Calculus

Name: Date:

# 2C: Product and Quotient Rule

So, we have *specific rules* for the differentiation of power functions and some trig. functions. We also have *general rules* for the derivatives of constant multiples of functions and the sum or difference of some functions. Now we need to investigate the case where we have a product or quotient of two functions.

## Product Rule

First, we need to figure out how to find the derivative of functions like $f(x) = x^2 \sin x$.

**Let's make a Rule!**

Ok, let's write our derivative as

$$u' = \lim_{\Delta x \to 0} \frac{\Delta u - u}{\Delta x}, \qquad \textit{and} \;\; v' = \lim_{\Delta x \to 0} \frac{\Delta v - v}{\Delta x}$$

Now let's look at the derivative $\frac{d}{dx}[uv]$ for functions $u(x)$ and $v(x)$.

$$(uv)' = \lim_{\Delta x \to 0} \frac{\Delta u \Delta v - uv}{\Delta x}$$

Now, I wish this fraction said $\frac{\Delta u - u}{\Delta x}$ or $\frac{\Delta v - v}{\Delta x}$ so we could write it as $u'$ and $v'$. Let's get creative…

**General Rule 4: Product Rule**

If $u$ and $v$ are differentiable functions, then $uv$ is a differentiable function and

$$\frac{d}{dx}[uv] = u'v + uv'$$

That is, the derivative of a product is equal to

"*The derivative of the 1st times the 2nd, plus the 1st times the derivative of the 2nd.*"

$$(uv)' = \lim_{\Delta x \to 0} \frac{\Delta u \Delta v - uv}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u \Delta v \boldsymbol{- u\Delta v + u\Delta v} - uv}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta u - u)\Delta v + u(\Delta v - v)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{(\Delta u - u)}{\Delta x} \Delta v + u \frac{(\Delta v - v)}{\Delta x} = u'v + uv'$$

**Examples** Find the derivative of the function.

a) $f(x) = x^3(5x^5 - x^2)$

b) $g(x) = 5x^3 \cos x$

c) $h(x) = 2x \cos x - 2 \sin x$

## Quotient Rule

If multiplication gets its own rule, division should get one too. Let's find a rule for division quotients.

So, we start with two differentiable functions $u(x)$ and $v(x)$.

$$\left(\frac{u}{v}\right)' = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\left(\frac{\Delta u}{\Delta v}\right) - \left(\frac{u}{v}\right)\right) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{\Delta u v - u\Delta v}{\Delta v \cdot v}\right) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{\Delta u v - uv - u\Delta v + uv}{\Delta v \cdot v}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{(\Delta u - u)v - u(\Delta v - v)}{\Delta v \cdot v}\right) = \lim_{\Delta x \to 0}\left(\frac{\frac{(\Delta u - u)}{\Delta x} v - u \frac{(\Delta v - v)}{\Delta x}}{\Delta v \cdot v}\right) = \frac{u'v - uv'}{v^2}$$

**General Rule 5: Quotient Rule**

If $u$ and $v$ are differential functions, so is $\left(\frac{u}{v}\right)'$, and

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

That is, the derivative of a quotient is equal to

$$\text{"}\textit{The derivative of the top times the botom, minus times the top the derivative of the bottom, all over the bottom squared.}\text{"}$$

**Examples** Use the quotient rule to find these derivatives.

a) $\frac{d}{dx}\left[\frac{4x^2-1}{x+3}\right]$

b) $\frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$

## Putting it all together… Combo time!

Now we have two very useful tools, let's put them together to find the derivative of some more complex functions.

**Try it!** Find the derivative of the following functions.

$y = \frac{x \sin x}{x+1}$

$g(x) = \left(\frac{x}{x+1}\right)(\sin x)$

How about some more trig. functions…

$\frac{d}{d\theta}\tan\theta = \frac{d}{d\theta}\left[\frac{\sin\theta}{\cos\theta}\right]$

$\frac{d}{d\theta}\cot\theta = \frac{d}{d\theta}\left[\frac{\cos\theta}{\sin\theta}\right]$

$\frac{d}{d\theta}\csc\theta = \frac{d}{d\theta}\left[\frac{1}{\sin\theta}\right]$

$\frac{d}{d\theta}\sec\theta = \frac{d}{d\theta}\left[\frac{1}{\cos\theta}\right]$

**Here's how the AP test will use these rules:**

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| $-1$ | 3 | 4 | $-2$ | 2 |
| 0 | 2 | $-3$ | 5 | $-1$ |

The table above gives values for two differentiable functions and their derivatives at selected values of $x$. Use the table to evaluate the following.

(a) $h'(0)$ if $h(x) = \dfrac{f(x)}{g(x)}$

(b) $h'(-1)$ if $h(x) = x \cdot f(x) \cdot g(x)$

## Higher Derivatives

Now that we have more ways to find derivatives, we can start to find **higher derivatives.** This just means we will take the derivative of the derivative.

__**Example**__ Find the second derivative of $s(t) = -.81t^2 + 2$ which is the position function for a falling object on the moon.

When working with physics applications, we have a very important relationship that we will work more with later:

$$s(t)$$ Position function

$$v(t) = s'(t)$$ Velocity function

$$a(t) = v'(t) = s''(t)$$ Acceleration function

In General, this table gives us the notation for higher derivatives:

| | | | | | |
|---|---|---|---|---|---|
| ***First derivative:*** | $y'$, | $f'(x)$, | $\dfrac{dy}{dx}$, | $\dfrac{d}{dx}[f(x)]$, | $D_x[y]$ |
| ***Second derivative:*** | $y''$, | $f''(x)$, | $\dfrac{d^2y}{dx^2}$, | $\dfrac{d^2}{dx^2}[f(x)]$, | $D_x^2[y]$ |
| ***Third derivative:*** | $y'''$, | $f'''(x)$, | $\dfrac{d^3y}{dx^3}$, | $\dfrac{d^3}{dx^3}[f(x)]$, | $D_x^3[y]$ |
| ***Fourth derivative:*** | $y^{(4)}$, | $f^{(4)}(x)$, | $\dfrac{d^4y}{dx^4}$, | $\dfrac{d^4}{dx^4}[f(x)]$, | $D_x^4[y]$ |
| $\vdots$ | | | | | |
| ***nth derivative:*** | $y^{(n)}$, | $f^{(n)}(x)$, | $\dfrac{d^ny}{dx^n}$, | $\dfrac{d^n}{dx^n}[f(x)]$, | $D_x^n[y]$ |