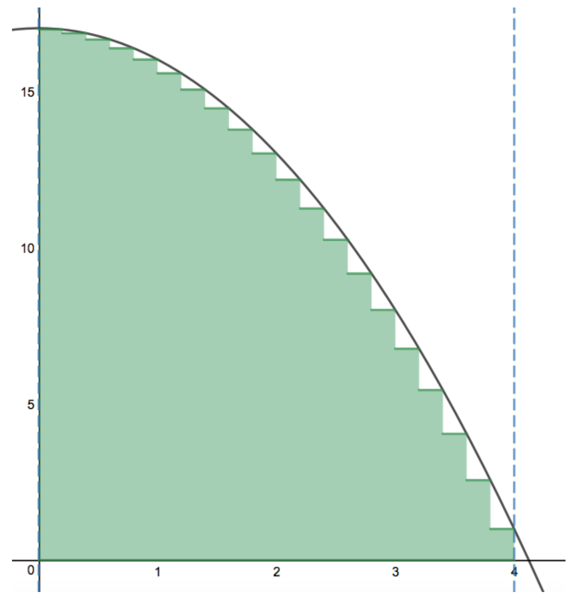
Calculus

Name: *Date:*

4B: Riemann Sums and Definite Integrals

In the previous lessons, we used rectangles to get an over-approximation or under-approximation of the area between a curve and the x-axis. If we want a better estimate, we can use more rectangles with smaller widths.



Let's revisit our function $f(x) = 17 - x^2$. What if we use 20 rectangles to find the area under the curve on the interval [0,4]. We would get the figure to the right.

Now let's consider the area if we used 100 rectangles, or how about 10,000? The limit of this area as $n \to \infty$ is called the **Definite Integral** of the function from $x = 0\ to\ 4$.

Let's us summations to find the limit of the area as a function of n.

Try it Out:

a) Use summation notation to find a function $s(n)$ that is the lower sum approximation for n subintervals of the interval [0,4] for the function $f(x) = 17 - x^2$

b) What is the area approximation for $n = 100$?

c) What is the area approximation for $n = 10{,}000$?

d) What is the limit of the area approximation as $n \to \infty$?

These sums are called Riemann sums after German mathematician Bernhard Riemann.

DEFINITION OF RIEMANN SUM

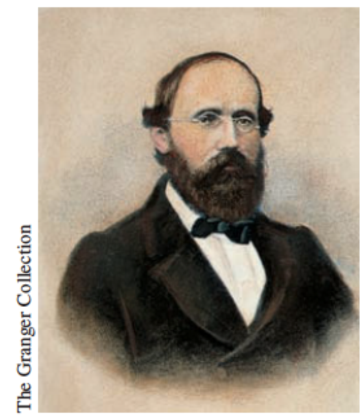
Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.



GEORG FRIEDRICH BERNHARD RIEMANN (1826–1866)

German mathematician Riemann did his most famous work in the areas of non-Euclidean geometry, differential equations, and number theory. It was Riemann's results in physics and mathematics that formed the structure on which Einstein's General Theory of Relativity is based.

Now, these subintervals do not need to be equal in width, but we will mostly focus on situations that have equal widths called **regular partitions.** When we have a regular partition, the width is given by

Regular Partition width of interval $[a, b]$ with n partitions:

$$\Delta x = \frac{b-a}{n}$$

Definite Integrals

We can now define a **definite integral** as the limit of the area as the width of the subintervals goes to zero.

Note: When $a \neq 0$ you will typically want to define

$$c_i = a + i\Delta x$$

DEFINITION OF DEFINITE INTEGRAL

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

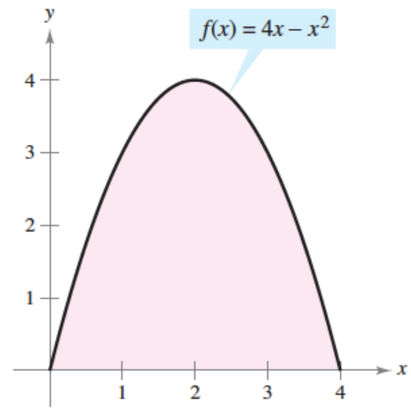
<u>Try it...</u> Use the previous example to find this definite integral:

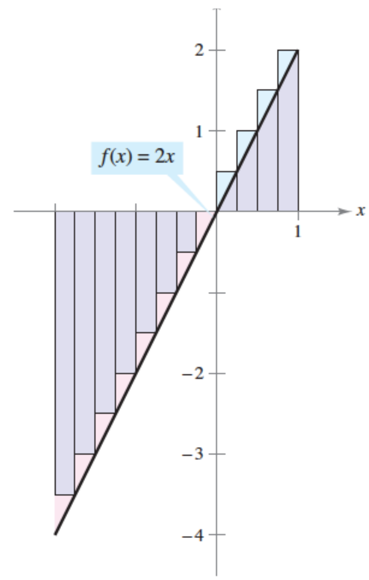
$$\int_0^4 17 - x^2\,dx =$$

<u>Examples</u> Use the limit definition of a definite Integral to find the following Integrals.

a) Find the area bounded by the function $f(x) = 4x - x^2$ and the $x - axis$ That is, find

$$\int_0^4 (4x - x^2)dx$$





b) Evaluate the following integral of $g(x) = 2x$ on the interval $[-2,1]$, and decide if it represents the area between the graph and th x-axis. Note, since this does not start at 0, you will need to make your function $f(c_i)$ where $c_i = -2 + \frac{3}{n}i$.

$$\int_{-2}^{1} 2x \ dx$$

Properties of Definite Integrals

If $f(x)$ is *<u>non-negative</u>* on interval $[a, b]$, then the integral represents the area.

> **THEOREM 4.4 CONTINUITY IMPLIES INTEGRABILITY**
>
> If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

<u>Example:</u> Find the integral using our work above.

$$\int_4^0 (17 - x^2) \ dx$$

> **DEFINITIONS OF TWO SPECIAL DEFINITE INTEGRALS**
>
> **1.** If f is defined at $x = a$, then we define $\int_a^a f(x)\,dx = 0$.
>
> **2.** If f is integrable on $[a, b]$, then we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

THEOREM 4.7 PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

2. $$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

Example: Find the integrals using area and properties

$$\int_0^5 |x-2|\,dx$$

$$\int_0^5 (3x+2)dx$$