Name: Date:

Exploring Integrals: the Road to Riemann

We have now answered the first big question of Calculus,

"How do you find the slope of the tangent line (i.e. the instantaneous change) of a function?"

Now it's time to tackle the second big question of Calculus,

"How do you find the area under a curve?"

In this activity, we will explore that question to find some good ways to approximate the area used by a German Mathematician Bernhard Riemann to develop his Riemann Integral.

Bernhard Riemann in 1863

Approximating Area

The Question: Find the best approximation for the area under the graph of $f(x) = 17 - x^2$ from 0 to 4.

Note that when we say the "area under the graph" we really mean the area between the graph and the x-axis. *For each area approximation, <u>try to write a single summation expression that will find your approximation.</u>*

Lower Sum

1. We need to approximate the area under the graph. Let's begin by breaking the area into *inscribed rectangles*. Draw these four inscribed rectangles with a width of 1 on the graph.
2. Find the heights of the rectangles and use these to find total area of these rectangles.

   Is this an *over-estimate* or an *under-estimate* of the exact area?

   The exact area is $\frac{140}{3}$, how does your estimate compare (use a ratio)?
   We call this the *lower sum.* (Or the *right-hand sum* for this function.)

3. How could you make this a better approximation for the area if you still use inscribed rectangles?

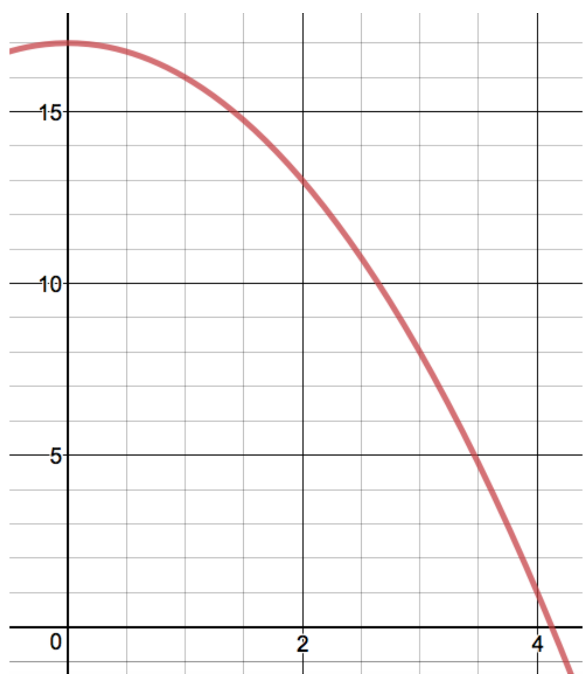
Upper Sum

4. Now let's try another method for approximation. This time, draw four *circumscribed* rectangles that have a width of 1 and a height equal to the greatest value of $f(x)$ the rectangle intersects.
5. Find the heights of these rectangles and use them to approximate the area of the curve. We call this the *upper sum*. (for this function it would also be the *left-hand* sum.)

6. Is this an *over approximation* or an *under approximation* of the actual area under the curve?

How does this compare to the actual area of $\frac{140}{3}$ (use a ratio)?

The "Average" Sum



7. Okay, we now have a lower sum and an upper sum. What if we change the height of the rectangles to equal the midpoint of the interval? Draw four rectangles on the interval [0,4] whose heights correspond to the $f(x)$ at the midpoint of each sub-interval.
8. Use these rectangles to find the area. This is called the *midpoint sum*.

9. Okay, we've done a lot of rectangles, can you think of any other geometric shapes that we could use to help find the area of the curve?

Trapezoidal Area

10. If you answered Trapezoids… that's a great idea! Now we will approximate the area under the curve by making trapezoids of width 1. Each endpoint of the "top" of the trapezoid should be a point on the graph of $f(x)$. Draw in the trapezoids to the right.

11. Now find an expression for the area approximation using the area of the trapezoids. Do you remember the area formula for a trapezoid?

12. How does your area approximation compare to the exact area of $\frac{140}{3}$?

13. Now, let's compare them all. List your four sums here:
    a. Lower Sum (LS): ____________________

    b. Upper Sum (US): ____________________

    c. Midpoint Sum (MS): ____________________

    d. Trapezoidal Sum (TS): ____________________

    e. Now fill in the inequality with LS, US, MS and TS

       _____________<_____________<_____________<_____________

So, we see that we can get the approximation many ways. We will soon find a way to get an exact equation by considering what happens as the rectangles get small, but we have more and more of them into the interval. This is called the Riemann Interval, and with Calculus we can find the exact area of $\frac{140}{3}$.