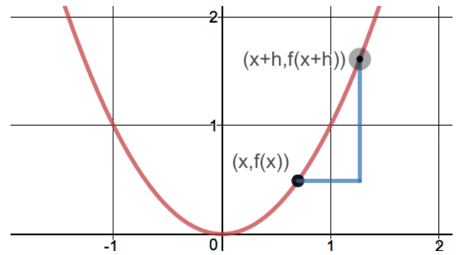
Calculus

Unit 2 Toolkit: Derivatives

2A: The Derivative and the Tangent Problem (2.1)



When we take the limit of the function we just found, we get the slope of the tangent line at the point $(x, f(x))$. This slope function is called the **derivative function** of $f(x)$. We call the function $f'(x)$, which is read "F prime of x".

The Limit Definition of the Derivative of a Function
If $f(x)$ is a differentiable function, it's derivative function is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Rates of Change

An important interpretation of the derivative is the "rate of change". There are many applications in which the rate of change is needed, so we have two very similar formulas that we need to be familiar with.

Average Rate of Change Using 2 points $(a, f(a)),\ and\ (b, f(b))$	Instantaneous rate of change Using 1 point $(c, f(c))$
The following are the equivalent: • Slope of the secant line • $\frac{\boldsymbol{\Delta x}}{\boldsymbol{\Delta y}} = \frac{\boldsymbol{f(b)-f(a)}}{\boldsymbol{b-a}}$ • Average rate of change on the interval $[a, b]$	**The following are the equivalent:** • Slope of the tangent line • $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ • Instantaneous rate of change at $x = c$

When finding the derivative at a certain value of $x = c$, use this definition

Is it Differentiable?

Finally, we need to decide if a function is even ***differentiable*** at all points on an interval. The key is that a function must have a ***smooth graph*** on the interval (a, b) to be differentiable on the interval.

$$\boldsymbol{sharp\ point = not\ differentiable}$$

Key things to remember/memorize

- If $f(x)$ is **differentiable** at $\mathrm{x} = c$, then $\mathrm{f}(x)$ **must be continuous** at $x = c$.
- The converse of the statement above [if $f(x)$ is continuous at $\mathrm{x} = c$, then it is differentiable at c] **is not always true**. See the example above.
- However, the contrapositive is true: if $f(x)$ is not continuous at $x = c$, then $f(x)$ is not differentiable at $x = c$.

2B: Basic Differentiation Rules and Rates of Change (2.2)

$\boldsymbol{Derivative\ of\ f(c)} \Leftrightarrow \boldsymbol{Slope\ of\ the\ Tangent}\ line\ at\ c \Leftrightarrow\ intantaneous\ \boldsymbol{Rate\ of\ Change}$

Also, remember that we will show the derivative in several forms:

$$f'(x) = \frac{d}{dx}f(x) = y' = \frac{d}{dx}y = \frac{dy}{dx}$$

General Rule 1: The Constant Rule

The derivative of a constant function is 0. If c is a real number, then

$$\frac{d}{dx}[c] = 0$$

Specific Rule 1: The Power Rule

If n is a rational number, then $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval

General Rule 2: The Constant Rule

For any constant c, if f and cf are differentiable, then

$$\frac{d}{dx}f(cx) = c\,f'(x)$$

Specific Rule 2: Derivatives of Sine and Cosine

$$\frac{d}{dx}\sin x = \cos x\,, \qquad \frac{d}{dx}\cos x = -\sin x$$

General Rule 3: The Sum and Difference Rules

For any differentiable functions f and g,

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

Important Physical Derivative Relationships:

Position	$s(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = v'(t) = s''(t)$

2C: Product and Quotient Rules and Higher-Order Derivatives (2.3)

General Rule 4: Product Rule

If u and v are differentiable functions, then uv is a differentiable function and

$$\frac{d}{dx}[uv] = u'v + uv'$$

That is, the derivative of a product is equal to

"*The derivative of the* 1*st times the* 2*nd*, *plus the* 1*st times the derivative of*

General Rule 5: Quotient Rule

If u and v are differential functions, so is $\left(\frac{u}{v}\right)'$, and

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

That is, the derivative of a quotient is equal to

"*The derivative of the top times the botom, minus times the top the derivative of the bottom, all over the bottom squared.*"

Specific Rule 3: Derivatives of Other Trig Functions

$$\frac{d}{dx}\tan x = \sec^2 x\,, \qquad \frac{d}{dx}\cot x = -\sec^2 x\,, \qquad \frac{\mathrm{d}}{\mathrm{dx}}\sec x = \sec x \cdot \tan x\,, \qquad \frac{\mathrm{d}}{\mathrm{dx}}\csc x = -\csc x \cdot \cot x$$

Higher Derivatives

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
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nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

2D: The Chain Rule (2.4)

General Rule 6: The Chain Rule

If $f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is differentiable, and '

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The Big Idea:
To find the derivative of a composite function $f(g(x))$, take:

$$f' = deriv.of\ the\ outside \cdot deriv.of\ inside$$

2E: Implicit Differentiation (2.5)

Just remember, when using implicit differentiation that y is just a function of x so we need to use the chain rule. However, since we don't really know what y is equal to, we end up with a y' or $\frac{dy}{dx}$ after differentiating.

<u>Steps for Implicit Differentiation:</u>

1. Take $\frac{d}{dx}$ of both sides of the equation.
2. Evaluate using the derivative rules
3. Solve for $\frac{dy}{dx}$.

2F: Derivatives of Logarithms and Exponentials (5.1, 5.4)

Secific Rule 4: Derivative of Natural Logarithms

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \qquad \frac{d}{du}[\ln u] = \frac{u'}{u}$$

Specific Rule 5: Derivatives of General Logarithms

For any integer base $a > 0$,

$$\boldsymbol{\frac{d}{dx}}[\log_a \boldsymbol{x}] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{x}{x \ln\ a}$$

Steps for Logarithmic Differentiation:

1. Take the natural log of both sides
2. Find the derivative of both sides (Implicitly)
3. Solve for $\frac{dy}{dx}$ by multiplying by y and substitute original function.

Specific Rule 6: Derivative of e^x

$$\frac{d}{dx}[e^x] = e^x, \qquad \frac{d}{dx}[e^u] = u'e\hat{\ }u$$

2G: Derivatives of trigonometric Functions (5.6)

Specific Rule 7: Derivative of Inverse Trigonometric Function.

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Here is a nice summary of all the Special Differentiation rules

BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS

1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$
9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u\, u'$
11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$
12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$
14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
22. $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$
23. $\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{|u|\sqrt{u^2-1}}$
24. $\frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{|u|\sqrt{u^2-1}}$