Name: Date:

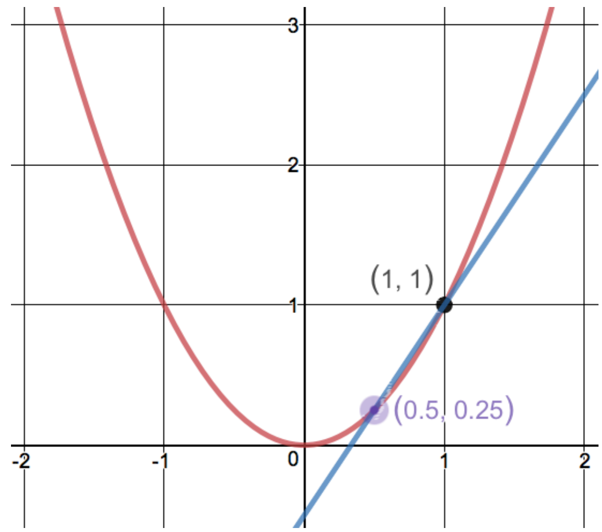
2A: Derivatives & The Tangent Line Problem

In the 1700s, mathematicians studied four major questions:

1. The tangent line problem (Section 1.1 and this section)
2. The velocity and acceleration problem (Sections 2.2 and 2.3)
3. The minimum and maximum problem (Section 3.1)
4. The area problem (Sections 1.1 and 4.2)

While many mathematicians made great progress on these problems, the credit for solving them usually goes to Gottfried Leibniz (1646-1716) and Isaac Newton (1642-1727). Working mostly independently Leibniz and Newton developed a whole new branch of mathematics called Calculus to answer these questions and prove some amazing discoveries like Newton's laws of motion and universal gravitation.

We will start by investigating the tangent line problem that will lead us to the definition of the derivative which is the foundation of Calculus.



Finding Slopes

Consider the graph of $f(x) = x^2$. Let's find the slope of the tangent line at (1,1) by finding the slope of secant lines near the point.

a) Compute these slopes:

$$\frac{f(1) - f(.5)}{1 - .5} = \qquad \frac{f(1) - f(.9)}{1 - .9} = \qquad \frac{f(1) - f(.99)}{1 - .99} =$$

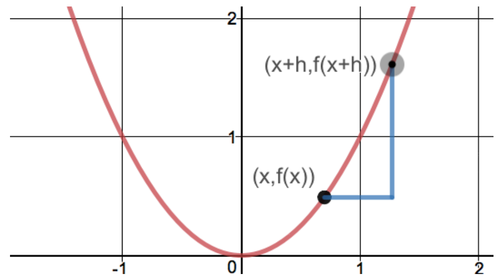
Based on these observations, what is the rate of change of the $f(x)$ at $x = 1$; that is, what is the slope of the tangent line at (1,1)?

b) Okay, let's generalize this by picking of two points that are h units apart. Find an expression for the slope of the secant line that passes through $\big(1, f(1)\big)$ and a point that is h units away $\big(1 + h, f(1 + h)\big)$. Simplify this function and call it $D(x)$.

c) What happens to this expression as h goes to 0?

Evaluate $\lim_{h \to 0} D(x)$

d) ***Generalize it!*** Suppose we want find the slope of the tangent line at any point on the function $f(x) = x^2$. Use the points $(x, f(x))$ and $(x + h, f(x + h))$ To find an expression for the slope of the secant line.



Formal (Limit) Definition of the Derivative

When we take the limit of the function we just found, we get the slope of the tangent line at the point $(x, f(x))$. This slope function is called the **derivative function** of $f(x)$. We call the function $f'(x)$, which is read "F prime of x".

The Formal Definition of the Derivative of a Function

If $f(x)$ is a differentiable function, it's derivative function is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note: We **will use other notations for the derivative such as**

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx} f(x)$$

Example Consider the function $f(x) = \sqrt{x - 2}$

a) Graph $f(x)$ and describe how the y-values change.

b) Find $f'(x)$ using the limit definition of the derivative.

c) Find the slope of the tangent lines at

$x = 3,$ $x = 10,$ $x = 27,$ $x = 2$

d) Describe the values for with $f(x)$ is continuous. For what values is $f(x)$ differentiable?

e) Find the equation of the tangent line at $x = 27$

Modified Limit Definition of the Derivative *at a point* (a nice shortcut!)

The **numerical value of** the derivative of $f(x)$ at point $(c, f(c))$ is defined as

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Example Use the modified limit definition to find $f'(1)$ for the function $f(x) = \frac{x}{x+1}$

Rates of Change

An important interpretation of the derivative is the "rate of change". There are many applications in which the rate of change is needed, so we have two very similar formulas that we need to be familiar with.

Average Rate of Change Using 2 points $(a, f(a)),\ and\ (b, f(b))$	**Instantaneous rate of change** Using 1 point $(c, f(c))$
The following are the equivalent: • Slope of the secant line • $\frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$ • Average rate of change on the interval $[a, b]$	**The following are the equivalent:** • Slope of the tangent line • $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ • Instantaneous rate of change at $x = c$

Consider this A skydiver is about to jump from an airplane at 5000 ft. when he drops his camera out of the airplane. Use the formula for the position of a falling object is $s(t) = -16t^2$ to find these:

a) Find the average rate of change (i.e. the average velocity) of the camera for the first 5 seconds of the freefall.

b) Find the instantaneous rate of change (i.e. the instantaneous velocity) of the camera at 5 seconds.

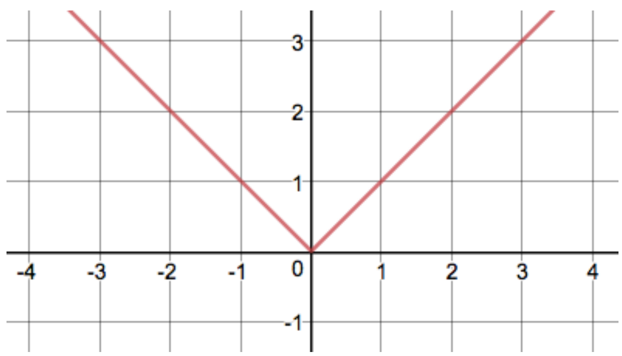
Is it Differentiable?

Finally, we need to decide if a function is even ***differentiable*** at all points on an interval. The key is that a function must have a ***smooth graph*** on the interval (a, b) to be differentiable on the interval.

A Function $f(x)$ is **differentiable** if and only if

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

That is, *the derivative is approaching the same value from the left and right.*



Consider the function $f(x) = |x|$.
Evaluate the following:

a) $\lim_{x \to 0} |x|$

b) What is the derivative at

$x = -2$ $\quad$ $x = 1$ $\quad$ $x = 0$

c) Is it differentiable at

Key things to remember/memorize

- If $f(x)$ is **differentiable** at $x = c$, then $f(x)$ **must be continuous** at $x = c$.
- The converse of the statement above [if $f(x)$ is continuous at $x = c$, then it is differentiable at c] **is not always true**. See the example above.
- However, the contrapositive is true: if $f(x)$ is not continuous at $x = c$, then $f(x)$ is not differentiable at $x = c$.