Name: Date:

3E: Derivatives with Limits and Inverses

Part 1: L'Hopital's Rule

At the beginning of our study of Calculus, we studied limits of functions and ran into several cases when we had problems finding the limit. These were cases like

$$\lim_{x\to 3}\frac{2x^2-18}{x-3}\Rightarrow\frac{0}{0},\qquad \lim_{x\to 2}\frac{\sqrt{x+14}\,-\,4}{x-2}\Rightarrow\frac{0}{0},\qquad \lim_{x\to 2}\frac{\frac{1}{x-2}-3}{x-2}\Rightarrow\frac{\infty}{0},\qquad \lim_{x\to\infty}\frac{3x^2+4}{2x^2-1}\Rightarrow\frac{\infty}{\infty}$$

All of these we can had algebraic techniques to solve, but sometimes this was a little work. We also have some difficulty limits like these which none of our techniques will work on:

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}\Rightarrow\frac{0}{0},\qquad \lim_{x\to\infty}\frac{\ln x}{x}\Rightarrow\frac{\infty}{\infty},\qquad \lim_{x\to-\infty}\frac{x^2}{e^{-x}}\Rightarrow\frac{\infty}{\infty},\qquad \lim_{x\to\infty} e^{-x}\sqrt{x}\Rightarrow 0\cdot\infty,\qquad \lim_{x\to 0^+} \mathrm{x}\cdot\sin x\Rightarrow 0\cdot 0$$

Indeterminate Forms

The key to recognizing these problematic situations is understanding that some expressions are **indeterminate** (which just means that the value of the expression is unknown). Notice all of these limits involve 0 and/or $\pm\infty$. The important thing to remember is that these values are ***approaching 0 or*** ∞. And the two numbers in the expression could be approaching 0 or $\pm\infty$ at different rates.

GUILLAUME L'HÔPITAL (1661–1704)

L'Hôpital's Rule is named after the French mathematician Guillaume François Antoine de L'Hôpital. L'Hôpital is credited with writing the first text on differential calculus (in 1696) in which the rule publicly appeared. It was recently discovered that the rule and its proof were written in a letter from John Bernoulli to L'Hôpital. "… I acknowledge that I owe very much to the bright minds of the Bernoulli brothers. … I have made free use of their discoveries …," said L'Hôpital.

A simple example is $\lim_{x\to\infty}\frac{2x}{x}\Rightarrow\frac{\infty}{\infty}$ when we use direct substitution.

However, when we stop to think about what is happening as the x-values get larger the top value is approaching an infinity that is twice as large, so we can say

$$\lim_{x\to\infty}\frac{2x}{x}=\lim_{x\to\infty}2=2.$$

These problems greatly interested a French mathematician by the name of Guillaume L'Hôpital in the late 1600's. However, he wasn't very mathematical genius, but he had a lot of money, so he paid Joseph Bernoulli to teach him. Then L'Hopital published the discoveries under his name.

Here is a list of the indeterminate forms we need to recognize:

$$\frac{\mathbf{0}}{\mathbf{0}},\qquad \frac{\pm\infty}{\pm\infty},\qquad \mathbf{0}\cdot\mathbf{0},\qquad \mathbf{0}\cdot\infty,$$

$$\mathbf{1}^{\infty},\qquad \infty^{\mathbf{0}},\qquad \mathbf{0}^{\mathbf{0}},\qquad \infty-\infty$$

L'Hôpital's rule in action

THEOREM 8.4 L'HÔPITAL'S RULE

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

Warning! Do not confuse this with the quotient rule. We are NOT saying…

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}$$

EXAMPLE 1 Indeterminate Form 0/0

Evaluate $\lim_{x\to 0} \frac{e^{2x} - 1}{x}$.

EXAMPLE 2 Indeterminate Form ∞/∞

Evaluate $\lim_{x\to \infty} \frac{\ln x}{x}$.

EXAMPLE 3 Applying L'Hôpital's Rule More Than Once

Evaluate $\lim_{x\to -\infty} \frac{x^2}{e^{-x}}$.

Some More Variations: Here are some other situations where we can apply L'Hôpital's rule.

EXAMPLE Indeterminate Form $0 \cdot \infty$

Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$.

(Hint: Rewrite with positive exponents.)

EXAMPLE Indeterminate Form $\infty - \infty$

Evaluate $\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

(Hint: Combine the fractions into one fraction)

EXAMPLE Indeterminate Form 1^∞

Evaluate $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$.

(Hint: let $y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$, take the natural log of both sides, and write as a fraction.)

EXAMPLE Indeterminate Form 0^0

Find $\lim_{x\to 0^+} (\sin x)^x$.

(Hint: let $y = \lim_{x\to 0^+} (\sin x)^x$, then take the natural log of both sides.)

Important note: These forms are *__not indeterminate!__*

$$\lim f(x) \to \infty \cdot \infty = \infty$$
$$\lim f(x) \to \infty + \infty = \infty$$

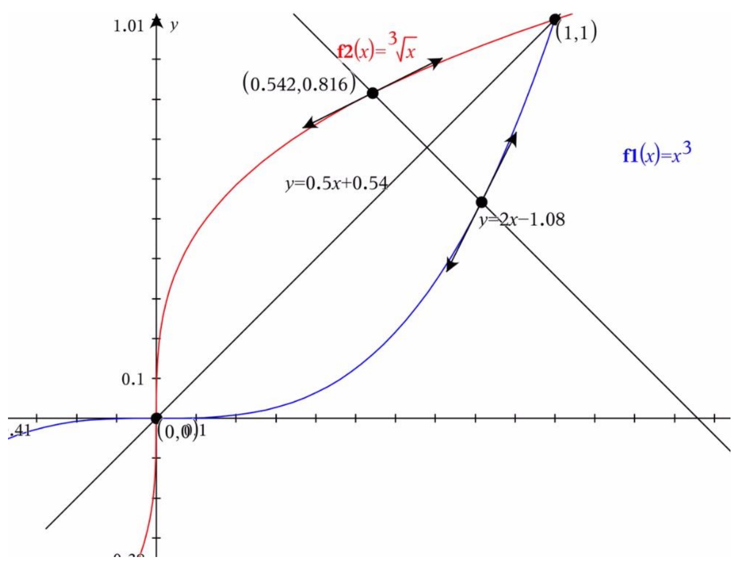
$$\lim f(x) \to -\infty - \infty = -\infty$$

$$\lim f(x) \to 0^{\infty} = 0$$
$$\lim f(x) \to 0^{-\infty} = \infty$$

Part 2: Derivatives of Inverse Functions

Sometimes we run into a function that is difficult to take the derivative of, but we know how to find the derivative of its inverse.

Review. Find the inverse of $f(x) = 2x - 3$.



Key Properties of inverse functions (let $\boldsymbol{g(x) = f^{-1}(x)}$):

- x-coordinates swap with y-coordinates
- $f\big(g(x)\big) = x = g\big(f(x)\big)$
- Domain and Range of $f(x)$ become the Range and Domain of $f^{-1}(x)$
- The graphs are reflected across the $y = x$ line
- If $f(a) = b$, then the slope of $f(a)$ is the reciprocal of the slope of $f'(b)$

So, if we can find the derivative of its inverse, we can use this simple rule:

The Derivative of an Inverse Function:

Let f be a function that is differentiable on an interval I. If g is the inverse function of f, then g is differentiable for any x for which $f'\big(g(x)\big) \neq 0$. Furthermore,

$$g'(x) = \frac{1}{f'\big(g(x)\big)}, \qquad f'\big(g(x)\big) \neq 0$$

Example If $f(x) = x^3$, a one-to-one function, find the following:

a) $f(2)$

b) $f'(x)$

c) $f'(2)$

d) $f^{-1}(8)$

e) $f^{-1}(x)$

f) $(f^{-1})'(x)$

g) $(f^{-1})'(8)$

e) $\frac{1}{f'(f^{-1}(8))}$

Example

x	$f(x)$	$f'(x)$	$g(x)$
-3	4	6	7
4	-8	-2	-3
7	-3	2	9

Selected values of one-to-one differentiable functions f and g (and f') are given in the table above. If $f(g(x)) = x = g(f(x))$, find $g'(-3)$.