Name: Date:

2B: Basic Differentiation & Rates of Change.

We now have a limit definition of the derivative that can be used to find the derivative of any function. However, this takes some time and tedious work to compute.

$$\boldsymbol{Derivative\ of\ f(c)} \Leftrightarrow \boldsymbol{Slope\ of\ the\ Tangent}\ line\ at\ c \Leftrightarrow\ intantaneous\ \boldsymbol{Rate\ of\ Change}$$

Also, remember that we will show the derivative in several forms:

$$f'(x) = \frac{d}{dx}f(x) = y' = \frac{d}{dx}y = \frac{dy}{dx}$$

Now, we will look for some rules to find the derivative of specific types of functions more easily. The function families that we will look at are:

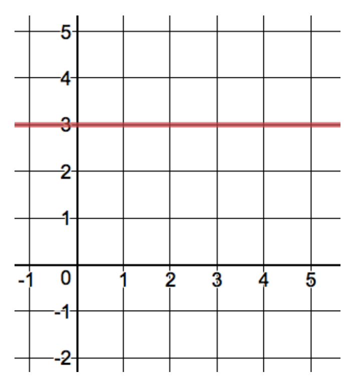
- Specific Rules: Find the derivatives of
$$f(x) = c, \qquad f(x) = x^n, \qquad f(x) = \sin x\,, \qquad f(x) = \cos x,$$
- General Rules: If $f(x)$ and $g(x)$ are differentiable, find
$$\frac{d}{dx}[c\ f(x)], \qquad \frac{d}{dx}[f(x) + g(x)]$$

A little clarification:

- $\frac{dy}{dx}$ is a noun. It means "The derivative of y with respect to x."
- $\frac{d}{dx}$ is a verb. It means "Take the derivative of the function with respect to x."

Constant Functions

Consider this, find the slope of the line $f(x) = 3$. What is it's derivative?



General Rule 1: The Constant Rule

The derivative of a constant function is 0. If c is a real number, then

$$\frac{d}{dx}[c] = 0$$

Power Functions

We have investigated the derivative of $y = x^2$ and found that $\frac{dy}{dx} = 2x$. We will use the limit definition to find the derivative of $f(x) = x^n$. Before we do this, we need to take a quick look into the ***binomial theorem*** (an important result that was ultimately generalized by Isaac Newton.)

Exploring Binomials: A useful side-track.

Our goal is to find a formula that will allow us to easily expand $(a+b)^n$ for any value of n.

$$(a+b)^1 = (a+b) = 1a + 1b$$

$$(a+b)^2 = (a+b)\cdot(1a+1b) = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = (a+b)\cdot(1a^2+2ab+1b^2) = \quad a^3 + \quad a^2b + \quad ab^2 + \quad b^3$$

$$(a+b)^4 = (a+b)\cdot(\quad a^3 + \quad a^2b + \quad ab^2 + \quad b^3)$$

$$= \quad a^4 + \quad a^3b + \quad a^2b^2 + \quad ab^3 + \quad b^4$$

We can look at this from a combinatorics perspective. Think of each term below as a collection of *a's* and *b's*.

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$$

$$= \quad a^5 + \quad a^4b + \quad a^3b^2 + \quad a^2b^3 + \quad ab^4 + \quad b^5$$

These coefficients are called the binomial coefficients, and they have a connection to **Pascal's Triangle** below.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Back to Derivatives

Use the definition of the derivative to find

$$\frac{d}{dx}[x^n] =$$

Specific Rule 1: The Power Rule

If n is a rational number, then $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing $x = 0$.

Try it. Find the derivative:

a) $\frac{d}{dx}[x^5]$

b) $\frac{d}{dx}\left[\frac{1}{x^2}\right]$

c) $\frac{d}{dx}[\sqrt{x}]$

The Constant Rule and Addition Rule

Now we have the power rule (which is very powerful indeed), what if we have a polynomial with coefficients and terms like

$$f(x) = 3x^5 - 2x^2 + x - 2$$

i. First, let's consider a constant multiple of a function like $3x^5$.

Find $\frac{d}{dx}[cf(x)]$ ***in terms of*** $f'(x)$

General Rule 2: The Constant Rule

For any constant c, if f and cf are differentiable, then

$$\frac{d}{dx}f(cx) = c\,f'(x)$$

ii. Now consider the sum or difference of two functions

Find $\frac{d}{dx}[f(x) + g(x)]$ ***in terms of*** $f'(x)$ ***and*** $g'(x)$.

Try it. Find the derivative.

General Rule 3: The Sum and Difference Rules

For any differentiable functions f and g,

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

a) $f(x) = 3x^5 - 2x^2 + x - 2$

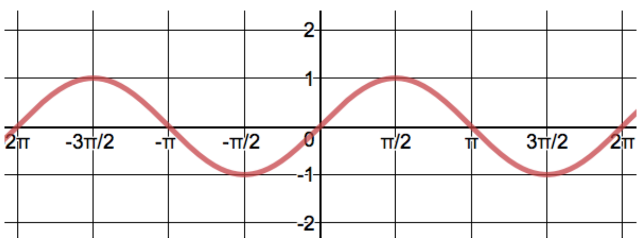
b) $\frac{d}{dx}[2x^4 + 12\sqrt[3]{x}]$

Trigonometric Functions

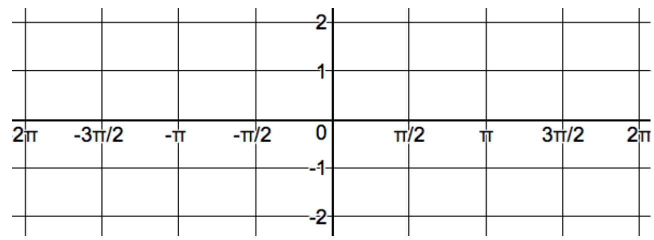
The last specific rule that we will look at for now is the derivative of $\sin x$ and $\cos x$.

Begin by trying to trace the graph of the derivative of $\sin x$ and $\cos x$.

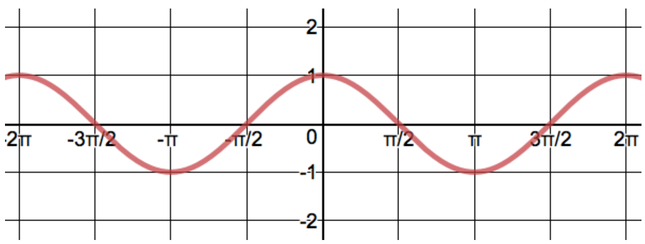
$f(x) = \sin x$



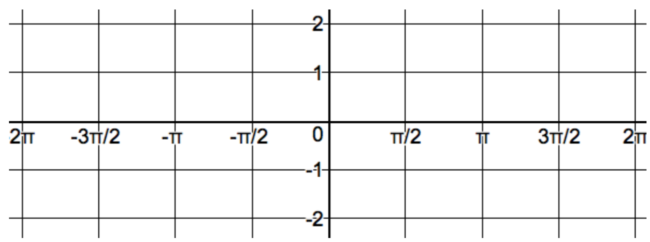
$f'(x) =$________________



$g(x) = \cos x$



$g'(x) =$________________



Ahhh-haaaa! Do these look familiar. We can prove these with the definition of the limit:

Use the definition of the limit to find:

$\frac{d}{dx}[\sin x] =$

$\frac{d}{dx}[\cos x] =$

Specific Rule 2: Derivatives of Sine and Cosine

$$\frac{d}{dx}\sin x = \cos x\,, \qquad \frac{d}{dx}\cos x = -\sin x$$

Try it. Find the Derivative.

a) $\frac{d}{dx}[5\sin x - 2\cos x]$

b) $\frac{d}{dt}[3t^5 + 2\cos t]$

The Derivative as the "Rate of Change"

Okay, here is some crazy important stuff!

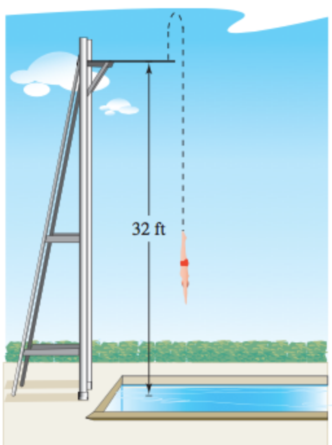
We know that a derivative at a point is...

1. **The slope of the tangent line at that point**
2. **The instantaneous rate of change at that point**

If we let a function, $s(t)$, represent an object position with respect to time, then **the objects velocity can be found by taking the derivative of s**! Wait there's more; if we take the **derivative of the object's velocity, then we have a function of acceleration**!

Position	$s(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = v'(t) = s''(t)$

<u>Let's try it</u>



At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32$$ Position function

where s is measured in feet and t is measured in seconds.

a) What is the diver's initial velocity?

b) What is the maximum height of diver?

c) When does the diver hit the water?

d) What is the diver's velocity at impact?

e) What is the diver's acceleration at t = 1/2 seconds? 1 sec?

Now that we know the power rule, we can circumvent the alternate form of the derivative to answer questions, like the one below, regarding differentiability.

<u>Example</u> Find the values of a and b so that $f(x)$ is differentiable for all x.

$$f(x) = \begin{cases} 3 - x, & x < 1 \\ ax^2 + bx, & x \geq 1 \end{cases}$$