Calculus

Name: Date:

1B: Evaluating Limits Analytically

Finding limits with tables and graphs gives us a good understanding of what a limit is, however it is sometimes very tedious or impossible to make a graph or table of a function. This is why we will usually prefer to solve limits ***analytically*** (or “algebraically) when possible.

Simple limits and basic properties

The simplest limits are ones that can be evaluated at the point $x = c$ and $f(c)$ is the limit. These are called “well-behaved”, continuous functions. For these functions we can use **direct substitution**

$$\lim_{x \to c} f(x) = f(c)$$

Some Elementary Limits

For real numbers b and c, and integer $n > 0$

$$\lim_{x \to c} b = b, \qquad \lim_{x \to c} x = c, \qquad \lim_{x \to c} x^n = c^n$$

Example Find the limit.

$$\lim_{x \to 2} x^2$$

Consider this: You haven’t seen any rules yet, but let’s take a crack at limits. Try to find this limit?

a) $\lim_{x \to 2} 2x^2$

b) $\lim_{x \to 2} 2x^2 + \log_2 x$

c) $\lim_{x \to 2} \frac{2x^2 + \log_2 x}{x - 1}$

d) $\lim_{x \to 2} \sqrt{\frac{2x^2 + \log_2 x}{x - 1}}$

With these limits, we can compute the limit of each part of the function separately with direct substitution, then put it all together to find the limit of the entire function. Here are some *simple* properties of limits that will help us to find limits of more complex functions.

Limits of Polynomial, radical, (simple) trigonometric, and Rational functions can be found by **direct substitution** as long as the denominator doesn't equal zero.

THEOREM 1.2 PROPERTIES OF LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. Scalar multiple: $\lim_{x\to c} [b f(x)] = bL$
2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

Limit Finding Strategies

The most important functions in Calculus that we will study will often result in the ration $\frac{0}{0}$ with direct substitution. So, we need to do some algebra to help evaluate the limit.

<u>*Consider This*</u>: Can this limit be evaluated by direct substitution?

$$\lim_{x\to 2} \frac{x^2 - 2x}{x - 2}$$

Find a function, $g(x)$ whose graph is identical to $f(x) = \frac{x^2-2x}{x-2}$ at all points but one?

Now, can you find $g(2)$ easily?

This is an example of a useful theorem we call the "**Limit replacement theorem**".

Limit Replacement Theorem.

If two functions $f(x) = g(x)$ for all values except for $x = c$ and $g(c)$ exists, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c)$$

So, the idea is that if we can find a simpler function that is almost identical (except at c), then we and use the simpler function to find the value $g(c)$ that fills the gap in $f(x)$.

<u>*Try it*</u> Find the graphical or numerical limit of the functions below, then we will find them analytically.

a)

$$\lim_{x\to 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} =$$

b)

$$\lim_{t \to 2} \frac{t^3 - 2t^2 - 5t + 10}{t^3 + 4t^2 - 4t - 16}$$

c)

$$\lim_{x \to 5} \frac{\frac{2}{x+3} - \frac{1}{4}}{x-5} =$$

d)

$$\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} =$$

e)

$$\lim_{x \to 3} \frac{\sqrt{x+6}-3}{x-3} =$$

Main Strategies for finding the limit of rational functions: (1) Try factoring, (2) Clear compound fractions, (3) Multiply by conjugate to rationalize.

So, this brings us to a 3 steps to use wen finding limits of functions.

Steps for evaluating the limit of $f(x)$ as $x \to c$

1. Can you evaluate the limit by direct substitution? i.e. Does $f(c)$ exist and is it the limit?
2. If $f(c)$ does not exist, is there a function $g(x) = f(x)$ for all values *except* $x = c$? Then
$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c)$$
3. Use a table or a graph to reinforce your conclusion.

Trigonometric Limits

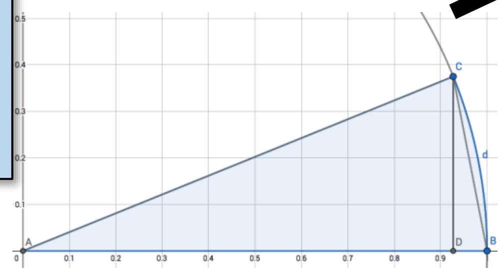
When working with simple trig. Functions, the limits for at all points in the domain of the function can be found by direct substitution. However, when they are combined with other functions, the result gets more complex. We have two special limits that we very important to our study of Calculus:

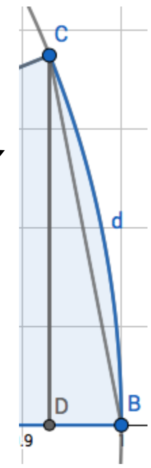
Special Trig. Limits

1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ 2. $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x} = 0$

bonus: $\lim\limits_{x \to 0} \dfrac{x}{\sin x} = 1$

https://www.geogebra.org/m/JWktjxQ3





These limits can show up in many different ways. Let's try some limits that will apply these special limits. They get a little tricky, so be persistent!

a) $\lim\limits_{x \to 0} \dfrac{\sin 5x}{5x} =$

b) $\lim\limits_{x \to 0} \dfrac{x + \sin x}{x}$

c) $\lim\limits_{x \to 0} \dfrac{x}{\tan x}$

d) $\lim\limits_{\theta \to 0} \dfrac{\sec \theta - 1}{\theta \sec \theta}$

e) $\lim\limits_{x \to 0} \dfrac{4 \sin 5x}{x \cos 2x}$

f) $\lim\limits_{x \to 0} \dfrac{\cot 11x}{\csc 25x}$