that gives a logarithm of zero or a negative number.

**EXAMPLE 10.38**

Solve: $2\log_5 x = \log_5 81.$

## Solution

| | |
|---|---|
| | $2\log_5 x = \log_5 81$ |
| Use the Power Property. | $\log_5 x^2 = \log_5 81$ |
| Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$. | $x^2 = 81$ |
| Solve using the Square Root Property. | $x = \pm 9$ |
| We eliminate $x = -9$ as we cannot take the logarithm of a negative number. | $x = 9$, ~~$x = -9$~~ |
| Check. | |

$x = 9$

$$\begin{aligned} 2\log_5 x &= \log_5 81 \\ 2\log_5 9 &\stackrel{?}{=} \log_5 81 \\ \log_5 9^2 &\stackrel{?}{=} \log_5 81 \\ \log_5 81 &= \log_5 81 \checkmark \end{aligned}$$

**TRY IT : :** 10.75 Solve: $2\log_3 x = \log_3 36$

**TRY IT : :** 10.76 Solve: $3\log x = \log 64$

Another strategy to use to solve logarithmic equations is to condense sums or differences into a single logarithm.

**EXAMPLE 10.39**

Solve: $\log_3 x + \log_3 (x - 8) = 2$.

## Solution

| | |
|---|---|
| | $\log_3 x + \log_3 (x-8) = 2$ |
| Use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$. | $\log_3 x(x-8) = 2$ |
| Rewrite in exponential form. | $3^2 = x(x-8)$ |
| Simplify. | $9 = x^2 - 8x$ |
| Subtract 9 from each side. | $0 = x^2 - 8x - 9$ |
| Factor. | $0 = (x-9)(x+1)$ |
| Use the Zero-Product Property. | $x - 9 = 0,\quad x + 1 = 0$ |
| Solve each equation. | $x = 9$, ~~$x = -1$~~ |
| Check. | |

$x = -1$

$$\begin{aligned} \log_3 x + \log_3 (x-8) &= 2 \\ \log_3 (-1) + \log_3 (-1-8) &\stackrel{?}{=} 2 \end{aligned}$$

We cannot take the log of a negative number.

$x = 9$

$$\begin{aligned} \log_3 x + \log_3 (x-8) &= 2 \\ \log_3 9 + \log_3 (9-8) &\stackrel{?}{=} 2 \\ 2 + 0 &\stackrel{?}{=} 2 \\ 2 &= 2 \checkmark \end{aligned}$$

**TRY IT : :** 10.77 Solve: $\log_2 x + \log_2 (x - 2) = 3$

**TRY IT : :** 10.78 Solve: $\log_2 x + \log_2 (x-6) = 4$

When there are logarithms on both sides, we condense each side into a single logarithm. Remember to use the Power Property as needed.

**EXAMPLE 10.40**

Solve: $\log_4 (x+6) - \log_4 (2x+5) = -\log_4 x$.

**Solution**

| | |
|---|---|
| | $\log_4 (x+6) - \log_4 (2x+5) = -\log_4 x$ |
| Use the Quotient Property on the left side and the Power Property on the right. | $\log_4 \left(\frac{x+6}{2x+5}\right) = \log_4 x^{-1}$ |
| Rewrite $x^{-1} = \frac{1}{x}$. | $\log_4 \left(\frac{x+6}{2x+5}\right) = \log_4 \frac{1}{x}$ |
| Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$. | $\frac{x+6}{2x+5} = \frac{1}{x}$ |
| Solve the rational equation. | $x(x+6) = 2x+5$ |
| Distribute. | $x^2 + 6x = 2x+5$ |
| Write in standard form. | $x^2 + 4x - 5 = 0$ |
| Factor. | $(x+5)(x-1) = 0$ |
| Use the Zero-Product Property. | $x+5=0, \quad x-1=0$ |
| Solve each equation. | $\cancel{x=-5}, \quad x=1$ |
| Check. We leave the check for you. | |

**TRY IT : :** 10.79 Solve: $\log(x+2) - \log(4x+3) = -\log x$.

**TRY IT : :** 10.80 Solve: $\log(x-2) - \log(4x+16) = \log\frac{1}{x}$.

## Solve Exponential Equations Using Logarithms

In the section on exponential functions, we solved some equations by writing both sides of the equation with the same base. Next we wrote a new equation by setting the exponents equal.

It is not always possible or convenient to write the expressions with the same base. In that case we often take the common logarithm or natural logarithm of both sides once the exponential is isolated.

**EXAMPLE 10.41**

Solve $5^x = 11$. Find the exact answer and then approximate it to three decimal places.

**Solution**

| | |
|---|---|
| | $5^x = 11$ |
| Since the exponential is isolated, take the logarithm of both sides. | $\log 5^x = \log 11$ |
| Use the Power Property to get the $x$ as a factor, not an exponent. | $x\log 5 = \log 11$ |
| Solve for $x$. Find the exact answer. | $x = \frac{\log 11}{\log 5}$ |
| Approximate the answer. | $x \approx 1.490$ |
| Since $5^1 = 5$ and $5^2 = 25$, does it makes sense that $5^{1.490} \approx 11$? | |

**TRY IT : :** 10.81 Solve $7^x = 43$. Find the exact answer and then approximate it to three decimal places.

**TRY IT : :** 10.82 Solve $8^x = 98$. Find the exact answer and then approximate it to three decimal places.

When we take the logarithm of both sides we will get the same result whether we use the common or the natural logarithm (try using the natural log in the last example. Did you get the same result?) When the exponential has base *e*, we use the natural logarithm.

**EXAMPLE 10.42**

Solve $3e^{x+2} = 24$. Find the exact answer and then approximate it to three decimal places.

### Solution

| | |
|---|---|
| | $3e^{x+2} = 24$ |
| Isolate the exponential by dividing both sides by 3. | $e^{x+2} = 8$ |
| Take the natural logarithm of both sides. | $\ln e^{x+2} = \ln 8$ |
| Use the Power Property to get the $x$ as a factor, not an exponent. | $(x+2)\ln e = \ln 8$ |
| Use the property $\ln e = 1$ to simplify. | $x + 2 = \ln 8$ |
| Solve the equation. Find the exact answer. | $x = \ln 8 - 2$ |
| Approximate the answer. | $x \approx 0.079$ |

**TRY IT : :** 10.83 Solve $2e^{x-2} = 18$. Find the exact answer and then approximate it to three decimal places.

**TRY IT : :** 10.84 Solve $5e^{2x} = 25$. Find the exact answer and then approximate it to three decimal places.

## Use Exponential Models in Applications

In previous sections we were able to solve some applications that were modeled with exponential equations. Now that we have so many more options to solve these equations, we are able to solve more applications.

We will again use the Compound Interest Formulas and so we list them here for reference.

**Compound Interest**

For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A* is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{when compounded } n \text{ times a year.}$$

$$A = Pe^{rt} \quad \text{when compounded continuously.}$$

**EXAMPLE 10.43**

Jermael's parents put $10,000 in investments for his college expenses on his first birthday. They hope the investments will be worth $50,000 when he turns 18. If the interest compounds continuously, approximately what rate of growth will they need to achieve their goal?

## Solution

| | |
|---|---|
| Identify the variables in the formula. | $A = \$50{,}000$<br>$P = \$10{,}000$<br>$r = ?$<br>$t = 17 \text{ years}$ |
| | $A = Pe^{rt}$ |
| Substitute the values into the formula. | $50{,}000 = 10{,}000e^{r \cdot 17}$ |
| Solve for $r$.Divide each side by 10,000. | $5 = e^{17r}$ |
| Take the natural log of each side. | $\ln 5 = \ln e^{17r}$ |
| Use the Power Property. | $\ln 5 = 17r \ln e$ |
| Simplify. | $\ln 5 = 17r$ |
| Divide each side by 17. | $\frac{\ln 5}{17} = r$ |
| Approximate the answer. | $r \approx 0.095$ |
| Convert to a percentage. | $r \approx 9.5\%$ |
| | They need the rate of growth to be approximately 9.5%. |

**TRY IT : :** 10.85

Hector invests $\$10{,}000$ at age 21. He hopes the investments will be worth $\$150{,}000$ when he turns 50. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal?

**TRY IT : :** 10.86

Rachel invests $\$15{,}000$ at age 25. She hopes the investments will be worth $\$90{,}000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

We have seen that growth and decay are modeled by exponential functions. For growth and decay we use the formula $A = A_0 e^{kt}$. Exponential growth has a positive rate of growth or growth constant, $k$, and exponential decay has a negative rate of growth or decay constant, *k*.

**Exponential Growth and Decay**

For an original amount, $A_0$, that grows or decays at a rate, *k*, for a certain time, *t*, the final amount, *A*, is:

$$A = A_0 e^{kt}$$

We can now solve applications that give us enough information to determine the rate of growth. We can then use that rate of growth to predict other situations.

**EXAMPLE 10.44**

Researchers recorded that a certain bacteria population grew from 100 to 300 in 3 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

## Solution

This problem requires two main steps. First we must find the unknown rate, *k*. Then we use that value of *k* to help us find the unknown number of bacteria.

| | |
|---|---|
| Identify the variables in the formula. | $A = 300$<br>$A_0 = 100$<br>$k = ?$<br>$t = 3$ hours<br>$A = A_0 e^{kt}$ |
| Substitute the values in the formula. | $300 = 100e^{k \cdot 3}$ |
| Solve for $k$. Divide each side by 100. | $3 = e^{3k}$ |
| Take the natural log of each side. | $\ln 3 = \ln e^{3k}$ |
| Use the Power Property. | $\ln 3 = 3k \ln e$ |
| Simplify. | $\ln 3 = 3k$ |
| Divide each side by 3. | $\frac{\ln 3}{3} = k$ |
| Approximate the answer. | $k \approx 0.366$ |
| We use this rate of growth to predict the number of bacteria there will be in 24 hours. | $A = ?$<br>$A_0 = 100$<br>$k = \frac{\ln 3}{3}$<br>$t = 24$ hours<br>$A = A_0 e^{kt}$ |
| Substitute in the values. | $A = 100e^{\frac{\ln 3}{3} \cdot 24}$ |
| Evaluate. | $A \approx 656{,}100$ |
| | At this rate of growth, they can expect 656,100 bacteria. |

**TRY IT : :** 10.87

Researchers recorded that a certain bacteria population grew from 100 to 500 in 6 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

**TRY IT : :** 10.88

Researchers recorded that a certain bacteria population declined from 700,000 to 400,000 in 5 hours after the administration of medication. At this rate of decay, how many bacteria will there be 24 hours from the start of the experiment?

Radioactive substances decay or decompose according to the exponential decay formula. The amount of time it takes for the substance to decay to half of its original amount is called the half-life of the substance.

Similar to the previous example, we can use the given information to determine the constant of decay, and then use that constant to answer other questions.

**EXAMPLE 10.45**

The half-life of radium-226 is 1,590 years. How much of a 100 mg sample will be left in 500 years?

## Solution

This problem requires two main steps. First we must find the decay constant $k$. If we start with 100-mg, at the half-life there will be 50-mg remaining. We will use this information to find $k$. Then we use that value of $k$ to help us find the amount of sample that will be left in 500 years.

| | |
|---|---|
| Identify the variables in the formula. | $A = 50$<br>$A_0 = 100$<br>$k = ?$<br>$t = 1590\text{years}$ |
| | $A = A_0 e^{kt}$ |
| Substitute the values in the formula. | $50 = 100e^{k \cdot 1590}$ |
| Solve for $k$. Divide each side by 100. | $0.5 = e^{1590k}$ |
| Take the natural log of each side. | $\ln 0.5 = \ln e^{1590k}$ |
| Use the Power Property. | $\ln 0.5 = 1590k \ln e$ |
| Simplify. | $\ln 0.5 = 1590k$ |
| Divide each side by 1590. | $\frac{\ln 0.5}{1590} = k$ exact answer |
| We use this rate of growth to predict the amount that will be left in 500 years. | $A = ?$<br>$A_0 = 100$<br>$k = \frac{\ln 0.5}{1590}$<br>$t = 500\text{years}$ |
| | $A = A_0 e^{kt}$ |
| Substitute in the values. | $A = 100e^{\frac{\ln 0.5}{1590} \cdot 500}$ |
| Evaluate. | $A \approx 80.4$ mg<br>In 500 years there would be approximately 80.4 mg remaining. |

**TRY IT : :** 10.89

The half-life of magnesium-27 is 9.45 minutes. How much of a 10-mg sample will be left in 6 minutes?

**TRY IT : :** 10.90

The half-life of radioactive iodine is 60 days. How much of a 50-mg sample will be left in 40 days?

**MEDIA : :**

Access these online resources for additional instruction and practice with solving exponential and logarithmic equations.

- Solving Logarithmic Equations (https://openstax.org/l/37Solvelog)
- Solving Logarithm Equations (https://openstax.org/l/37sSolvelogeqs2)
- Finding the rate or time in a word problem on exponential growth or decay (https://openstax.org/l/37Solveforrate)
- Finding the rate or time in a word problem on exponential growth or decay (https://openstax.org/l/37Solveforrate2)

# 10.5 EXERCISES

## Practice Makes Perfect

### Solve Logarithmic Equations Using the Properties of Logarithms

*In the following exercises, solve for x.*

**288.** $\log_4 64 = 2\log_4 x$

**289.** $\log 49 = 2\log x$

**290.** $3\log_3 x = \log_3 27$

**291.** $3\log_6 x = \log_6 64$

**292.** $\log_5 (4x - 2) = \log_5 10$

**293.** $\log_3 \left(x^2 + 3\right) = \log_3 4x$

**294.** $\log_3 x + \log_3 x = 2$

**295.** $\log_4 x + \log_4 x = 3$

**296.** $\log_2 x + \log_2 (x - 3) = 2$

**297.** $\log_3 x + \log_3 (x + 6) = 3$

**298.** $\log x + \log(x + 3) = 1$

**299.** $\log x + \log(x - 15) = 2$

**300.** $\log(x + 4) - \log(5x + 12) = -\log x$

**301.** $\log(x - 1) - \log(x + 3) = \log\frac{1}{x}$

**302.** $\log_5 (x + 3) + \log_5 (x - 6) = \log_5 10$

**303.** $\log_5 (x + 1) + \log_5 (x - 5) = \log_5 7$

**304.** $\log_3 (2x - 1) = \log_3 (x + 3) + \log_3 3$

**305.** $\log(5x + 1) = \log(x + 3) + \log 2$

### Solve Exponential Equations Using Logarithms

*In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.*

**306.** $3^x = 89$

**307.** $2^x = 74$

**308.** $5^x = 110$

**309.** $4^x = 112$

**310.** $e^x = 16$

**311.** $e^x = 8$

**312.** $\left(\frac{1}{2}\right)^x = 6$

**313.** $\left(\frac{1}{3}\right)^x = 8$

**314.** $4e^{x+1} = 16$

**315.** $3e^{x+2} = 9$

**316.** $6e^{2x} = 24$

**317.** $2e^{3x} = 32$

**318.** $\frac{1}{4}e^x = 3$

**319.** $\frac{1}{3}e^x = 2$

**320.** $e^{x+1} + 2 = 16$

**321.** $e^{x-1} + 4 = 12$

*In the following exercises, solve each equation.*

**322.** $3^{3x+1} = 81$

**323.** $6^{4x-17} = 216$

**324.** $\frac{e^{x^2}}{e^{14}} = e^{5x}$

**325.** $\frac{e^{x^2}}{e^x} = e^{20}$

**326.** $\log_a 64 = 2$

**327.** $\log_a 81 = 4$

**328.** $\ln x = -8$

**329.** $\ln x = 9$

**330.** $\log_5(3x - 8) = 2$

**331.** $\log_4(7x + 15) = 3$

**332.** $\ln e^{5x} = 30$

**333.** $\ln e^{6x} = 18$

**334.** $3\log x = \log 125$

**335.** $7\log_3 x = \log_3 128$

**336.** $\log_6 x + \log_6(x - 5) = 24$

**337.** $\log_9 x + \log_9(x - 4) = 12$

**338.** $\log_2(x + 2) - \log_2(2x + 9) = -\log_2 x$

**339.** $\log_6(x + 1) - \log_6(4x + 10) = \log_6 \frac{1}{x}$

*In the following exercises, solve for x, giving an exact answer as well as an approximation to three decimal places.*

**340.** $6^x = 91$

**341.** $\left(\frac{1}{2}\right)^x = 10$

**342.** $7e^{x-3} = 35$

**343.** $8e^{x+5} = 56$

## Use Exponential Models in Applications

*In the following exercises, solve.*

**344.** Sung Lee invests $\$5,000$ at age 18. He hopes the investments will be worth $\$10,000$ when he turns 25. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?

**345.** Alice invests $\$15,000$ at age 30 from the signing bonus of her new job. She hopes the investments will be worth $\$30,000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

**346.** Coralee invests $\$5,000$ in an account that compounds interest monthly and earns $7\%$. How long will it take for her money to double?

**347.** Simone invests $\$8,000$ in an account that compounds interest quarterly and earns $5\%$. How long will it take for his money to double?

**348.** Researchers recorded that a certain bacteria population declined from 100,000 to 100 in 24 hours. At this rate of decay, how many bacteria will there be in 16 hours?

**349.** Researchers recorded that a certain bacteria population declined from 800,000 to 500,000 in 6 hours after the administration of medication. At this rate of decay, how many bacteria will there be in 24 hours?

**350.** A virus takes 6 days to double its original population $(A = 2A_0)$. How long will it take to triple its population?

**351.** A bacteria doubles its original population in 24 hours $(A = 2A_0)$. How big will its population be in 72 hours?

**352.** Carbon-14 is used for archeological carbon dating. Its half-life is 5,730 years. How much of a 100-gram sample of Carbon-14 will be left in 1000 years?

**353.** Radioactive technetium-99m is often used in diagnostic medicine as it has a relatively short half-life but lasts long enough to get the needed testing done on the patient. If its half-life is 6 hours, how much of the radioactive material form a 0.5 ml injection will be in the body in 24 hours?

## Writing Exercises

**354.** Explain the method you would use to solve these equations: $3^{x+1} = 81, \quad 3^{x+1} = 75.$ Does your method require logarithms for both equations? Why or why not?

**355.** What is the difference between the equation for exponential growth versus the equation for exponential decay?

## Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

| I can... | Confidently | With some help | No-I don't get it! |
|---|---|---|---|
| solve logarithmic equations using the properties of logarithms. | | | |
| solve exponential equations using logarithms. | | | |
| use exponential models in applications. | | | |

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

# CHAPTER 10 REVIEW

## KEY TERMS

**asymptote** A line which a graph of a function approaches closely but never touches.

**common logarithmic function** The function $f(x) = \log x$ is the common logarithmic function with base $10,$ where $x > 0.$

$$y = \log x \text{ is equivalent to } x = 10^y$$

**exponential function** An exponential function, where $a > 0$ and $a \neq 1,$ is a function of the form $f(x) = a^x.$

**logarithmic function** The function $f(x) = \log_a x$ is the logarithmic function with base $a,$ where $a > 0,$ $x > 0,$ and $a \neq 1.$

$$y = \log_a x \text{ is equivalent to } x = a^y$$

**natural base** The number *e* is defined as the value of $\left(1 + \frac{1}{n}\right)^n,$ as *n* gets larger and larger. We say, as *n* increases without bound, $e \approx 2.718281827...$

**natural exponential function** The natural exponential function is an exponential function whose base is *e*: $f(x) = e^x.$ The domain is $(-\infty, \infty)$ and the range is $(0, \infty).$

**natural logarithmic function** The function $f(x) = \ln x$ is the natural logarithmic function with base $e,$ where $x > 0.$

$$y = \ln x \text{ is equivalent to } x = e^y$$

**one-to-one function** A function is one-to-one if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

## KEY CONCEPTS

### 10.1 Finding Composite and Inverse Functions

- **Composition of Functions:** The composition of functions $f$ and $g,$ is written $f \circ g$ and is defined by

$$(f \circ g)(x) = f(g(x))$$

  We read $f(g(x))$ as $f$ of $g$ of $x.$

- **Horizontal Line Test:** If every horizontal line, intersects the graph of a function in at most one point, it is a one-to-one function.
- **Inverse of a Function Defined by Ordered Pairs:** If $f(x)$ is a one-to-one function whose ordered pairs are of the form $(x, y),$ then its inverse function $f^{-1}(x)$ is the set of ordered pairs $(y, x).$
- **Inverse Functions:** For every $x$ in the domain of one-to-one function $f$ and $f^{-1},$

$$\begin{aligned} f^{-1}(f(x)) &= x \\ f\left(f^{-1}(x)\right) &= x \end{aligned}$$

- **How to Find the Inverse of a One-to-One Function:**

  Step 1. Substitute *y* for $f(x).$

  Step 2. Interchange the variables *x* and *y*.

  Step 3. Solve for *y*.

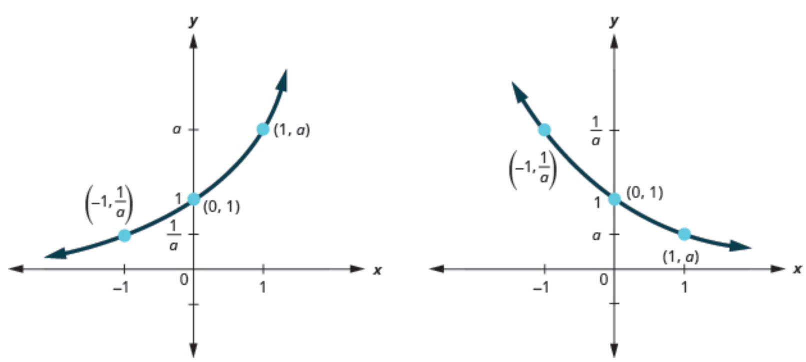
  Step 4. Substitute $f^{-1}(x)$ for $y.$

  Step 5. Verify that the functions are inverses.

## 10.2 Evaluate and Graph Exponential Functions

- **Properties of the Graph of** $f(x) = a^x$ :

| when $a > 1$ | | when $0 < a < 1$ | |
|---|---|---|---|
| Domain | $(-\infty, \infty)$ | Domain | $(-\infty, \infty)$ |
| Range | $(0, \infty)$ | Range | $(0, \infty)$ |
| $x$-intercept | none | $x$-intercept | none |
| $y$-intercept | $(0, 1)$ | $y$-intercept | $(0, 1)$ |
| Contains | $(1, a), \left(-1, \frac{1}{a}\right)$ | Contains | $(1, a), \left(-1, \frac{1}{a}\right)$ |
| Asymptote | $x$-axis, the line $y = 0$ | Asymptote | $x$-axis, the line $y = 0$ |
| Basic shape | increasing | Basic shape | decreasing |




- **One-to-One Property of Exponential Equations:**
For $a > 0$ and $a \neq 1$,

$$A = A_0 e^{rt}$$

- **How to Solve an Exponential Equation**
Step 1. Write both sides of the equation with the same base, if possible.
Step 2. Write a new equation by setting the exponents equal.
Step 3. Solve the equation.
Step 4. Check the solution.
- **Compound Interest:** For a principal, $P,$ invested at an interest rate, $r,$ for $t$ years, the new balance, $A,$ is

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ when compounded $n$ times a year.

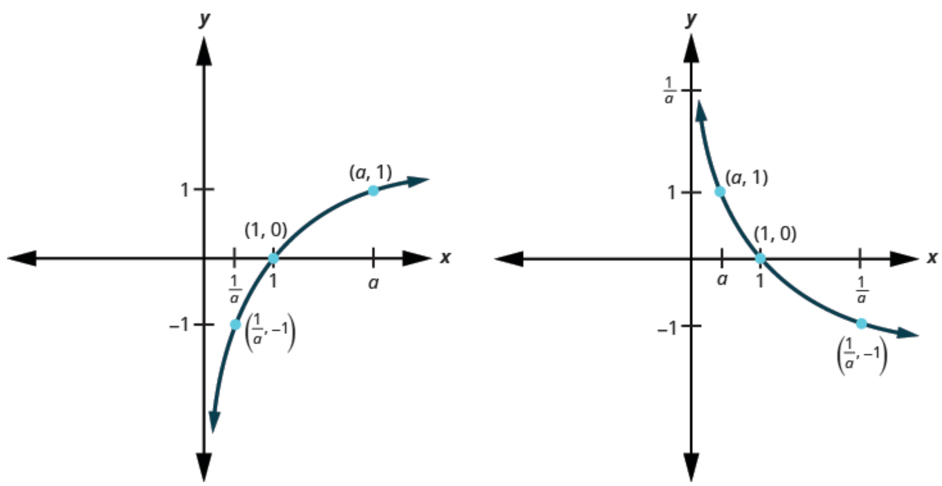
$A = Pe^{rt}$ when compounded continuously.

- **Exponential Growth and Decay:** For an original amount, $A_0$ that grows or decays at a rate, $r,$ for a certain time $t,$ the final amount, $A,$ is $A = A_0 e^{rt}.$

## 10.3 Evaluate and Graph Logarithmic Functions

- **Properties of the Graph of** $y = \log_a x$ :

| when $a > 1$ | | when $0 < a < 1$ | |
|---|---|---|---|
| Domain | $(0, \infty)$ | Domain | $(0, \infty)$ |
| Range | $(-\infty, \infty)$ | Range | $(-\infty, \infty)$ |
| *x*-intercept | $(1, 0)$ | *x*-intercept | $(1, 0)$ |
| *y*-intercept | none | *y*-intercept | none |
| Contains | $(a, 1), \left(\frac{1}{a}, -1\right)$ | Contains | $(a, 1), \left(\frac{1}{a}, -1\right)$ |
| Asymptote | *y*-axis | Asymptote | *y*-axis |
| Basic shape | increasing | Basic shape | decreasing |




- **Decibel Level of Sound:** The loudness level, $D$, measured in decibels, of a sound of intensity, $I$, measured in watts per square inch is $D = 10\log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude $R$ of an earthquake is measured by $R = \log I,$ where $I$ is the intensity of its shock wave.

## 10.4 Use the Properties of Logarithms

- **Properties of Logarithms**

$$\log_a 1 = 0 \qquad \log_a a = 1$$

- **Inverse Properties of Logarithms**
  - For $a > 0,\ x > 0$ and $a \neq 1$

$$a^{\log_a x} = x \qquad \log_a a^x = x$$

- **Product Property of Logarithms**
  - If $M > 0, N > 0, \text{a} > 0$ and $\text{a} \neq 1,$ then,

$$\log_a M \cdot N = \log_a M + \log_a N$$

  The logarithm of a product is the sum of the logarithms.
- **Quotient Property of Logarithms**

- If $M>0,\ N>0, \mathrm{a}>0$ and $\mathrm{a}\neq 1,$ then,

$$\log_a \frac{M}{N}=\log_a M-\log_a N$$

The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**
  - If $M>0,\ \mathrm{a}>0,\ \mathrm{a}\neq 1$ and $p$ is any real number then,

$$\log_a M^p = p\log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

- **Properties of Logarithms Summary**
If $M>0,\ \mathrm{a}>0,\ \mathrm{a}\neq 1$ and $p$ is any real number then,

| Property | Base $a$ | Base $e$ |
|---|---|---|
| | $\log_a 1=0$ | $\ln 1=0$ |
| | $\log_a a=1$ | $\ln e=1$ |
| **Inverse Properties** | $a^{\log_a x}=x$<br>$\log_a a^x=x$ | $e^{\ln x}=x$<br>$\ln e^x=x$ |
| **Product Property of Logarithms** | $\log_a(M\cdot N)=\log_a M+\log_a N$ | $\ln(M\cdot N)=\ln M+\ln N$ |
| **Quotient Property of Logarithms** | $\log_a \frac{M}{N}=\log_a M-\log_a N$ | $\ln\frac{M}{N}=\ln M-\ln N$ |
| **Power Property of Logarithms** | $\log_a M^p=p\log_a M$ | $\ln M^p=p\ln M$ |

- **Change-of-Base Formula**
For any logarithmic bases *a* and *b*, and $M>0,$

$$\log_a M=\frac{\log_b M}{\log_b a} \qquad \log_a M=\frac{\log M}{\log a} \qquad \log_a M=\frac{\ln M}{\ln a}$$

$$\text{new base } b \qquad \text{new base } 10 \qquad \text{new base } e$$

## 10.5 Solve Exponential and Logarithmic Equations

- **One-to-One Property of Logarithmic Equations:** For $M>0,\ N>0,\ \mathrm{a}>0,$ and $\mathrm{a}\neq 1$ is any real number:

$$\text{If } \log_a M=\log_a N,\ \text{ then } M=N.$$

- **Compound Interest:**
For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A*, is:

$$A=P\left(1+\frac{r}{n}\right)^{nt} \qquad \text{when compounded } n \text{ times a year.}$$

$$A=Pe^{rt} \qquad \text{when compounded continuously.}$$

- **Exponential Growth and Decay:** For an original amount, $A_0$ that grows or decays at a rate, *r*, for a certain time *t*, the final amount, *A*, is $A=A_0e^{rt}$.

# REVIEW EXERCISES

## 10.1 Section 10.1 Finding Composite and Inverse Functions

### Find and Evaluate Composite Functions

*In the following exercises, for each pair of functions, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.*

**356.** $f(x) = 7x - 2$ and $g(x) = 5x + 1$

**357.** $f(x) = 4x$ and $g(x) = x^2 + 3x$

*In the following exercises, evaluate the composition.*

**358.** For functions $f(x) = 3x^2 + 2$ and $g(x) = 4x - 3$, find

ⓐ $(f \circ g)(-3)$

ⓑ $(g \circ f)(-2)$

ⓒ $(f \circ f)(-1)$

**359.** For functions $f(x) = 2x^3 + 5$ and $g(x) = 3x^2 - 7$, find

ⓐ $(f \circ g)(-1)$

ⓑ $(g \circ f)(-2)$

ⓒ $(g \circ g)(1)$

### Determine Whether a Function is One-to-One

*In the following exercises, for each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.*

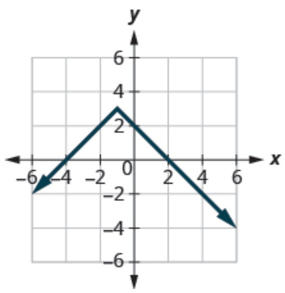
**360.** $\{(-3, -5), (-2, -4), (-1, -3), (0, -2), (-1, -1), (-2, 0), (-3, 1)\}$

**361.** $\{(-3, 0), (-2, -2), (-1, 0), (0, 1), (1, 2), (2, 1), (3, -1)\}$

**362.** $\{(-3, 3), (-2, 1), (-1, -1), (0, -3), (1, -5), (2, -4), (3, -2)\}$

*In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.*

**363.** ⓐ

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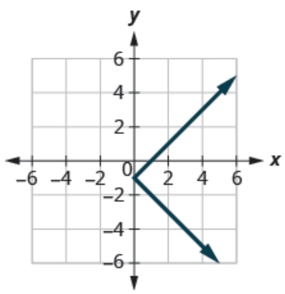
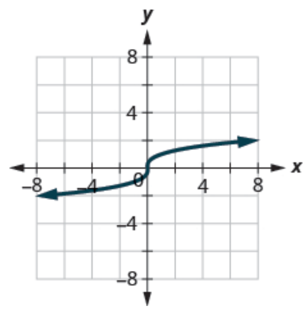



**364.** ⓐ

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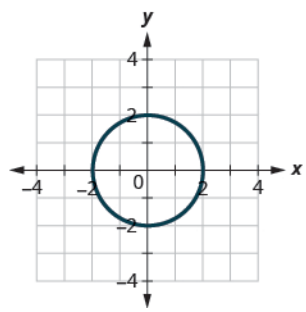


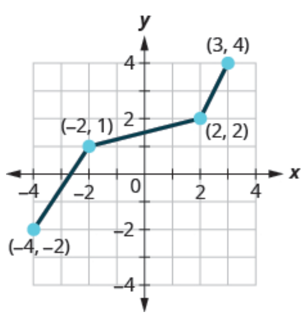

## Find the Inverse of a Function

*In the following exercise, find the inverse of the function. Determine the domain and range of the inverse function.*

**365.** $\{(-3,\ 10),\ (-2,\ 5),\ (-1,\ 2),\ (0,\ 1)\}$

*In the following exercise, graph the inverse of the one-to-one function shown.*

**366.**




*In the following exercises, verify that the functions are inverse functions.*

**367.** $f(x) = 3x + 7$ and $g(x) = \frac{x-7}{3}$

**368.** $f(x) = 2x + 9$ and $g(x) = \frac{x+9}{2}$

*In the following exercises, find the inverse of each function.*

**369.** $f(x) = 6x - 11$

**370.** $f(x) = x^3 + 13$

**371.** $f(x) = \frac{1}{x+5}$

**372.** $f(x) = \sqrt[5]{x-1}$

## 10.2 Section 10.2 Evaluate and Graph Exponential Functions

### Graph Exponential Functions

*In the following exercises, graph each of the following functions.*

**373.** $f(x) = 4^x$

**374.** $f(x) = \left(\frac{1}{5}\right)^x$

**375.** $g(x) = (0.75)^x$

**376.** $g(x) = 3^{x+2}$

**377.** $f(x) = (2.3)^x - 3$

**378.** $f(x) = e^x + 5$

**379.** $f(x) = -e^x$

### Solve Exponential Equations

*In the following exercises, solve each equation.*

**380.** $3^{5x-6} = 81$

**381.** $2^{x^2} = 16$

**382.** $9^x = 27$

**383.** $5^{x^2+2x} = \frac{1}{5}$

**384.** $e^{4x} \cdot e^7 = e^{19}$

**385.** $\frac{e^{x^2}}{e^{15}} = e^{2x}$

### Use Exponential Models in Applications

*In the following exercises, solve.*

**386.** Felix invested $\$12,000$ in a savings account. If the interest rate is $4\%$ how much will be in the account in 12 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously.

**387.** Sayed deposits $\$20,000$ in an investment account. What will be the value of his investment in 30 years if the investment is earning $7\%$ per year and is compounded continuously?

**388.** A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. She starts her experiment with 150 of the bacteria that grows at a rate of $15\%$ per hour. She will check on the bacteria every 24 hours. How many bacteria will he find in 24 hours?

**389.** In the last five years the population of the United States has grown at a rate of $0.7\%$ per year to about 318,900,000. If this rate continues, what will be the population in 5 more years?

## 10.3 Section 10.3 Evaluate and Graph Logarithmic Functions

### Convert Between Exponential and Logarithmic Form

*In the following exercises, convert from exponential to logarithmic form.*

**390.** $5^4 = 625$

**391.** $10^{-3} = \frac{1}{1,000}$

**392.** $63^{\frac{1}{5}} = \sqrt[5]{63}$

**393.** $e^y = 16$

*In the following exercises, convert each logarithmic equation to exponential form.*

**394.** $7 = \log_2 128$

**395.** $5 = \log 100{,}000$

**396.** $4 = \ln x$

### Evaluate Logarithmic Functions

*In the following exercises, solve for x.*

**397.** $\log_x 125 = 3$

**398.** $\log_7 x = -2$

**399.** $\log_{\frac{1}{2}} \frac{1}{16} = x$

*In the following exercises, find the exact value of each logarithm without using a calculator.*

**400.** $\log_2 32$

**401.** $\log_8 1$

**402.** $\log_3 \frac{1}{9}$

### Graph Logarithmic Functions

*In the following exercises, graph each logarithmic function.*

**403.** $y = \log_5 x$

**404.** $y = \log_{\frac{1}{4}} x$

**405.** $y = \log_{0.8} x$

### Solve Logarithmic Equations

*In the following exercises, solve each logarithmic equation.*

**406.** $\log_a 36 = 5$

**407.** $\ln x = -3$

**408.** $\log_2 (5x - 7) = 3$

**409.** $\ln e^{3x} = 24$

**410.** $\log(x^2 - 21) = 2$

### Use Logarithmic Models in Applications

**411.** What is the decibel level of a train whistle with intensity $10^{-3}$ watts per square inch?

## 10.4 Section 10.4 Use the Properties of Logarithms

### Use the Properties of Logarithms

*In the following exercises, use the properties of logarithms to evaluate.*

**412.** ⓐ $\log_7 1$ ⓑ $\log_{12} 12$

**413.** ⓐ $5^{\log_5 13}$ ⓑ $\log_3 3^{-9}$

**414.** ⓐ $10^{\log \sqrt{5}}$ ⓑ $\log 10^{-3}$

**415.** ⓐ $e^{\ln 8}$ ⓑ $\ln e^5$

*In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.*

**416.** $\log_4(64xy)$

**417.** $\log 10{,}000m$

*In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.*

**418.** $\log_7 \frac{49}{y}$

**419.** $\ln \frac{e^5}{2}$

*In the following exercises, use the Power Property of Logarithms to expand each logarithm. Simplify, if possible.*

**420.** $\log x^{-9}$

**421.** $\log_4 \sqrt[7]{z}$

*In the following exercises, use properties of logarithms to write each logarithm as a sum of logarithms. Simplify if possible.*

**422.** $\log_3\left(\sqrt{4}x^7 y^8\right)$

**423.** $\log_5 \frac{8a^2 b^6 c}{d^3}$

**424.** $\ln \frac{\sqrt{3x^2 - y^2}}{z^4}$

**425.** $\log_6 \sqrt[3]{\frac{7x^2}{6y^3 z^5}}$

*In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.*

**426.** $\log_2 56 - \log_2 7$

**427.** $3\log_3 x + 7\log_3 y$

**428.** $\log_5\left(x^2 - 16\right) - 2\log_5(x + 4)$

**429.** $\frac{1}{4}\log y - 2\log(y - 3)$

### Use the Change-of-Base Formula

*In the following exercises, rounding to three decimal places, approximate each logarithm.*

**430.** $\log_5 97$

**431.** $\log_{\sqrt{3}} 16$

## 10.5 Section 10.5 Solve Exponential and Logarithmic Equations

### Solve Logarithmic Equations Using the Properties of Logarithms

*In the following exercises, solve for x.*

**432.** $3\log_5 x = \log_5 216$

**433.** $\log_2 x + \log_2(x - 2) = 3$

**434.** $\log(x - 1) - \log(3x + 5) = -\log x$

**435.** $\log_4(x - 2) + \log_4(x + 5) = \log_4 8$

**436.** $\ln(3x - 2) = \ln(x + 4) + \ln 2$

### Solve Exponential Equations Using Logarithms

*In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.*

**437.** $2^x = 101$

**438.** $e^x = 23$

**439.** $\left(\frac{1}{3}\right)^x = 7$

**440.** $7e^{x+3} = 28$

**441.** $e^{x-4} + 8 = 23$

### Use Exponential Models in Applications

**442.** Jerome invests $\$18,000$ at age 17. He hopes the investments will be worth $\$30,000$ when he turns 26. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?

**443.** Elise invests $\$4500$ in an account that compounds interest monthly and earns $6\%$. How long will it take for her money to double?

**444.** Researchers recorded that a certain bacteria population grew from 100 to 300 in 8 hours. At this rate of growth, how many bacteria will there be in 24 hours?

**445.** Mouse populations can double in 8 months $\left(A = 2A_0\right)$. How long will it take for a mouse population to triple?

**446.** The half-life of radioactive iodine is 60 days. How much of a 50 mg sample will be left in 40 days?

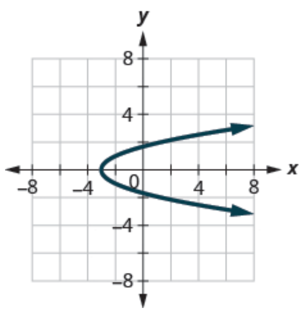
## PRACTICE TEST

**447.** For the functions, $f(x) = 6x + 1$ and $g(x) = 8x - 3$, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

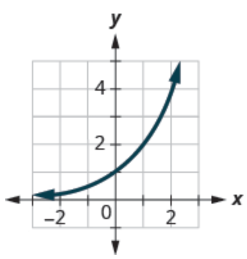
**448.** Determine if the following set of ordered pairs represents a function and if so, is the function one-to-one. $\{(-2, 2), (-1, -3), (0, 1), (1, -2), (2, -3)\}$

**449.** Determine whether each graph is the graph of a function and if so, is it one-to-one.

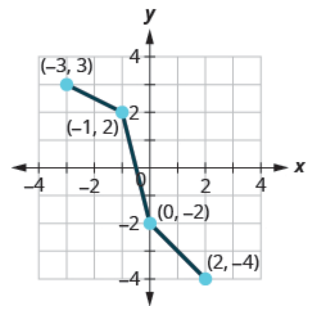
ⓐ




ⓑ




**450.** Graph, on the same coordinate system, the inverse of the one-to-one function shown.




**451.** Find the inverse of the function $f(x) = x^5 - 9$.

**452.** Graph the function $g(x) = 2^{x-3}$.

**453.** Solve the equation $2^{2x-4} = 64$.

**454.** Solve the equation $\frac{e^{x^2}}{e^4} = e^{3x}$.

**455.** Megan invested $\$21{,}000$ in a savings account. If the interest rate is $5\%$, how much will be in the account in 8 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously.

**456.** Convert the equation from exponential to logarithmic form: $10^{-2} = \frac{1}{100}$.

**457.** Convert the equation from logarithmic equation to exponential form: $3 = \log_7 343$

**458.** Solve for *x*: $\log_5 x = -3$

**459.** Evaluate $\log_{11} 1$.

**460.** Evaluate $\log_4 \frac{1}{64}$.

**461.** Graph the function $y = \log_3 x$.

**462.** Solve for *x*: $\log(x^2 - 39) = 1$

**463.** What is the decibel level of a small fan with intensity $10^{-8}$ watts per square inch?

**464.** Evaluate each. ⓐ $6^{\log_6 17}$ ⓑ $\log_9 9^{-3}$

*In the following exercises, use properties of logarithms to write each expression as a sum of logarithms, simplifying if possible.*

**465.** $\log_5 25ab$

**466.** $\ln \frac{e^{12}}{8}$

**467.** $\log_2 \sqrt[4]{\frac{5x^3}{16y^2 z^7}}$

*In the following exercises, use the Properties of Logarithms to condense the logarithm, simplifying if possible.*

**468.** $5\log_4 x + 3\log_4 y$

**469.** $\frac{1}{6}\log x - 3\log(x + 5)$

**470.** Rounding to three decimal places, approximate $\log_4 73$.

**471.** Solve for *x*: $\log_7 (x + 2) + \log_7 (x - 3) = \log_7 24$

*In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.*

**472.** $\left(\frac{1}{5}\right)^x = 9$

**473.** $5e^{x-4} = 40$

**474.** Jacob invests $14,000 in an account that compounds interest quarterly and earns $4\%$. How long will it take for his money to double?

**475.** Researchers recorded that a certain bacteria population grew from 500 to 700 in 5 hours. At this rate of growth, how many bacteria will there be in 20 hours?

**476.** A certain beetle population can double in 3 months $(A = 2A_0)$. How long will it take for that beetle population to triple?