Calculus

Name: Date:

5A: Integrating Transcendental Functions

So, we have learned what it means to integrate a function and how it applies to the area between a curve and the x-axis. We have also learned how to integrate many different functions using antiderivatives. So far we have used two basic types of anti-derivative rules:

- Power Rule: $\int x^n dx = \frac{x^{n+1}}{n+1}$
- General Power Rule: $\int \left((n-1)u^{n-1}\right)u^n dx = \frac{u^{n+1}}{n+1}$
- Trig. Rules: $\int \sin x \;\; dx = \cos x$, etc

We also have a couple theorems and techniques to help us find definite integrals and work with more complex functions

- First Fundamental Theorem of Calculus $\int_a^b f(x)dx = F(b) - F(a)$
- U-Substitution

Natuaral Logarithms and Integrals

The Natural logarithm is a function that is actually defined by an integral. The definition is here to the right.

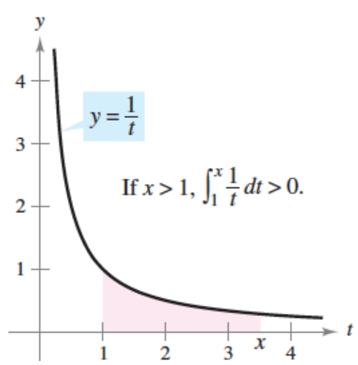
DEFINITION OF THE NATURAL LOGARITHMIC FUNCTION

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

Graphically, we see that the shaded area below is the value of the natural log.



Now , it would be nice if there was a number that you could substitute for this x in $\ln x$ that would equal exactly 1. There is! This is the natural number e.

DEFINITION OF e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\, dt = 1.$$

Earlier in the year, we learned that the derivative of the natural log function is $\frac{1}{x}$

THEOREM 5.3 DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$
2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

This definition brings us to Log rule for Integration to the right.

This resolves the problem that the power rule cannot solve.

THEOREM 5.5 LOG RULE FOR INTEGRATION

Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,dx = \ln|x| + C$ 2. $\int \frac{1}{u}\,du = \ln|u| + C$

That is, $\int \frac{1}{x}\,dx = \int x^{-1}\,dx \neq \frac{x^0}{0} + C$ as the power rule would imply.

Try These: Evaluate the Integrals

a) $\int \frac{5}{x}\,dx$

b) $\int \frac{2x+1}{x^2+x}\,dx$

c) $\int \frac{\cos x}{\sin x}\,dx$

As we can see in problems (b) and (c), if the numerator is the derivative of the denominator, the we can quickly integrate using this **general log integral rule:**

$$\int \frac{u'}{u}dx = \ln|u| + C$$

Integrals of Exponentials

When studying derivatives, we found that $\frac{d}{dx}e^x = e^x$. Since e^x is the only function that is its own derivative, it is also its own antiderivative.

THEOREM 5.12 INTEGRATION RULES FOR EXPONENTIAL FUNCTIONS

Let u be a differentiable function of x.

1. $\int e^x\,dx = e^x + C$ 2. $\int e^u\,du = e^u + C$

Try It: Find the integrals

a) $\int 2xe^{x^2}\,dx$

b) $\int 5^x\,dx$

(hint: $5 = e^{\ln 5}$)

Using the properties of exponents and logarithms, we can write any exponential expression in terms of e.

$$e^{\ln a} = a$$

This gives us the rule:

$$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$$

You can remember this new integral rule, or remember the change of base property above and use u-substitution.

Integrals of Trig. Functions

Now we can use the integral of logarithms to find the integral of trig. ratios. You are familiar with these:

$$\int \sin x\,dx = -\cos x + c, \qquad \int \cos x\,dx = \sin x + c.$$

Let's consider the other trig ratios.

Try It: Evaluate the integrals using trig identities to rewrite.

a) $\int \tan x\ dx$

b) $\int \cot x\ dx$

c) (hint: multiply by $\frac{(\sec x+\tan x)}{(\sec x+\tan x)}$) $\int \sec x\,dx$

d) $\int \csc x\,dx$

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

Integrals of Inverse Trig. Functions

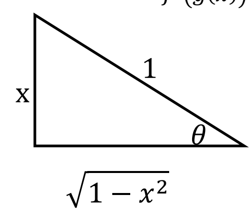
When finding the derivative of an inverse, we have this rule to remember:

Derivative of Inverse Theorem

If $g(x) = f^{-1}(x)$ is the inverse of $f(x)$ then we have $g'(x) = \frac{1}{f'(g(x))}$

So, $\frac{\boldsymbol{d}}{\boldsymbol{dx}}[\arcsin(x)] = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$

Using the triangle to the right.



The other inverse trig derivatives were found the same way

Now, we need to reverse this to find the antiderivatives.

When we do this, we notice that half of these derivatives are just the opposite of each other, so we choose to use the left column for antiderivatives

THEOREM 5.16 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

THEOREM 5.17 INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$ **2.** $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

<u>Try it:</u> Evaluate the integrals.

a) $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$

b) $\displaystyle\int \frac{4}{25+16x^2}\,dx$

BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin\dfrac{u}{a} + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\arctan\dfrac{u}{a} + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\operatorname{arcsec}\dfrac{|u|}{a} + C$

BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS

1. $\dfrac{d}{dx}[cu] = cu'$
2. $\dfrac{d}{dx}[u \pm v] = u' \pm v'$
3. $\dfrac{d}{dx}[uv] = uv' + vu'$
4. $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{vu' - uv'}{v^2}$
5. $\dfrac{d}{dx}[c] = 0$
6. $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\dfrac{d}{dx}[x] = 1$
8. $\dfrac{d}{dx}[|u|] = \dfrac{u}{|u|}(u'), \quad u \neq 0$
9. $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$
10. $\dfrac{d}{dx}[e^u] = e^u\,u'$
11. $\dfrac{d}{dx}[\log_a u] = \dfrac{u'}{(\ln a)u}$
12. $\dfrac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\dfrac{d}{dx}[\sin u] = (\cos u)u'$
14. $\dfrac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\dfrac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\dfrac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\dfrac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\dfrac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\dfrac{d}{dx}[\arcsin u] = \dfrac{u'}{\sqrt{1 - u^2}}$
20. $\dfrac{d}{dx}[\arccos u] = \dfrac{-u'}{\sqrt{1 - u^2}}$
21. $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1 + u^2}$
22. $\dfrac{d}{dx}[\operatorname{arccot} u] = \dfrac{-u'}{1 + u^2}$
23. $\dfrac{d}{dx}[\operatorname{arcsec} u] = \dfrac{u'}{|u|\sqrt{u^2 - 1}}$
24. $\dfrac{d}{dx}[\operatorname{arccsc} u] = \dfrac{-u'}{|u|\sqrt{u^2 - 1}}$