4D2: 2nd Fundamental Theorem of Calculus

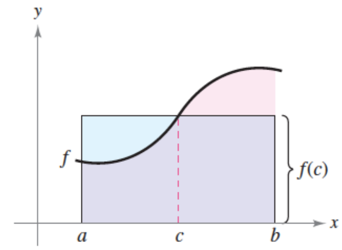
The 1st Fundamental Theorem of Calculus defines a definite integral of $f(x)$ by its antiderivatives $F(x)$.

FTC1: $\int_a^b f(x) = F(b) - F(a)$. Now we will flip it around and define the function by its derivative. But first we will need a couple simple theorems regarding the area under a curve.

A Couple Simple Area Theorems

Mean Value Theorem for Integrals

> **THEOREM 4.10 MEAN VALUE THEOREM FOR INTEGRALS**
>
> If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that
>
> $$\int_a^b f(x)\,dx = f(c)(b-a).$$



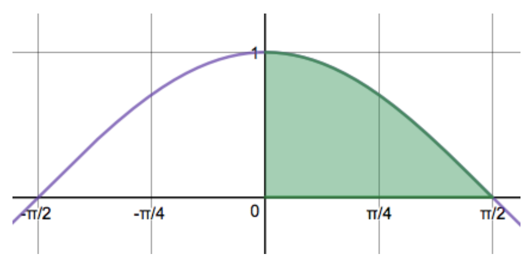
Mean value rectangle:

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

This theorem is useful because it tells us that there exists some value c on the interval $[a, b]$ that will give us a simple rectangle that has the exact area as the integral.

__**Example**__ Find the value of c on $[0, \frac{\pi}{2}]$ that satisfies the MVT for Integrals for $f(x) = \cos x$. That is,

$$f(c)\left(\frac{\pi}{2} - 0\right) = \int_0^{\frac{\pi}{2}} \cos x\ dx$$

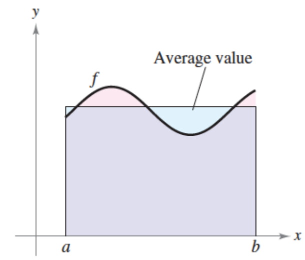


Average value of a function on an interval.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Since an integral is a summation of infinitely narrow rectangles, we can think of the integral $\int_a^b f(x)\,dx$ as the "sum of all the values of $f(x)$ from a to b". Furthermore, when we divide by $b-a$, we are dividing by the "number of infinitely thin rectangles". Therefore, we get a nice average of all the heights.

<u>**Example**</u> Find the average value of $g(x) = (2x+4)^2$ on the interval $[-4, -1]$.

2nd Fundamental Theorem of Calculus

<u>**2nd Fundamental Theorem of Calculus**</u> (FTC2)

If f is continuous on an open interval I , then, for every x on the interval, $G(x) = \int_a^x f(t)\ dt$

Then

$$G'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$

<u>**Example 1**</u> Compute $G'(x)$ by finding the integral first using FTC1.

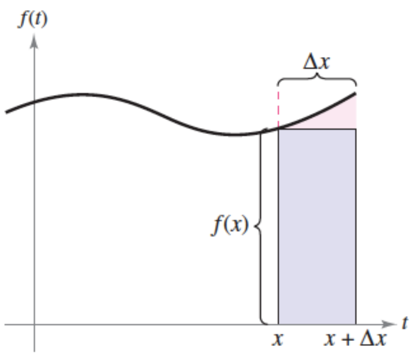
$$G(x) = \int_2^x 6t^2 - 2t\ dt$$

<u>**Example 2**</u> Compute G'(x) using FTC2.

$$G(x) = \int_0^x \frac{dt}{1+t}$$

Example 3 Compute $F'(x)$ using FTC2.

$$F(x) = \int_0^{x^2+3} \sin(\ln(t+7))\; dt$$



Proof of FTC2

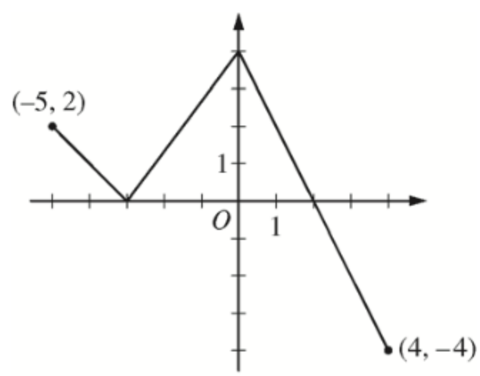
We define $G(x) = \int_a^x f(t)\; dt$. Now if we increase x to $x + \Delta x$, we get an additional area of

$$\Delta G = f(c)\Delta x \;\rightarrow \frac{\Delta G}{\Delta x} = f(c)$$

for some c in $[x, x+\Delta x]$ (which exists thanks to the MVT for Intervals). Taking the limit as $\Delta x \rightarrow 0$, we get

$$G'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta G}{\Delta x} = f(x)$$

AP Application Problem (2014-NC)



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\, dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.