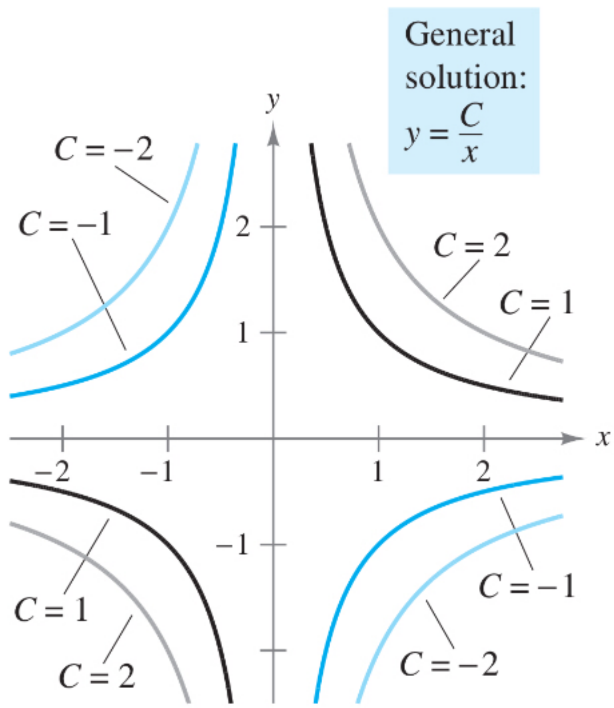
Name: Date:

5B: Differential Equations

A **differential equation** (also known as a "diff EQ", or an ODE for "ordinary differential Equation") is a function of x, y, and derivatives of y. A differential equation has a **General solution** that represents a family of functions. We can use the general solution of a differential equation to find a **particular solution** using given **initial conditions**. The **order** of a differential equation is the highest order derivative in the equation.

Some differential equations have a **Singular solution** that cannot be written as a special case of the general solution. However, we will not study these in this class.

Since the general solution of a differential equation represents a countless number of functions that differ by a constant, we often want to consider the graphs of all these functions. To do this, we make a graph of all the **solution curves** like the one to the right that shows the graph of the solutions for different constants.



Solution curves for $xy' + y = 0$

Try These:

a) Show that $y = \frac{C}{x}$ is a general solution of $xy' + y = 0$.

b) For the differential equation $xy' - 3y = 0$, verify that $y = Cx^3$ is a solution.

Now find the particular solution determined by the initial condition $y = 2$ wen $x = -3$.

Slope Fields

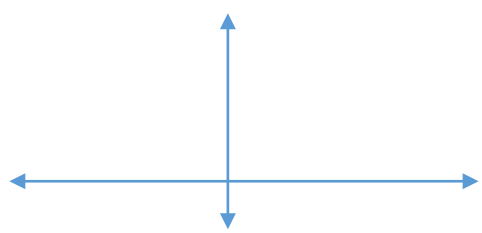
Solving a differential equation analytically can be difficult or even impossible. However, there is a graphical approach you can use to learn a lot about the solution of a differential equation. Consider a differential equation of the form

$y' = F(x, y)$ Differential equation

where $F(x, y)$ is some expression in x and y. At each point (x, y) in the xy-plane where F is defined, the differential equation determines the slope $y' = F(x, y)$ of the solution at that point. If you draw short line segments with slope $F(x, y)$ at selected points (x, y) in the domain of F, then these line segments form a **slope field,** or a *direction field*, for the differential equation $y' = F(x, y)$. Each line segment has the same slope as the solution curve through that point. A slope field shows the general shape of all the solutions and can be helpful in getting a visual perspective of the directions of the solutions of a differential equation.

Example

Sketch a slope field for the differential equation $y' = x - y$ for the points $(-1, 1)$, $(0, 1)$, and $(1, 1)$.



Exmaple

Match each slope field with its differential equation.

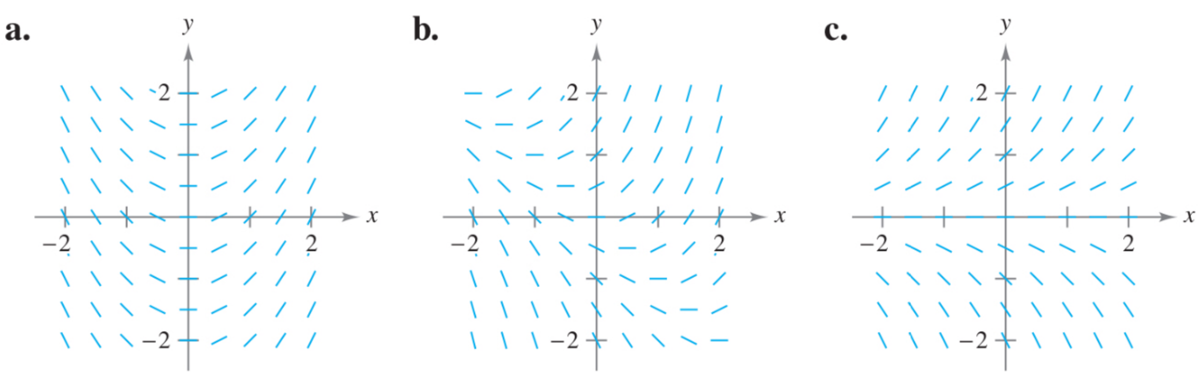


Figure 6.3

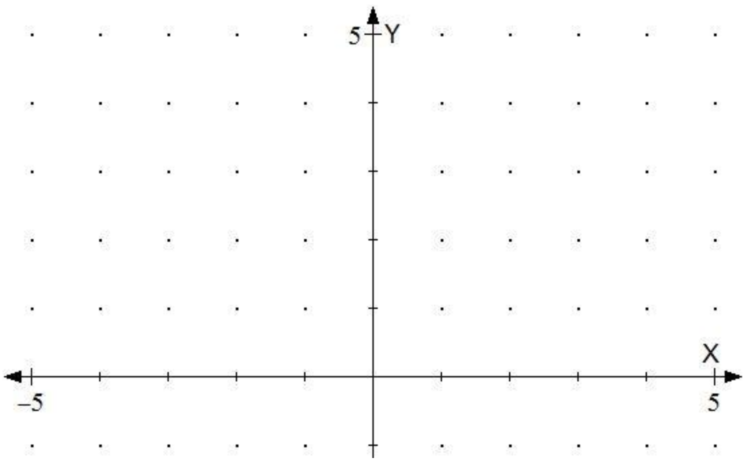
i. $y' = x + y$ **ii.** $y' = x$ **iii.** $y' = y$

Example

Sketch a slope field for the differential equation

$$y' = 2x + y.$$

Use the slope field to sketch the solution that passes through the point (1, 1).



Seperable Differential Equations

A **Separable Differential Equation** is one in which the $x's$ and the $dx's$ can be separated from the $y's$ and $dy's$.

A **first-order separable differential equation** is one that has the form $\frac{dy}{dx} = f(x)g(y)$. In this equation, the right side is just a product of a function of x and a different function of y.

When we are separating the variables, we must show this factorization, then we will treat dy and dx as infinitesimal variables that can be separated and moved about the equation.

Example

Solve the equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ by separating the variables, then find the particular solution that satisfies the given initial conditions below.

a) $y(0) = 2$

b) $y(0) = -2$

c) $y(0) = 0$

Example

Find the general and particular solution to the seperable differential equation $\frac{dy}{dx} = x^2y$ given the initial conditions given

a) $f(0) = 1$

b) $f(0) = -6$

AP Example

AP 1998-4

Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f, the slope is given by $\frac{dy}{dx} = \frac{3x^2+1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation of the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

(c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2+1}{2y}$ with the initial condition $f(1) = 4$.

(d) Use your solution from part (c) to find the exact value of $f(1.2)$.

(e) What would the solution equation be if the initial condition were $f(1) = -4$?

MEMORIZE. MEMORIZE. MEMORIZE.

If $\frac{dy}{dt} = ky$, then $y = Ce^{kt}$, where C is the initial amount present (y-intercept of the graph).

AP Example

Radium-226 ($^{226}_{88}Ra$) loses its mass at a rate that is directly proportional to its mass. If its half-life is 1590 years, and if we start with a sample of radium-226 with a mass of 100 mg,

(a) Find the formula for the mass, $M(t)$ that remains after t years.

(b) How many mg of the original sample remains after 100 years?

(c) How many years (exact answer) will it take for the sample to have only 3 mg remaining?

AP Example

Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated. **(Note: Use the axes provided in the pink test booklet.)**

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the *xy*-plane. Describe all points in the *xy*-plane for which the slopes are positive.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$.