Calculus

Unit 4 Toolkit: Integration

4A: Area Approximations (4.2)

Properties of Summations

Using the distributive and associative properties of addition, we can prove these important properties (k is a constant).

$$\sum_{i=1}^{n} k\, a_i = k \sum_{i=1}^{n} a_i \qquad \textit{and} \qquad \sum_{i=1}^{n} (a_i + b_i)$$

n n

THEOREM 4.2 SUMMATION FORMULAS

1. $\displaystyle\sum_{i=1}^{n} c = cn$

2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

4B: Summation Notation and Area (4.3)

DEFINITION OF RIEMANN SUM

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\, \Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

Limit Definition of a Definite Integral

For a continuous function $f(x)$, to calculate

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $c_i = a + i\Delta x$.

THEOREM 4.4 CONTINUITY IMPLIES INTEGRABILITY

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

DEFINITIONS OF TWO SPECIAL DEFINITE INTEGRALS

1. If f is defined at $x = a$, then we define $\displaystyle\int_a^a f(x)\,dx = 0$.

2. If f is integrable on $[a, b]$, then we define $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

THEOREM 4.7 PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

4C: Antiderivatives and Indefinite Integrals (4.1)

Definition of Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I

BASIC INTEGRATION RULES

Differentiation Formula	*Integration Formula*	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$	
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$	
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$	

Ordinary Differential Equation: $\frac{dy}{dx} = f(x, y)$

- A **General Solution** is found by integrating both sides. The resulting function will have a general constant, c.
- A **Particular Solution** is found by integrating, then applying *initial conditions*.

4D: The Fundamental Theorem of Calculus (4.4)

First Fundamental Theorem of Calculus (FTC1)

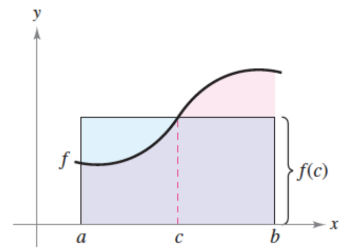
If f is continuous on $[a, b]$ and F is the antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

THEOREM 4.10 MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

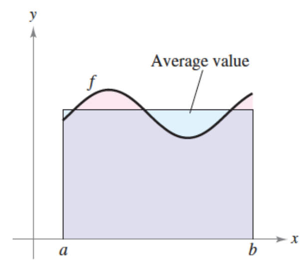
$$\int_a^b f(x)\,dx = f(c)(b-a).$$



DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

2nd **Fundamental Theorem of Calculus** (FTC2)

If f is continuous on an open interval I , then, for every x on the interval, $G(x) = \int_a^x f(t)\ dt$

Then

$$G'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$

4E: Integration by Substitution (4.5)

$$\int f\big(g(x)\big)g'(x)\ dx$$

Inside Function

Derivative of Inside Function

THEOREM 4.14 THE GENERAL POWER RULE FOR INTEGRATION

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

THEOREM 4.16 INTEGRATION OF EVEN AND ODD FUNCTIONS

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0.$