Specific Rules

$$\frac{d}{dx}[c]=0$$

$$\frac{d}{dx}[x^n]=nx^{n-1}$$

General Rules:

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$$

Calculus Name: Date:

2C: Product and Quotient Rule

So, we have *specific rules* for the differentiation of power functions and some trig. functions. We also have *general rules* for the derivatives of constant multiples of functions and the sum or difference of some functions. Now we need to investigate the case where we have a product or quotient of two functions.

Product Rule

First, we need to figure out how to find the derivative of functions like $f(x) = x^2 \sin x$.

Let's make a Rule!

Ok, let's write our derivative as $u' = \lim_{\Delta x \to 0} \frac{\Delta u - u}{\Delta x}$, *and* $v' = \lim_{\Delta x \to 0} \frac{\Delta v - v}{\Delta x}$

Now let's look at the derivative $\frac{d}{dx}[uv]$ for functions $u(x)$ and $v(x)$.

$$(uv)' = \lim_{\Delta x \to 0} \frac{\Delta u \Delta v - uv}{\Delta x}$$

Now, I wish this fraction said $\frac{\Delta u - u}{\Delta x}$ or $\frac{\Delta v - v}{\Delta x}$ so we could write it as u' and v'. Let's get creative…

$$(uv)' = \lim_{\Delta x \to 0} \frac{\Delta u \Delta v - uv}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta u \Delta v - u\Delta v) + (u\Delta v - uv)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta u - u)\Delta v + u(\Delta v - v)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{(\Delta u - u)}{\Delta x}\overset{v}{\Delta v} + u\frac{(\Delta v - v)}{\Delta x} = \boxed{u'v + uv'}$$

<u>Quotient Rule</u>

If u and v are differentiable functions, then uv is a differentiable function and

$$\frac{d}{dx}[uv] = u'v + uv'$$

That is, the derivative of a product is equal to

"*The derivative of the 1st times the 2nd, plus the 1st times the derivative of the 2nd.*"

<u>Examples</u> Find the derivative of the function.

a) $f(x) = \overset{u}{x^3}\overset{v}{(5x^5 - x^2)}$

$$f'(x) = u'v + u\cdot v'$$

$$= 3x^2(5x^5 - x^2) + x^3(25x^4 - 2x)$$

$$= 15x^7 - 3x^4 + 25x^7 - 2x^4$$

$$= \boxed{40x^7 - 5x^4}$$

b) $g(x) = \underbrace{5x^3}_{u}\underbrace{\cos x}_{v}$

$\frac{d}{dx}(uv) = u'v + uv'$

$g'(x) = 15x^2\cos x + 5x^3(-\sin x)$

$\boxed{15x^2\cos x - 5x^3\sin x}$

c) $h(x) = 2x\cos x - 2\sin x$

$h'(x) = \cancel{2\cos x} + 2x(-\sin x) - \cancel{2\cos x}$

$\boxed{h'(x) = -2x\sin x}$

Quotient Rule

If multiplication gets its own rule, division should get one too. Let's find a rule for division quotients.

So, we start with two differentiable functions $u(x)$ and $v(x)$.

$$\left(\frac{u}{v}\right)' = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\left(\frac{\Delta u}{\Delta v}\right) - \left(\frac{u}{v}\right)\right) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{\Delta uv - u\Delta v}{\Delta v \cdot v}\right) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{\Delta uv - uv - u\Delta v + uv}{\Delta v \cdot v}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{(\Delta u - u)v - u(\Delta v - v)}{\Delta v \cdot v}\right) = \lim_{\Delta x \to 0}\left(\frac{\frac{(\Delta u - u)}{\Delta x}v - u\frac{(\Delta v - v)}{\Delta x}}{\Delta v \cdot v}\right) = \frac{u'v - uv'}{v^2}$$

$\frac{d}{dx}\frac{u}{v}$

Quotient Rule

If u and v are differential functions, so is $\left(\frac{u}{v}\right)'$, and

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

That is, the derivative of a quotient is equal to

"The derivative of the top times the botom, minus times the top the derivative of the bottom, all over the bottom squared."

Examples Use the quotient rule to find these derivatives.

a) $\frac{d}{dx}\left[\frac{4x^2-1}{x+3}\right] = \frac{u'v - uv'}{v^2}$

$= \frac{8x(x+3) - (4x^2-1)1}{(x+3)^2}$

$= \frac{8x^2 + 24x - 4x^2 + 1}{(x+3)^2} = \boxed{\frac{8x^2 + 20x + 1}{(x+3)^2}}$

Putting it all together… Combo time!

Now we have two very useful tools, let's put them together to find the derivative of some more complex functions.

Try it! Find the derivative of the following functions.

$$y = \frac{x \sin x}{x+1}$$

$$\frac{d}{dx}\left(\frac{x\sin x}{x+1}\right) = \frac{\overbrace{\frac{d}{dx}(x\sin x)}^{u'}\overbrace{(x+1)}^{v} - \overbrace{(x\sin x)}^{u}\overbrace{(1)}^{v'}}{(x+1)^2}$$

$$= \frac{(1\sin x + x\cos x)(x+1) - x\sin x}{(x+1)^2}$$

$$= \frac{\cancel{x\sin x} + \sin x + x^2\cos x + x\cos x - \cancel{x\sin x}}{(x+1)^2}$$

$$\boxed{\frac{\sin x + x^2\cos x + x\cos x}{(x+1)^2}}$$

$$g(x) = \left(\frac{x}{x+1}\right)(\sin x)$$

How about some more trig. functions…

$$\frac{d}{d\theta}\tan\theta = \frac{d}{d\theta}\left[\frac{\sin\theta}{\cos\theta}\right] = \frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \left(\frac{1}{\cos\theta}\right)^2$$

$$\boxed{\frac{d}{dx}\tan\theta = \sec^2\theta}$$

$$\frac{d}{d\theta}\cot\theta = \frac{d}{d\theta}\left[\frac{\cos\theta}{\sin\theta}\right] = \frac{-\sin\theta\sin\theta - \cos\theta\cos\theta}{\sin^2\theta}$$

$$= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}$$

$$= \frac{-1(\sin^2\theta + \cos^2\theta)}{\sin^2\theta} = \frac{-1}{\sin^2\theta} = \boxed{-\csc^2\theta = \frac{d}{dx}\cot\theta}$$

$$\frac{d}{d\theta}\csc\theta = \frac{d}{d\theta}\left[\frac{1}{\sin\theta}\right]$$

$$= \frac{\cancel{0\cdot\sin\theta} - 1\cos\theta}{\sin^2\theta}$$

$$= \frac{-\cos\theta}{\sin^2\theta} = -\cot\theta\csc\theta$$

$$\frac{d}{d\theta}\sec\theta = \frac{d}{d\theta}\left[\frac{1}{\cos\theta}\right] = \frac{\cancel{0\cos\theta} - 1(-\sin\theta)}{\cos^2\theta}$$

$$= \frac{\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$$

$$= \sec\theta\cdot\tan\theta$$

$$\boxed{\frac{d}{dx}\sec\theta = \sec\theta\tan\theta}$$

Here's how the AP test will use these rules:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	3	4	-2	2
0	2	-3	5	-1

The table above gives values for two differentiable functions and their derivatives at selected values of x. Use the table to evaluate the following.

(a) $h'(0)$ if $h(x)=\dfrac{f(x)}{g(x)}$

$h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

$h'(0) = \dfrac{f'(0)g(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{(-3)(5) - 2(-1)}{5^2} = \boxed{\dfrac{-12}{25}}$

(b) $h'(-1)$ if $h(x)=x\,f(x)\cdot g(x)$

$h'(x) = 1(f(x)\cdot g(x)) + x\cdot \frac{d}{dx}(f(x)g(x))$

$h'(x) = f(x)g(x) + x(f'(x)g(x) + f(x)g'(x))$

$h'(x) = 1f(x)g(x) + x f'(x)g(x) + x f(x)g'(x)$

$h'(-1) = 1f(-1)g(-1) + (-1)f'(-1)g(-1) + (-1)f(-1)g'(-1)$

$= 1\cdot 3(-2) + (-1)4(-2) + (-1)\cdot 3\cdot 2$

$= -6 - 8 - 6 = \boxed{-20}$

Higher Derivatives

Now that we have more ways to find derivatives, we can start to find **higher derivatives.** This just means we will take the derivative of the derivative.

Example Find the second derivative of $s(t) = -.81t^2 + 2$ which is the position function for a falling object on the moon.

$s'(t) = -1.62t$ $\qquad \frac{ds}{dt} = -1.62t$

$\boxed{s''(t) = -1.62}$ $\qquad \frac{d}{dt}\left(\frac{ds}{dt}\right) = \boxed{\frac{d^2s}{dt^2} = -1.62}$

When working with physics applications, we have a very important relationship that we will work more with later:

$$s(t) \quad \text{Position function}$$
$$v(t) = s'(t) \quad \text{Velocity function}$$
$$a(t) = v'(t) = s''(t) \quad \text{Acceleration function}$$

In General, this table gives us the notation for higher derivatives:

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
	$\vdots$				
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$