Unit 1 Toolkit: Limits

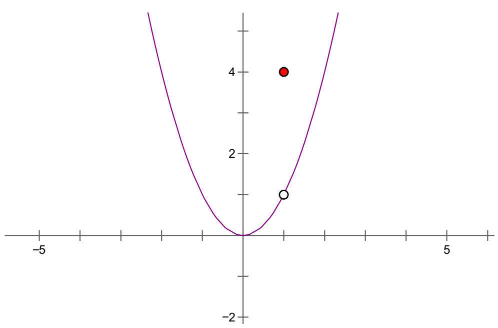
1A: Finding Limits Graphically and Numerically (1.2)

Definition: If $f(x)$ becomes arbitrarily close (i.e. really close) to a single number L as x approaches c from *either side*, then the **limit of $f(x)$**, as x approaches c, is L. This is written as

$$\lim_{x \to \infty} f(x) = L$$

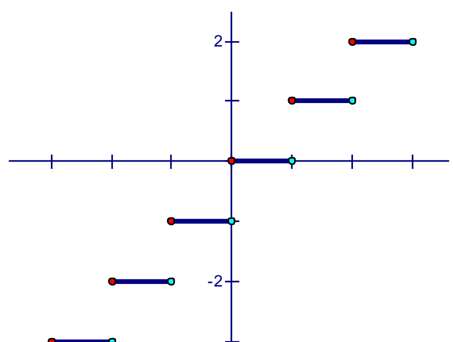
<u>Types of discontinuity</u>

Removable discontinuity



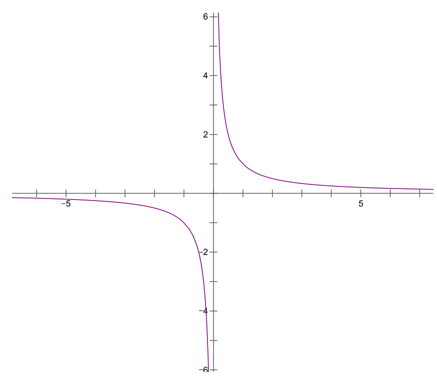
$\lim_{x \to 1} f(x) = 1$

Jump discontinuity



$\lim_{x \to 1} f(x)$ Does Not Exist because it's not the same from both sides

Infinite discontinuity



$\lim_{x \to 0} f(x)$ Does Not Exist because it's not the same from both sides

Epsilon-Delta ($\boldsymbol{\varepsilon - \delta}$) Definition of a Limit:

Let $f(x)$ be a function defined on an open interval including c (except possibly at the value c, and let L be a real number. Then we can say

$$\lim_{x \to c} f(x) = L$$

if for any small number $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \qquad then \ |f(x) - L| < \varepsilon.$$

1B: Evaluating Limits Analytically (1.3)

Some General Limits

$$\lim_{x\to b} \boldsymbol{c} = \boldsymbol{c}\ , \qquad \lim_{x\to b} x = b\ , \qquad \lim_{x\to c} x^n = c^n$$

THEOREM 1.2 PROPERTIES OF LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. Scalar multiple: $\lim_{x\to c} [b\,f(x)] = bL$
2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

THEOREM 1.3 LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

THEOREM 1.4 THE LIMIT OF A FUNCTION INVOLVING A RADICAL

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

THEOREM 1.5 THE LIMIT OF A COMPOSITE FUNCTION

If f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

Steps for evaluating the limit of $f(x)$ as $x \to c$

1. Can you evaluate the limit by direct substitution? i.e. Does $f(c)$ exist and is it the limit?
2. If $f(c)$ does not exist, is there a function $g(x) = f(x)$ for all values *except* $x = c$? Then
$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c)$$
3. Use a table or a graph to reinforce your conclusion.

Two Special Trigonometry Limits

The following are two special limits that we will use as we study calculus.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\ , \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

THEOREM 1.8 THE SQUEEZE THEOREM

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L.

1C: Continuity and One-Sided Limits (1.4)

Continuity Requirements. A function f is continuous at $x = c$ if these conditions are met:

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ *exists*
3. $\lim_{x \to c} f(x) = f(c)$

Continuous on an interval $(\boldsymbol{a}, \boldsymbol{b})$ implies that $f(x)$ is continuous on all points on the interval.

Continuous everywhere implies that $f(x)$ is continuous on the entire real line $(-\infty, \infty)$.

One sided limits:

- Limit from the right: $\lim_{x \to c^+} f(x) = L$
- Limit from the left: $\lim_{x \to c^-} f(x) = L$
  - Note $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = L$ ***and*** $\lim_{x \to c^+} f(x) = L$

Properties of Continuity. If b is real number and functions f and g are continuous at $x = c$, then the following are also continuous:

$$b \cdot f, \qquad f \pm g, \qquad fg, \qquad \frac{f}{g}$$

Functions that are continuous everywhere *on their domain*:

Polynomial, Rational, Radical, Trigonometric (sin, cos, tan, cot, sec, csc)

Intermediate Value Theorem (IVT)

If $f(x)$ is continuous on the closed interval $[a, b]$, and $f(a) \neq f(b)$, and k is any real number such that $f(a) < k < f(b)$, then there exists at least one number c in $[a, b]$ such that $f(c) = k$.

1D: Infinite Limits (1.4)

Vertical Asymptotes ⇔ Infinite Limits

Vertical asymptotes occur on a function $R(x) = f(x)/g(x)$ when $f(x) \neq 0$ and $g(x) = 0$.

Here is the technical definition of an infinite limit…

DEFINITION OF INFINITE LIMITS

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x \to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.40). Similarly, the statement

$$\lim_{x \to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$.

To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.

Basically, for any large number M, there is a value of $f(x)$ that is greater than M for some x that is very close to c (more specifically, for *all* values of x that are less than δ away from c).

***Here's the most important part*:**

Properties of Infinite Limits. Let $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$, then

1. $\lim_{x \to c}[f(x) \pm g(x)] = \infty$
2. $\lim_{x \to c} f(x)g(x) = \infty$ if $L > 0$,
   $\lim_{x \to c} f(x)g(x) = -\infty$ if $L < 0$
3. $\lim_{x \to c} \frac{g(x)}{f(x)} = \infty$