Unit 3 Toolkit: Applications of Derivatives

3A: Extrema on an Interval (3.1)

To find extrema on an interval, we need to find critical numbers where $f'(x) = 0$ or where $f'(x)$ is undefined.

Remember: There is only a *guaranteed* min/max if the function is continuous

DEFINITION OF EXTREMA

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum,** or the **global minimum** and **global maximum,** on the interval.

THEOREM 3.1 THE EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

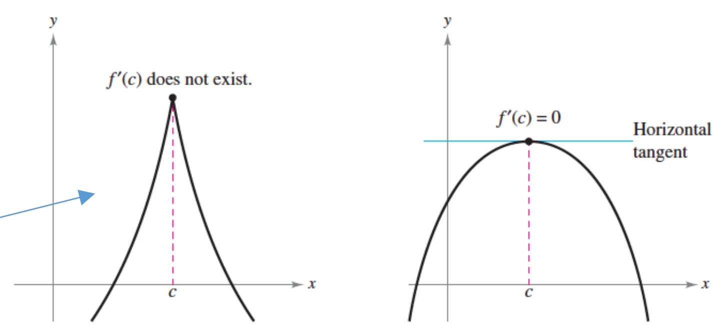
To find critical numbers:

1. Find $f'(x)$
2. Find values when $f'(x)$ is undefined
3. Find values that solve $f'(x) = 0$

DEFINITION OF A CRITICAL NUMBER

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.

If the graph is "Pointy" at $x = c$, then it is not differentiable there!



c is a critical number of f.

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

THEOREM 3.2 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL NUMBERS

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

3B: Intervals of Increasing and Decreasing Change (3.3)

Between critical values, a function must be increasing, decreasing, or constant.

If $f(x)$ is increasing, then $f'(x)$ is **positive**
If $f(x)$ is decreasing, then $f'(x)$ is **negative**
If $f(x)$ is constant, then $f'(x)$ is **zero**

Key Technique:
Use numberlines to

1. Mark Critical values
2. Evaluate Derivative between all Critical values.

THEOREM 3.5 TEST FOR INCREASING AND DECREASING FUNCTIONS

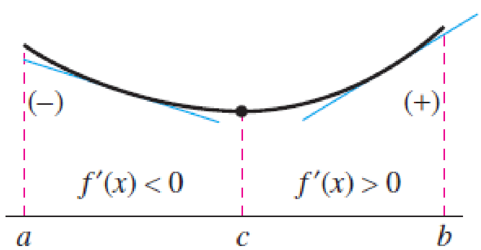
Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

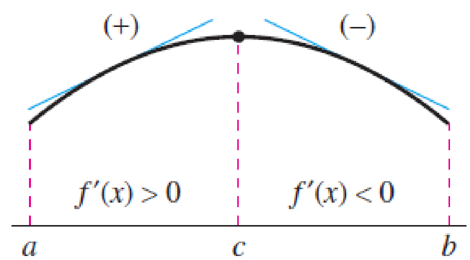
THEOREM 3.6 THE FIRST DERIVATIVE TEST

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

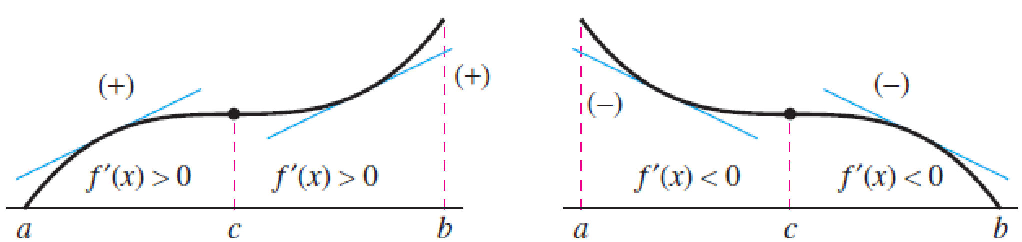
1. If $f'(x)$ changes from negative to positive at c, then f has a *relative minimum* at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a *relative maximum* at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.



Relative minimum



Relative maximum



Neither relative minimum nor relative maximum

3C: Concavity, the Second Derivative Test, and Curve Sketching (3.4, 3.6)

Definition: Concavity

If f is differentiable on an open interval I, then the graph of f is
Concave upward on I if f' is increasing on the interval, and
Concave downward on I if f' is decreasing on the interval.

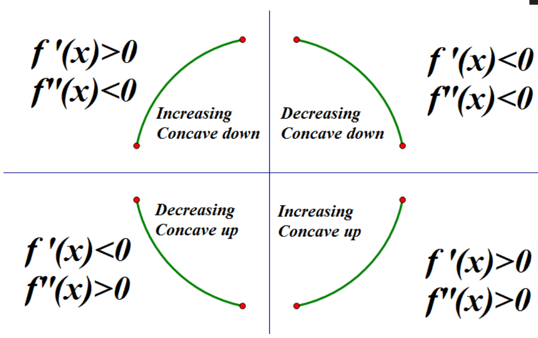
Want to find concavity?

Use the second derivative!
Make another number line!

THEOREM 3.7 TEST FOR CONCAVITY

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.



DEFINITION OF POINT OF INFLECTION

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

THEOREM 3.9 SECOND DERIVATIVE TEST

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

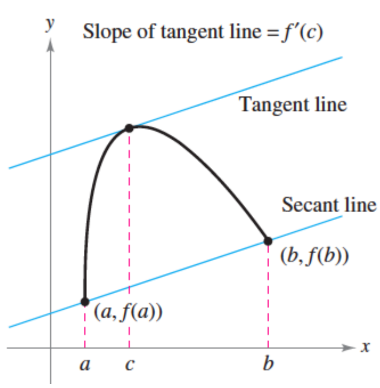
1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Steps for Curve Sketching:

1. Pre-Calc stuff:
   Find the intercepts, asymptotes, end behavior, and symmetry of the graph.
2. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist.
3. Use this to locate the extrema and inflection points.
4. Sketch the curve to "fill in the gaps".

3D: Rolle's Theorem and the Mean Value Theorem (3.3)



THEOREM 3.4 THE MEAN VALUE THEOREM

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The key thing to verify here is the hypotheses:

1. Is $f(x)$ continuous on $[a, b]$?
2. Is $f(x)$ differentiable on (a, b)?

If so, then we can conclude that there is a point on the interval where the
$\boldsymbol{tangent\ line\ slope = secant\ line\ slope}$

3E: Limits at Infinity and L'Hopital's rule. (3.5)

THEOREM 8.4 L'HÔPITAL'S RULE

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

Warning! Do not confuse this with the quotient rule. We are NOT saying

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}$$

Important note: These forms are ***not indeterminate!***

$$\lim f(x) \to \infty \cdot \infty = \infty$$
$$\lim f(x) \to \infty + \infty = \infty$$

$$\lim f(x) \to -\infty - \infty = -\infty$$

$$\lim f(x) \to 0^{\infty} = 0$$
$$\lim f(x) \to 0^{-\infty} = \infty$$

3F: Optimization Problems (3.7)

Key Steps:

1. **Identify the formulas that apply to the situation.**
2. **Define the variables and create one equation with one variable.**
3. **Find the critical points.**
4. **Write solution from critical points.**

3G: Motion along a line

Physics relationships to know:

- **Speed = |Velocity|**
- **Displacement on interval [a,b]**
  $\boldsymbol{D = s(b) - s(a}$
- **Speed is increasing at time t if**
  $$\boldsymbol{sign(\,v(t)\,) = sign(\,a(t)\,) \neq 0}$$

Important Physical Derivative Relationships:	
Position	$s(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = v'(t) = s''(t)$

3H: Related Rates (2.6)

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.