Geometry

Unit 5: Similarity and Trigonometry

Unit 5
Similarity and Right Triangle Trigonometry

Unit 5, Lesson 1

Photocopy Faux Pas

Learning Focus

Describe the essential features of a dilation transformation.

How do I use a dilation to enlarge or shrink a figure?

How are distance and area in a dilated image related to corresponding distances and area in the original image?

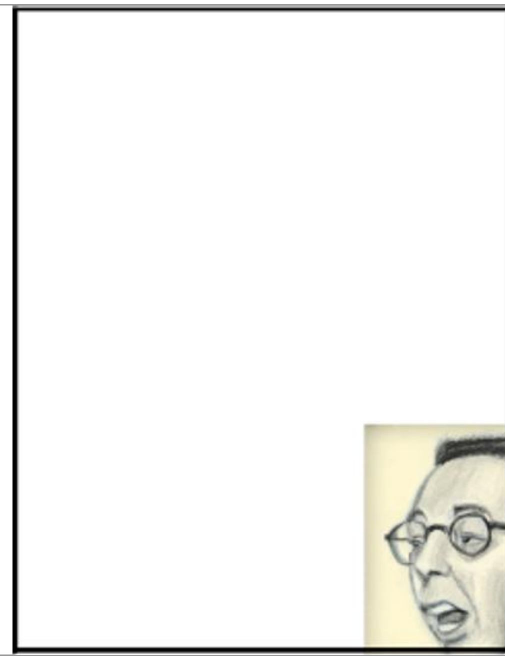
Open Up the Math: Launch, Explore, Discuss

Burnell has a new job at a copy center helping people use the photocopy machines. Burnell thinks he knows everything about making photocopies, and so he didn't complete his assignment to read the training manual.

Mr. and Mrs. Donahue are making a scrapbook for Mr. Donahue's grandfather's 75th birthday party, and they want to enlarge a sketch of their grandfather which was drawn when he was in World War II. They have purchased some very expensive scrapbook paper, and they would like this image to be centered on the page. Because they are unfamiliar with the process of enlarging an image, they have come to Burnell for help.

"We would like to make a copy of this image that is twice as big, and centered in the middle of this very expensive scrapbook paper," Mrs. Donahue says. "Can you help us with that?"

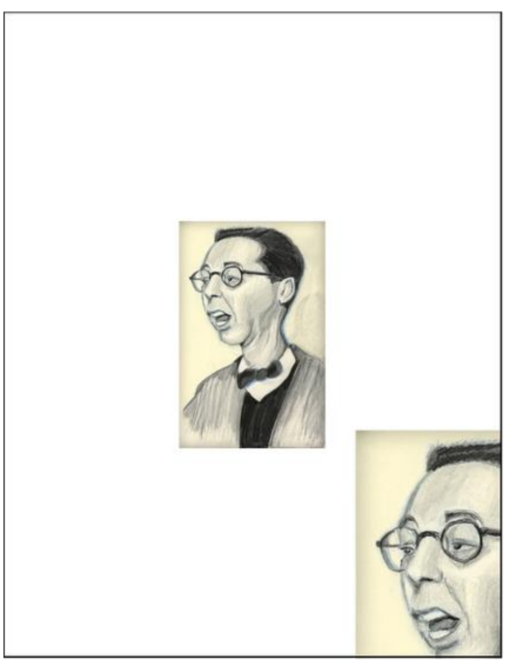
"Certainly," says Burnell. "Glad to be of service."

Burnell taped the original image in the middle of a white piece of paper, placed it on the glass of the photocopy machine, inserted the expensive scrapbook paper into the paper tray, and set the enlargement feature at 200%. In a moment, this image was produced. “You’ve ruined our expensive paper,” cried Mrs. Donahue. “Much of the image is off the paper instead of being centered.” “And this image is more than twice as big,” Mr. Donahue complained. “One-fourth of Grandpa’s picture is taking up as much space as the original.”	

In the diagram provided in problem 2, both the original image—which Burnell taped in the middle of a sheet of paper—and the copy of the image have been included on the same figure.

1. Explain how the photocopy machine produced the partial copy of the original image.

2. Using a “rubber band stretcher” finish the rest of the enlarged sketch.



Pause and Reflect

3. Where should Burnell have placed the original image if he wanted the final image to be centered on the paper?

4. Mr. Donahue complained that the copy was four times bigger than the original. What do you think? Did Burnell double the image or quadruple it? What evidence would you use to support your claim?

5. Transforming a figure by shrinking or enlarging it in this way is a called dilation. Based on your thinking about how the photocopy was produced, list all of the things you need to pay attention to when dilating an image.

Ready for More?

Consider the following question: Is it possible for Burnell to locate a position to place the original photo on the paper so the enlarged image will be located anywhere on the paper that Mr. and Mrs. Donahue may select?

Takeaways

Transforming a figure by shrinking or enlarging it is called a dilation. Based on your thinking about how the photocopy was produced, list all of the things you need to pay attention to when dilating an image.

We observed the following relationships between the pre-image and image figures of a dilation:

- ______________________________
- ______________________________

Adding Notation, Vocabulary, and Conventions

Words we use when describing a dilation, and what they tell us about the dilation:

Center of dilation: ______________________________

Scale factor of dilation: ______________________________

Vocabulary

- **center of dilation**
- collinear, collinearity
- **dilation**
- scale factor

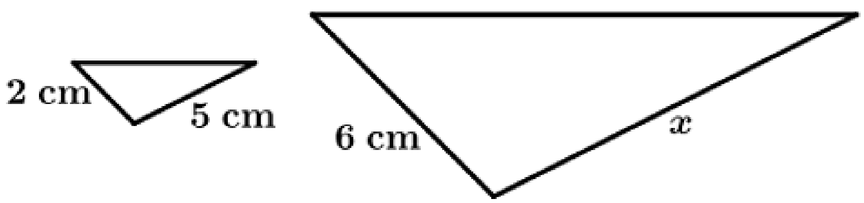
Bold terms are new in this lesson.

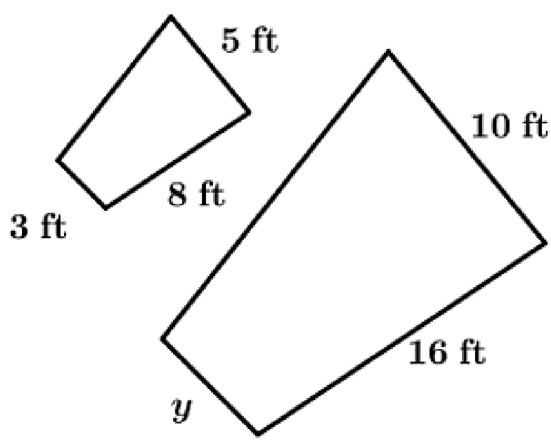
Lesson Summary

In this lesson, we observed the key features of a dilation transformation while figuring out how a photocopy machine enlarges an image. We learned how to locate points on a dilated image by using the center and scale factor that define a specific dilation. We observed that “the same shape, different size” relationship between the pre-image and image figures are a consequence of the way dilations are defined.

Retrieval

The pairs of figures below are similar to one another. Find the scale factor between the figures, and find the unknown measure indicated by the letter.

1. 

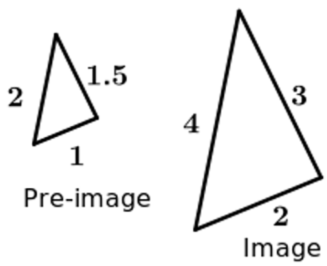
2. 

3. When figures are similar, what will you know besides the fact that they have a scale factor?

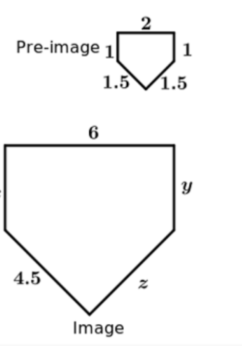
Unit 5, Lesson 1 – Ready, Set, Go

Ready

Give the factor by which each pre-image was multiplied to create the image. Use the scale factor to find any missing lengths.

4. 

Scale Factor:

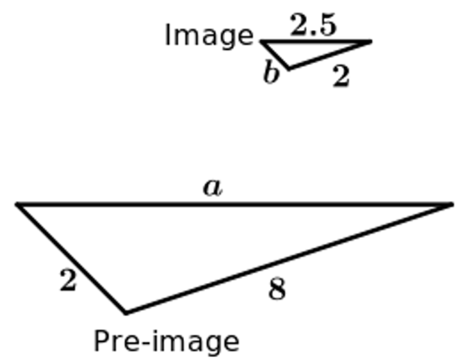
5. 

Scale Factor: _______

$x =$ _______

$y =$ _______

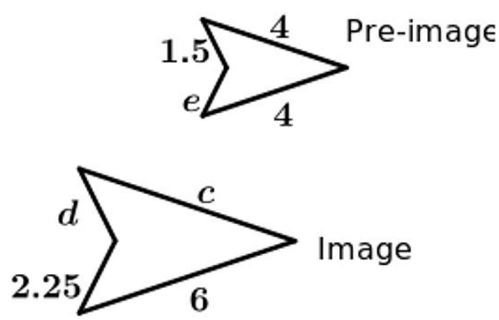
$z =$ _______

6. 

Scale Factor: _______

$a =$ _______

$b =$ _______

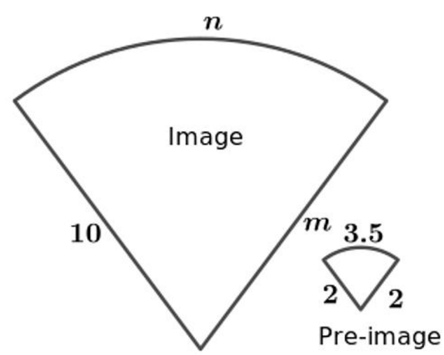
7. 

Scale Factor: _______

$c =$ _______

$d =$ _______

$e =$ _______

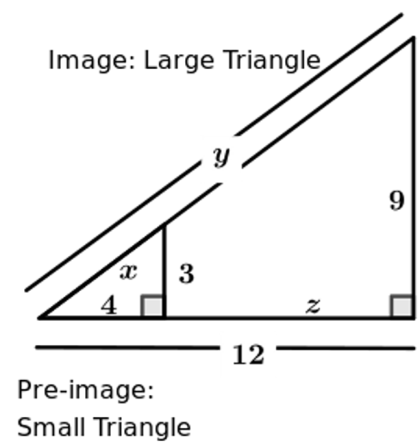


8.

Scale Factor: _______

$n =$ _______

$m =$ _______



9.

Scale Factor: _______

$x =$ _______

$y =$ _______

$z =$ _______

Set

For each real-world context or circumstance, determine the center of the dilation and an approximate scale factor or way to determine the scale factor.

7. Fran walks backward to a distance that will allow her family to all show up in the photo she is about to take.

8. The theatre technician plays with the zoom in and out buttons in an effort to fill the entire movie screen with the image.

9. Melanie zooms in on a picture she has on her phone by touching with two fingers and spreading them apart until the picture is enlarged to the desired size.

10. A digital animator creates artistic works on her computer. She is currently doing an animation that has several telephone poles along a street that goes off into the distance.

11. Ms. Sunshine is having her class do a project with a rubber-band tracing device that uses three rubber bands.

12. A copy machine is set at 300% for making a photocopy.

Go

Determine whether the given representation is representative of a linear, exponential, or quadratic function. Classify each as such and justify your reasoning.

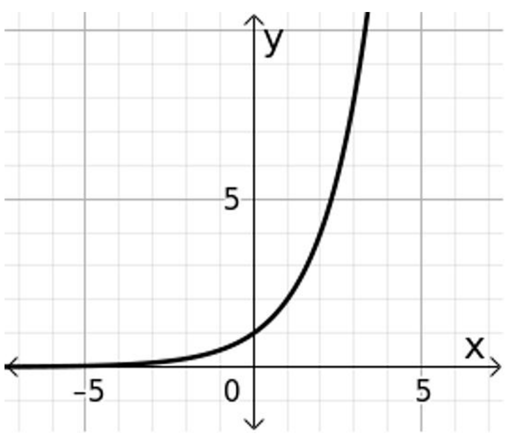
13.

x	y
2	7
3	12
4	19
5	28

Type of function: _______

Justification: _______

14.



Type of function: _______

Justification: _______

15. $y = 3x^2 + 3x$

Type of function: _______

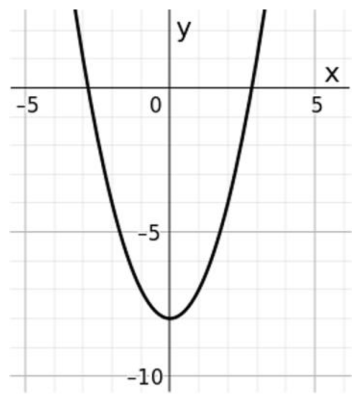
Justification: _______

16. $y = 7x - 10$

Type of function: _______

Justification: ______

17.



Type of function: _______

Justification: _______

18.

x	y
2	-5
7	5
14	19
25	41

Type of function: _______

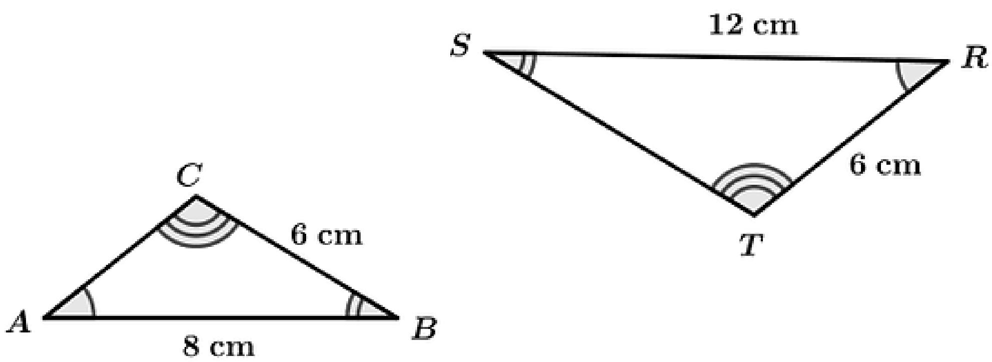
Justification: ____

Unit 5, Lesson 2

Triangle Dilations

Jump Start

Given $\triangle RST \sim \triangle ABC$, find the missing sides of each triangle.



Learning Focus

Create similar figures by dilation given the scale factor.

Prove a theorem about the midline of a triangle using dilations.

How do I know if two geometric figures are similar?

How do I know if two geometric figures are similar by dilation?

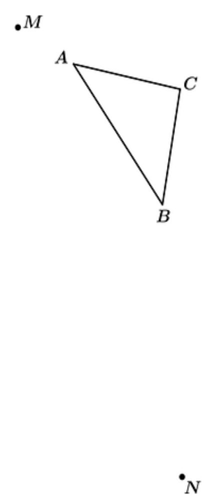
What interesting characteristics of an image are produced by dilating a polygon centered at one of the vertices of the polygon?

Open Up the Math: Launch, Explore, Discuss

1.

   a. Given $\triangle ABC$, use point M as the center of a dilation with a scale factor of 3 to locate the vertices of a new triangle, $A'B'C'$.

   b. Now use point N as the center of a dilation with a scale factor of $\frac{1}{2}$ to locate the vertices of a new triangle, $A''B''C''$.



2. Label the vertices in the two triangles you created in the diagram above. Based on this diagram, write several proportionality statements you believe are true. First write your proportionality statements using the names of the sides of the triangles in your ratios. Then verify that the proportions are true by replacing the side names with their measurements, measured to the nearest millimeter. (Try to find at least 10 proportionality statements you believe are true.)

   My list of proportions:

3. Based on your work, under what conditions are the corresponding line segments in an image and its pre-image parallel after a dilation? That is, which word best completes this statement:

   After a dilation, corresponding line segments in an image and its pre-image are _______ parallel.

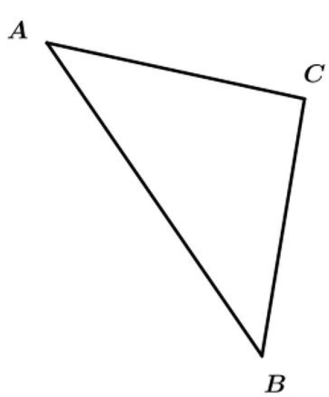
   A. never

   B. sometimes

   C. always

4. Give reasons for your answer. If you choose “sometimes,” be very clear in your explanation how to tell when the corresponding line segments before and after the dilation are parallel and when they are not.

5. Given $\triangle ABC$, use point A as the center of a dilation with a scale factor of 2 to locate the vertices of a new triangle, $A'B'C'$.



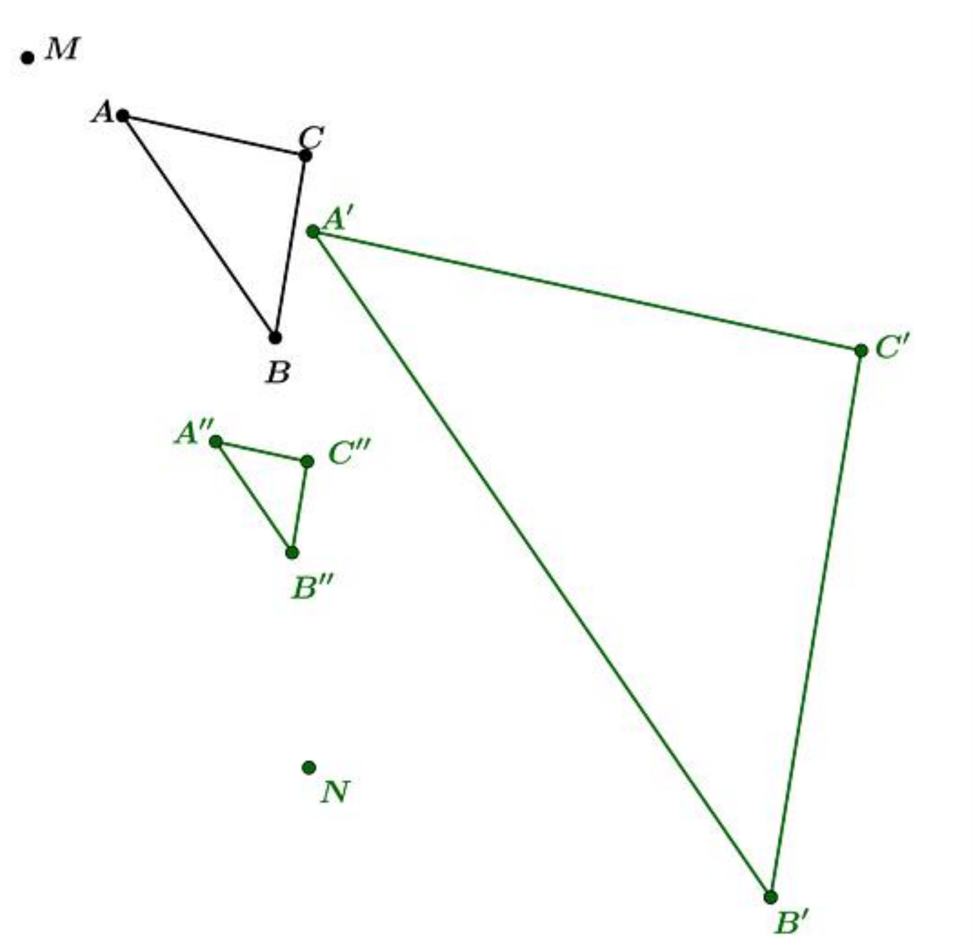
6. Explain how the diagram you created above can be used to prove the following theorem:

The segment joining midpoints of two sides of a triangle is parallel to the third side and half the length.

Ready for More?

In problem 1, $\triangle ABC$ was transformed using the two centers of dilation, points M and N, to create two new triangles, one larger than $\triangle ABC$ and one that was smaller. Locate a third center of dilation for the small and large triangles formed by the dilations described in the task.

1. Complete:



2. What is the scale factor for this new dilation?

Takeaways

Key relationships between an image and its pre-image after a dilation:

Adding Notation, Vocabulary, and Conventions

A dilation is a transformation of the plane, such that if O is the center of the dilation and a nonzero number k is the scale factor, then P' is the image of point P if ________________________________

A two-dimensional figure is similar to another if ________________________________

Vocabulary

- **midline of a triangle**
- **midline of a triangle theorem**

Bold terms are new in this lesson.

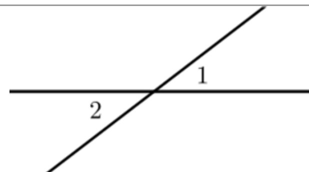
Lesson Summary

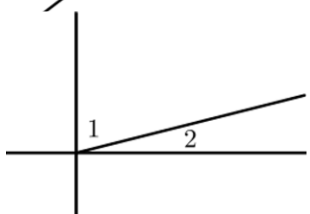
In this lesson, we extended our understanding of similar figures. Since corresponding segments of similar figures are proportional, and dilations produce similar figures, corresponding parts of an image and its pre-image after a dilation are proportional. We also learned that corresponding line segments in a dilation are parallel. These two observations provided a tool for proving a theorem about a midline of a triangle, a segment connecting the midpoints of two sides of a triangle.

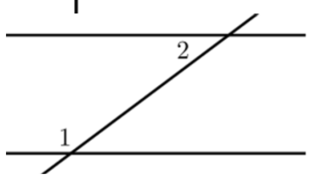
Unit 5, Lesson 2 – Ready, Set, Go

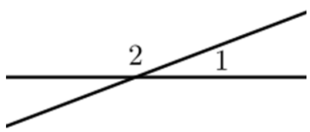
Ready

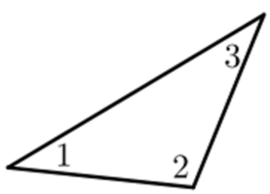
3. Match the diagrams below with the best name or phrase that describes the angles.

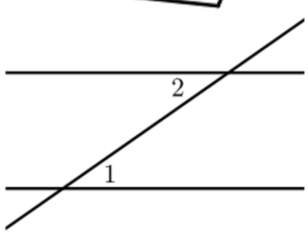
A. 

B. 

C. 

D. 

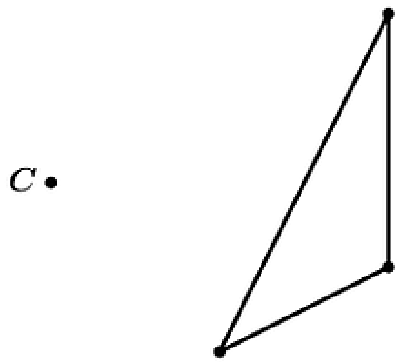
E. 

F. 

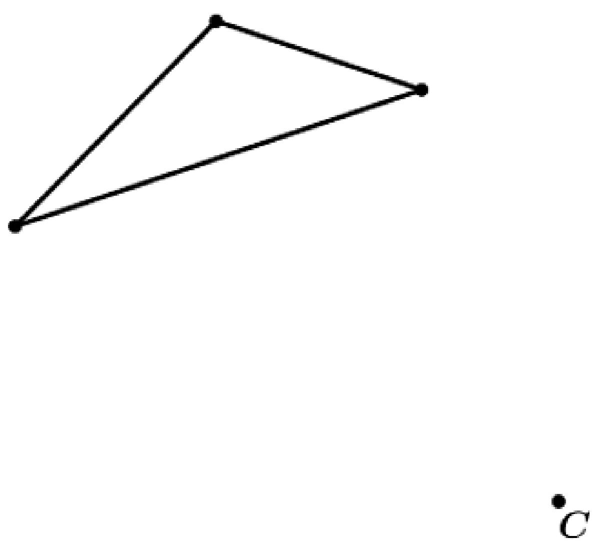
1. Alternate Interior Angles
2. Vertical Angles
3. Complementary Angles
4. Triangle Sum Theorem
5. Linear Pair
6. Same Side Interior Angles

Set

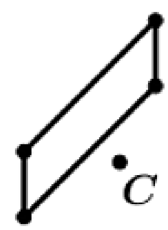
7. Create an image by applying a dilation with point C as the center of dilation and a scale factor of 2.



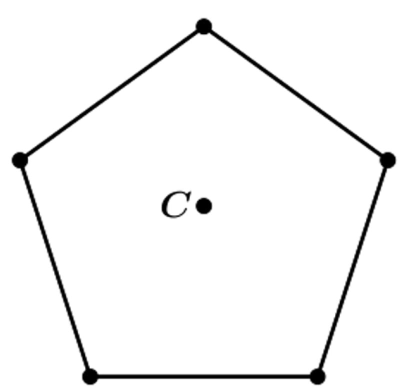
8. Create an image by applying a dilation with point C as the center of dilation and a scale factor of $\frac{1}{2}$.



9. Create an image by applying a dilation with point C as the center of dilation and a scale factor of 3.

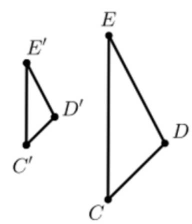


10. Create an image by applying a dilation with point C as the center of dilation and a scale factor of $\frac{1}{4}$.

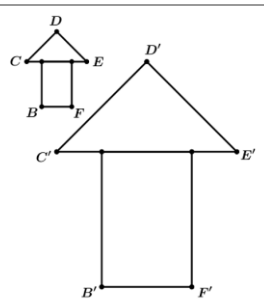


Decide if the two given figures are similar. Include as many details as possible about the transformations used to justify the similarity. For example, if a dilation is used, state details such as the center of the dilation and the scale factor.

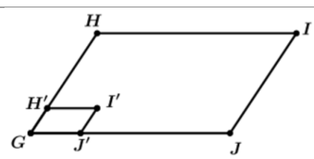
6. Complete:



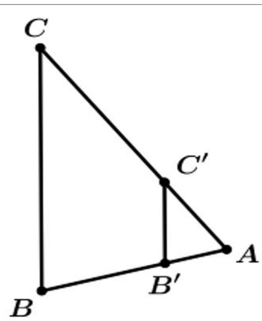
7. Complete:



8. Complete:



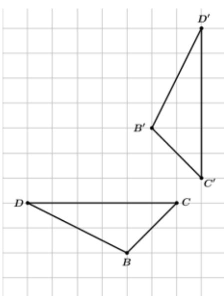
9. Complete:



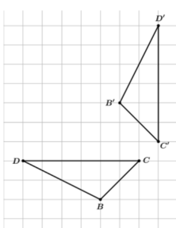
Go

Based on the given image and pre-image, determine the transformation that occurred. Justify the transformation occurred by showing evidence of some kind.

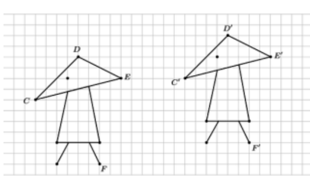
(For example, if the transformation was a reflection, show the line of reflection exists and prove it is the perpendicular bisector of all segments that connect corresponding points from the image and pre-image. Provide similar evidence for rotations, translations, and dilations.Complete:



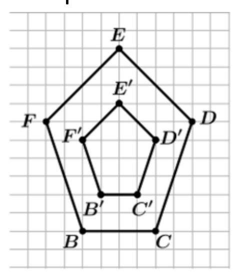
10. Complete:



11. Complete:



12. Complete:



Unit 5, Lesson 3

Similar Triangles & Other Figures

Learning Focus

Determine criteria for triangle similarity.

What is the difference between the common usage of the word *similar* (e.g., rectangles are more similar to each other than rectangles and triangles are), and the mathematical conventions for the word? What does it mean for two polygons to be similar?

How can I prove (or disprove) that two triangles are similar?

Open Up the Math: Launch, Explore, Discuss

Two figures are said to be congruent if the second can be obtained from the first by a sequence of rotations, reflections, and translations. Previously we found that we only needed three pieces of information to guarantee that two triangles were congruent: SSS, ASA, or SAS.

What about AAA? Are two triangles congruent if all three pairs of corresponding angles are congruent?

In this task we will consider what is true about triangles that are similar, but not congruent.

Definition of Similarity: *Two figures are similar if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations.*

Mason and Mia are testing out conjectures about similar polygons. Here is a list of their conjectures.

Conjecture 1: *All rectangles are similar.*

Conjecture 2: *All equilateral triangles are similar.*

Conjecture 3: *All isosceles triangles are similar.*

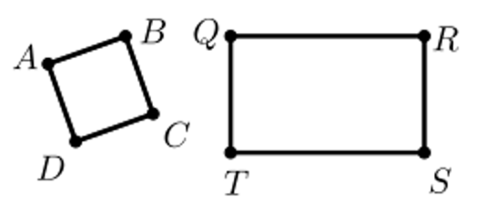
Conjecture 4: *All rhombuses are similar.*

Conjecture 5: *All squares are similar.*

1. Which of these conjectures do you think are true? Why?

Mason is explaining to Mia why he thinks conjecture 1 is true using the diagram.

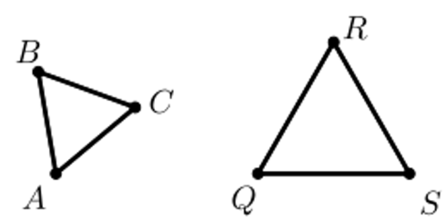
"All rectangles have four right angles. I can translate and rotate rectangle $ABCD$ until vertex A coincides with vertex Q in rectangle $QRST$. Since A and Q are both right angles, side AB will lie on top of side QR, and side AD will lie on top of side QT. I can then dilate rectangle $ABCD$ with point A as the center of dilation, until points B, C, and D coincide with points R, S, and T."



2. Does Mason's explanation convince you that rectangle $ABCD$ is similar to rectangle $QRST$ based on the definition of similarity given above? Does his explanation convince you that all rectangles are similar? Why or why not?

Mia is explaining to Mason why she thinks conjecture 2 is true using the diagram.

"All equilateral triangles have three 60° angles. I can translate and rotate $\triangle ABC$ until vertex A coincides with vertex Q on $\triangle QRS$. Since A and Q are both 60° angles, side AB will lie on top of side QR, and side AC will lie on top of side QS. I can then dilate $\triangle ABC$ with point A as the center of dilation, until points B and C coincide with points R and S."



3. Does Mia's explanation convince you that $\triangle ABC$ is similar to $\triangle QRS$ based on the definition of similarity given above? Does her explanation convince you that all equilateral triangles are similar? Why or why not?

4. For each of the other three conjectures, write an argument like Mason's and Mia's to convince someone that the conjecture is true, or explain why you think it is not always true.

   a. Conjecture 3: *All isosceles triangles are similar.*

   b. Conjecture 4: *All rhombuses are similar.*

   c. Conjecture 5: *All squares are similar.*

Mason has another conjecture: *Scaled drawings of polygons are similar figures*.

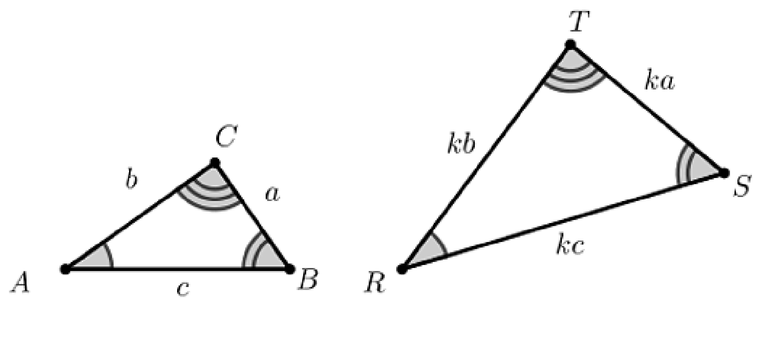
Here is what Mason knows about scaled drawings from previous work:

- Corresponding angles of scaled drawings are congruent.
- Corresponding sides of scaled drawings are proportional.

Mia proposes they try to justify this conjecture for scaled drawings of triangles. She has suggested the following diagram.

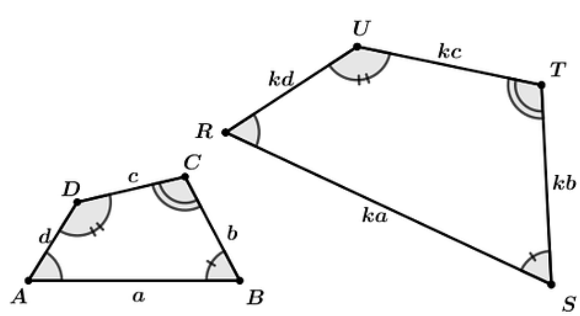
5. Explain how you can use the definition of similar figures—two figures are similar if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations—to show the following scaled drawings of triangles are similar.

   Given: Corresponding angles of $\triangle ABC$ and $\triangle RST$ are congruent, and corresponding sides are proportional by a scale factor k.



6. How can you extend Mia and Mason's justification that scaled drawings of triangles are similar to show that scaled drawings of quadrilaterals are similar figures?

   Given: Corresponding angles of quadrilateral $ABCD$ and quadrilateral $RSTU$ are congruent, and corresponding sides are proportional by a scale factor k.



While the definition of similarity given at the beginning of the task works for all similar figures, including figures with nonlinear boundaries, an alternative definition of similarity can be given for polygons: *Two polygons are similar if all corresponding angles are congruent and all corresponding pairs of sides are proportional*.

7. How does this definition help you find the error in Mason's thinking about conjecture 1?

8. How does this definition help confirm Mia's thinking about conjecture 2?

9. How might this definition help you think about the other three conjectures?

    a. Conjecture 3: *All isosceles triangles are similar.*

    b. Conjecture 4: *All rhombuses are similar.*

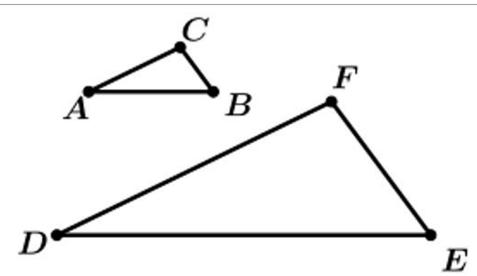
    c. Conjecture 5: *All squares are similar.*

Pause and Reflect

AAA, SAS, and SSS Similarity

From our work with rectangles, it is obvious that knowing that all rectangles have four right angles (an example of AAAA for quadrilaterals) is not enough to claim that all rectangles are similar. What about triangles? In general, are two triangles similar if all three pairs of corresponding angles are congruent?

10.

Explain why the following conjecture is true. Conjecture: *Two triangles are similar if their corresponding angles are congruent.* Use the diagram to support your reasoning. Remember to start by marking what you are given to be true (AAA) in the diagram.	

11. Mia thinks the following conjecture is true. She calls it "*AA Similarity for Triangles*." What do you think? Is it true? Why?

    Conjecture: *Two triangles are similar if they have two pairs of corresponding congruent angles.*

12. Using the diagram given in problem 10, how might you modify your proof that $\triangle ABC \sim \triangle DEF$ if you are given the following information about the two triangles:

    a. $\angle A \cong \angle D$, $DE = k \cdot AB,\ DF = k \cdot AC$; that is, $\frac{DE}{AB} = \frac{DF}{AC}$

    b. $DE = k \cdot AB,\ DF = k \cdot AC$ and $EF = k \cdot BC$; that is, $\frac{DE}{AB} = \frac{DF}{AC} = \frac{EF}{BC}$

Ready for More?

Compare and contrast the ways it can be proven that two triangles are congruent to the ways it can be proven that two triangles are similar. Are there other triangle similarity theorems that can be stated and proved?

Takeaways

To prove that two polygons are similar, we need to show that all corresponding angles are congruent, and that all corresponding pairs of sides are proportional. However, if the polygons are triangles, we can show using less information by applying one of the following theorems:

- ______________________________
- ______________________________
- ______________________________

Reflecting on the work with triangle similarity theorems, I learned or was reminded of the following insights about the proof process:

Adding Notation, Vocabulary, and Conventions

In the English language, the word *similar* means: ______________________________

In a mathematical context, the word *similar* means: ______________________________

In an alternative definition, similarity of polygons means: ______________________________

Vocabulary

- **AA similarity theorem**
- **SAS triangle similarity**
- **SSS triangle similarity**
- **similarity**
- **triangle similarity**

Bold terms are new in this lesson.

Lesson Summary

In this lesson, we examined what it means to say that two figures are similar geometrically, and we examined conditions under which two triangles will be similar. We wrote and justified several theorems for triangle similarity criteria.

Retrieval

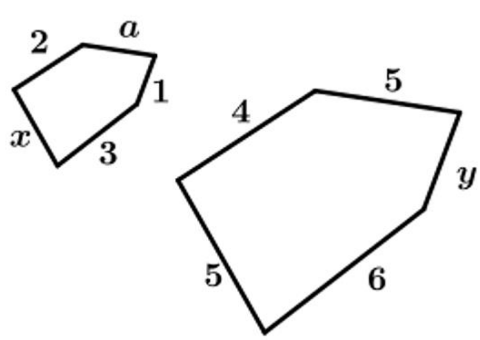
Solve each proportion. Show your work and check your solution.

1. $\frac{2}{5} = \frac{x}{20}$

2. $\frac{3}{7} = \frac{18}{b}$

3. $\frac{4}{x+2} = \frac{1}{3}$

4. Create three equivalent ratios for the similar polygons.



Unit 5, Lesson 3 – Ready, Set, Go

Ready

Solve each proportion. Show your work and check your solution.

5. $\frac{7}{12} = \frac{a}{24}$

6. $\frac{x}{7} = \frac{18}{21}$

7. $\frac{9}{c} = \frac{6}{10}$

8. $\frac{a}{2} = \frac{13}{20}$

9. $\frac{3}{b+2} = \frac{6}{5}$

10. $\frac{\sqrt{3}}{2} = \frac{\sqrt{12}}{c}$

11. $\frac{3}{4} = \frac{x}{20}$

12. $\frac{3}{4} = \frac{b+3}{20}$

13. $\frac{3}{6} = \frac{8}{x}$

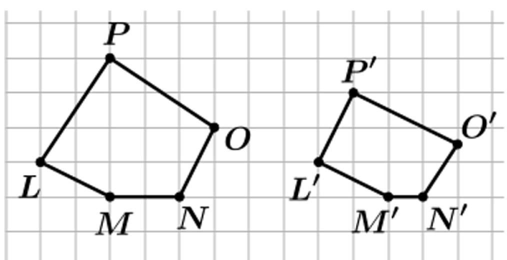
Set

Provide an argument to prove each conjecture, or provide a counterexample to disprove it.

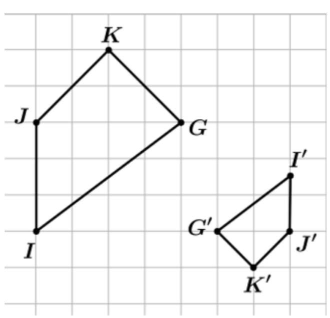
10. All right triangles are similar.

11. All regular polygons are similar to other regular polygons with the same number of sides.

12. The polygons on the grid are similar.

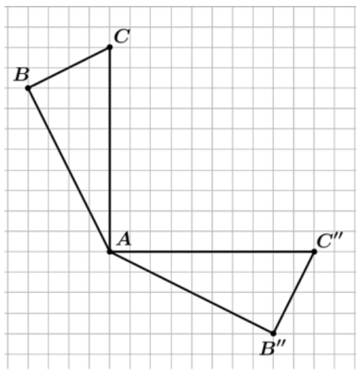


13. The polygons on the grid are similar.

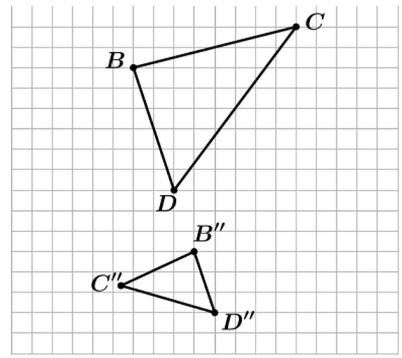


A sequence of transformations occurred to create the two similar polygons. Provide a specific set of steps that can be used to create the image from the pre-image with one, two, or three transformations.

14.



16.



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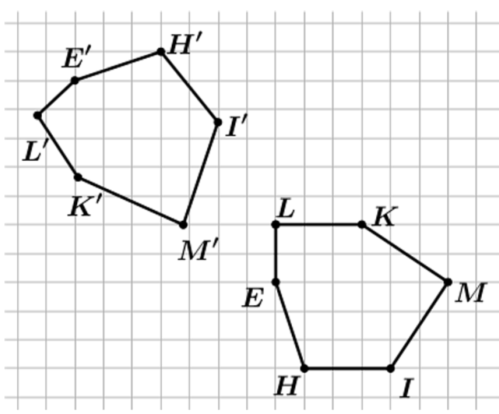

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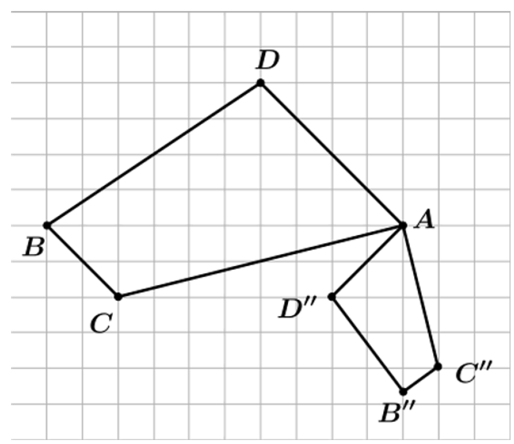
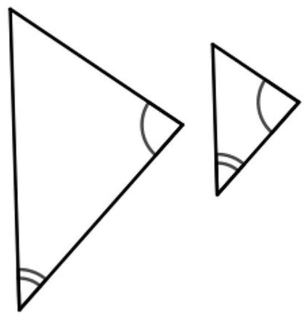


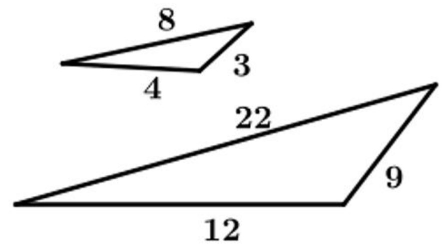
18. One definition of similar triangles is based on transformations. However, there are other criteria for determining whether or not triangles are similar. What are these other criteria?

19. Determine whether or not the pairs of triangles are similar and explain why.

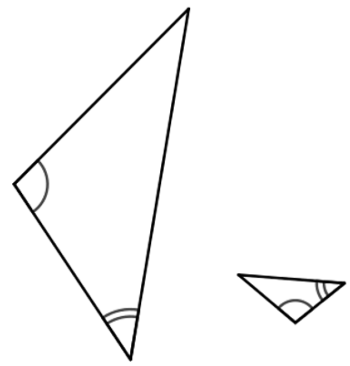
a.



b.



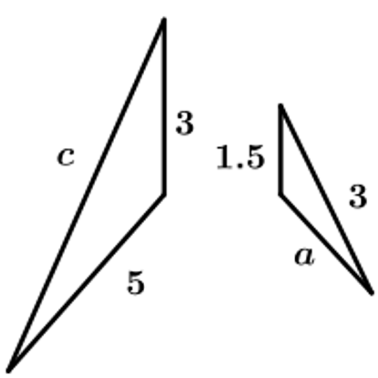
c.

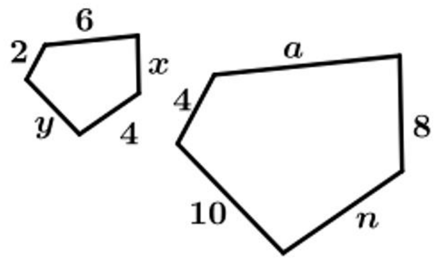


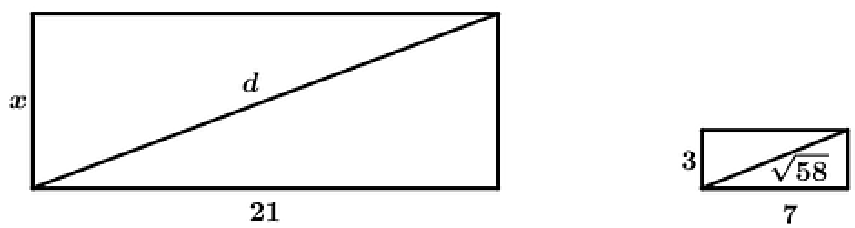
Go

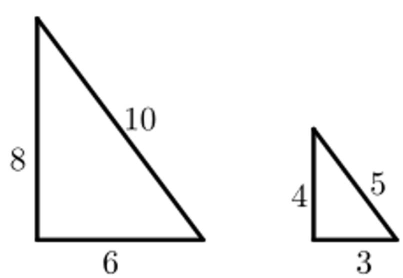
For each pair of similar polygons, give three ratios that would be equivalent.

20.



21. 

22. 

23. 

Unit 5, Lesson 4

Cut by a Transversal

Jump Start

Each of the following problems is designed to help you identify valid and invalid strategies for changing the form of fractions. Students have suggested that the following equations are true. Examine each problem to determine if the two fractions are equivalent. (You can verify your prediction numerically by calculating decimal equivalents of the fractions on each side of the equation.) If you determine the fractions are not equivalent, what errors in reasoning might have led students to say the fractions are equivalent?

1. $\frac{2+5}{6} = \frac{5}{3}$

2. $\frac{2+5}{6} = \frac{1+5}{3}$

3. $\frac{2+5}{6} = \frac{1}{3} + \frac{5}{6}$

Learning Focus

Prove that a line drawn parallel to one side of a triangle that intersects the other two sides divides the other two sides proportionally.

What observations can I make about the segments formed on two sides of a triangle by a line drawn parallel to the third side?

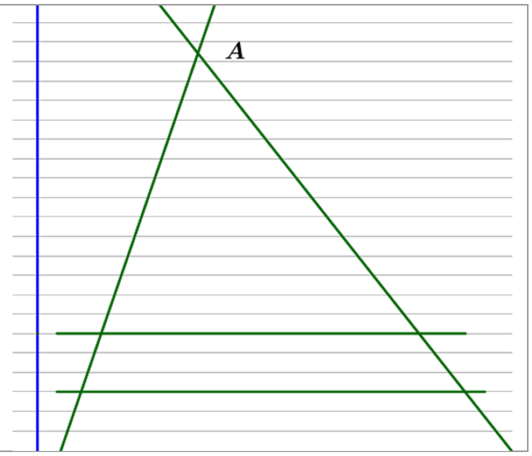
What if the parallel line is below the base?

How might I justify my observations algebraically?

Open Up the Math: Launch, Explore, Discuss

Draw two intersecting transversals on a sheet of lined paper, as in the following diagram. Label the point of intersection of the transversals A. Select any two of the horizontal lines to form the third side of two different triangles. Label the endpoints of the third side of the smaller triangle B and C, and the endpoints of the third side of the larger triangle D and E.

1.

What convinces you that the two triangles formed by the transversals and the horizontal lines are similar? (Note: We can assume that the horizontal lines on a sheet of lined paper are parallel.)	

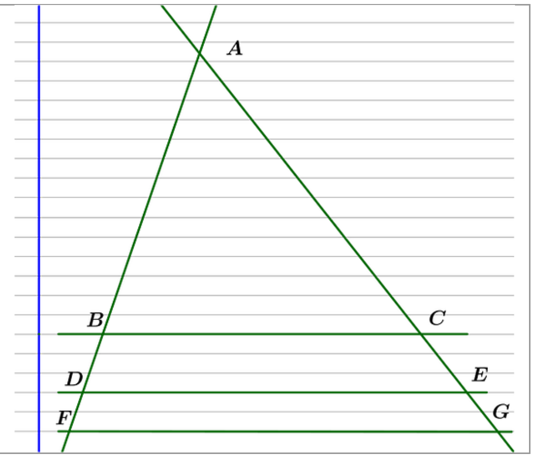
2. Write some proportionality statements about the sides of the triangles you have drawn, and then verify the proportionality statements by measuring the sides of the triangles.

3. Select a third horizontal line segment to form a third triangle that is similar to the other two. Write some additional proportionality statements and verify them with measurements.

Tristan has written this proportion for problem 3, based on his diagram below: $\frac{BD}{AB} = \frac{CE}{AC}$

Tia thinks Tristan's proportion is wrong because some of the segments in his proportion are not sides of a triangle.

4.

Check out Tristan's idea using measurements of the segments in his diagram.	

5. Now check out this same idea using proportions of segments from your own diagram. Test at least two different proportions, including segments that do not have A as one of their endpoints.

6. Based on your examples, do you think Tristan or Tia is correct?

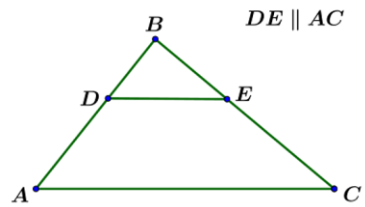
7. Tia still isn't convinced, since Tristan is basing his work on a single diagram. She decides to start with a proportion she knows is true: $\frac{AD}{AB} = \frac{AE}{AC}$. (Why is this true?)

   Tia realizes that she can rewrite this proportion as $\frac{AB+BD}{AB} = \frac{AC+CE}{AC}$. (Why is this true?)

   Can you use Tia's proportion to prove algebraically that Tristan is correct?

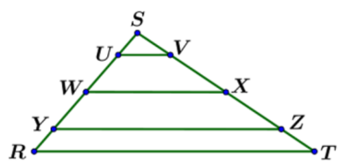
Ready for More?

Explore how the area of the smaller triangle formed by drawing an interior segment parallel to one of the sides of a triangle is related to the area of the original triangle.



Takeaways

The triangle midline theorem is a special case of a theorem sometimes referred to as "the side-splitter theorem."



Given: $\overline{UV} \parallel \overline{WX} \parallel \overline{YZ} \parallel \overline{RT}$

Here are some proportionality statements that can be written based on the side-splitter theorem:

Vocabulary

- **proportion: proportionality statement**
- **ratio**
- **side-splitter theorem**

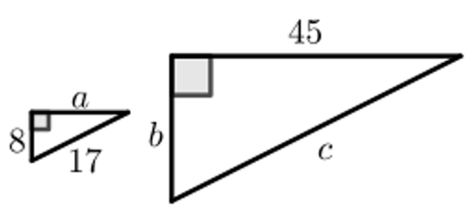
Bold terms are new in this lesson.

Lesson Summary

In a previous lesson, we learned that a midline of a triangle, a line that passes through the midpoints of two of the sides, is parallel to the third side and half its length. In this lesson, we extended this theorem to include other segments that cut the sides of a triangle proportionally. We also proved a non-intuitive "side-splitting" theorem about the multiple segments formed when multiple lines parallel to a side of a triangle cut the other two sides of the triangle.

Retrieval

Find all of the missing side lengths in the right triangles. The two triangles are similar to one another.

1. 

2. The line shown has several triangles that can be used to determine the slope. Draw in three slope-defining triangles of different sizes for the line and then create the ratio of rise to run for each of the triangles you draw.

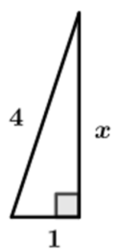
ratios: ________

Unit 5, Lesson 4 – Ready, Set, Go

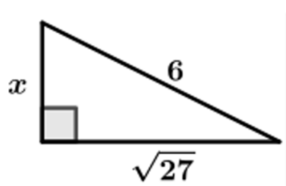
Ready

Solve for x to find the missing side in each right triangle.

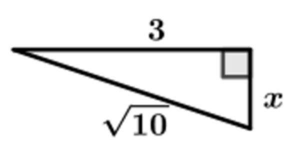
3.



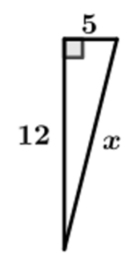
4.



5.

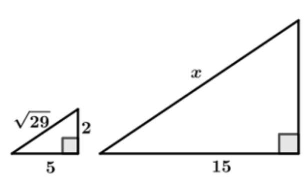


6.

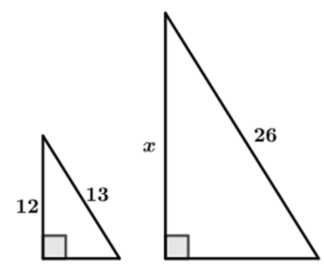


Create a proportion for each set of similar triangles that can be used to find the missing side length indicated. Then solve the proportion.

7.



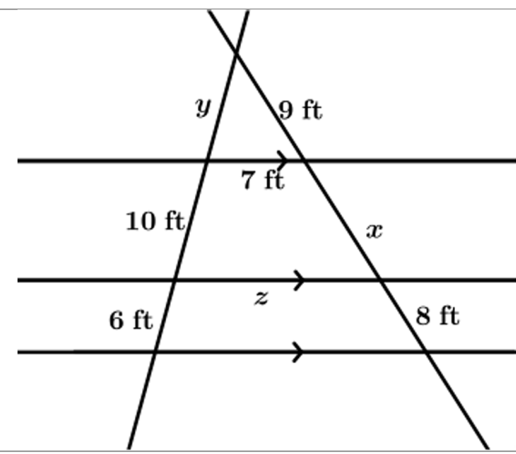
8.



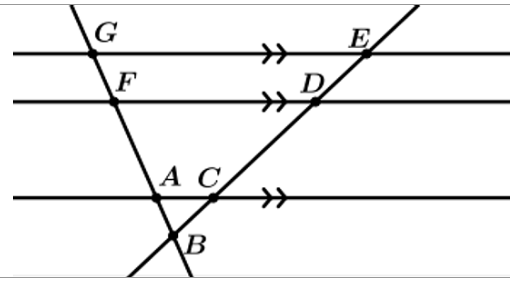
Set

13.

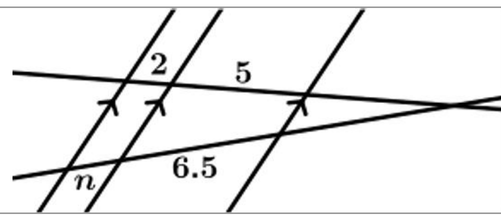
Jack and Diane are designing a trellis for some special plants they are planning to put in their backyard garden. Use the values provided on the sketch of the trellis to find the missing values for x, y, and z.



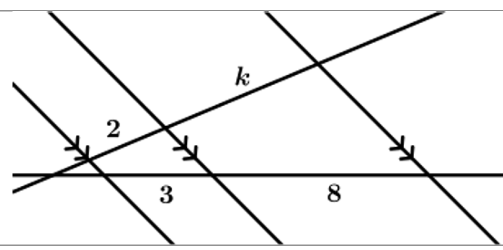
14.

Use the diagram to prove $\frac{FA}{AB} = \frac{DC}{CB}$.	

15.

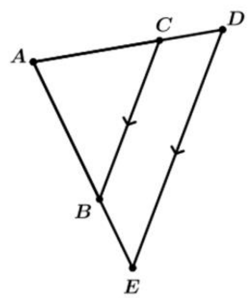
Write and solve a proportion that will provide the missing length.	

16.

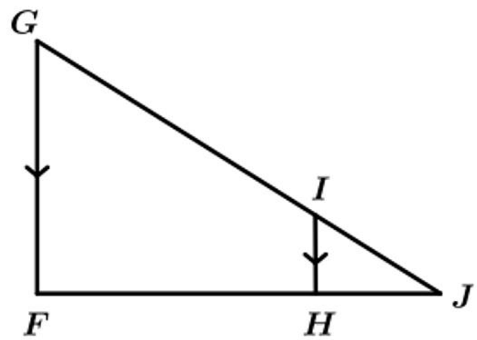
Write and solve a proportion that will provide the missing length.	

For problems 11–14, find the parallel line segments in the diagram and write a mathematical statement showing the parallel line segments. Then write a similarity statement for the triangles that are similar (i.e. $\triangle ABC \sim \triangle XYZ$). Explain why those triangles would be similar.

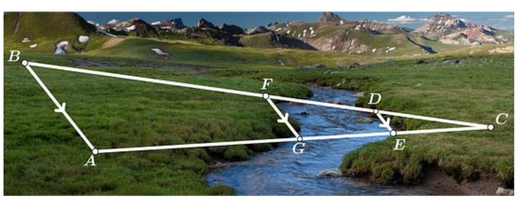
11.



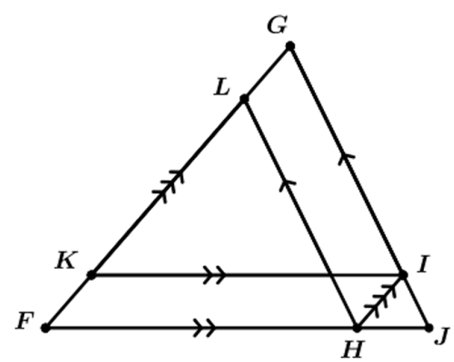
13.



12.



14.



Go

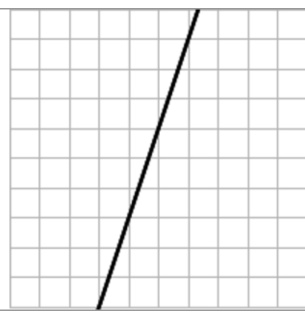
Each line below has several triangles that can be used to determine the slope. Draw in three slope-defining triangles of different sizes for each line and then create the ratio of rise to run for each.

15. Complete:

ratios: _______

16. Complete:

ratios: _______



17. Complete:

ratios: _______

18. Complete:

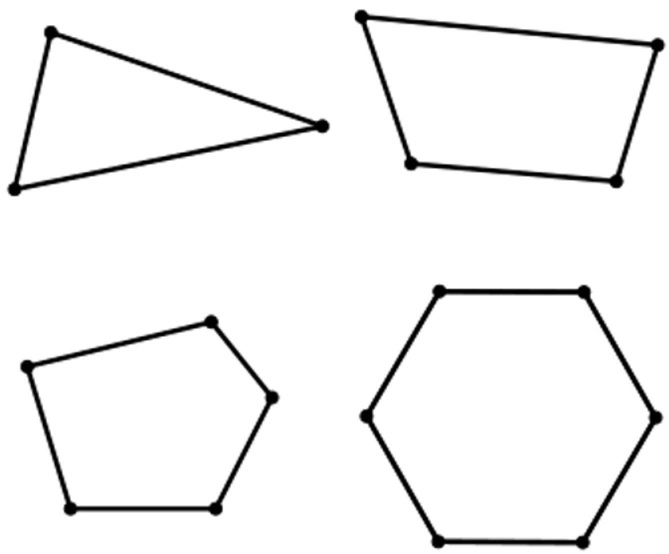
ratios: _______

Unit 5, Lesson 5

Measured Reasoning

Jump Start

1. For each of the following polygons, pick one vertex and draw all of the possible diagonals of the polygon that include that vertex.



2. What conjecture might you make about the relationship between the number of diagonals drawn from a vertex and the number of sides of the polygon. Can you explain why this is so?

Learning Focus

Practice using geometric reasoning in computational work.

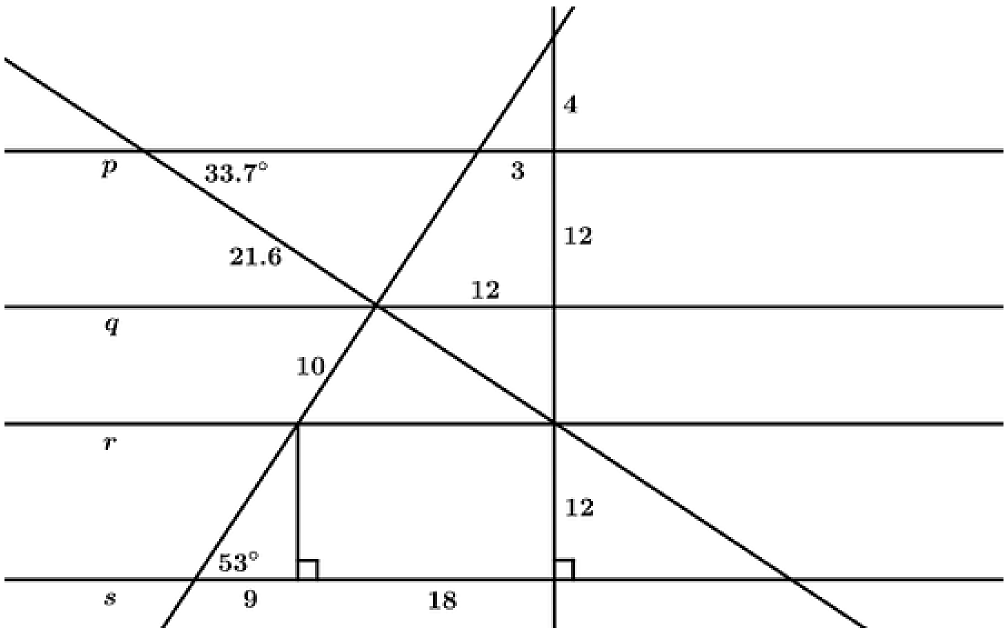
How do I look for structure in a diagram so I can use familiar features of the diagram to find the measures of unknown sides and angles?

What measurements do I need to calculate first, in order to calculate additional measurements?

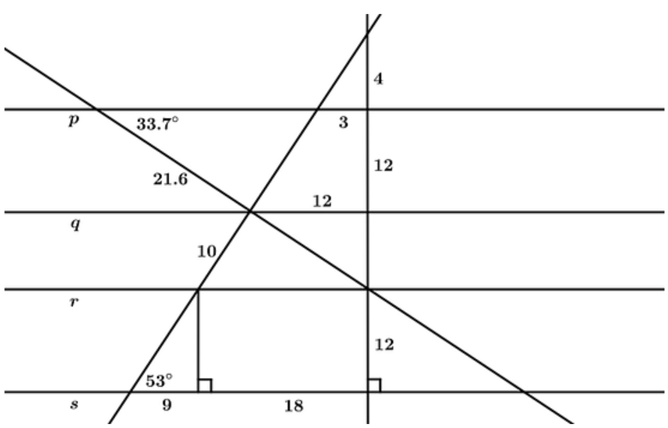
Open Up the Math: Launch, Explore, Discuss

In the diagram, lines p, q, r, and s are all parallel.

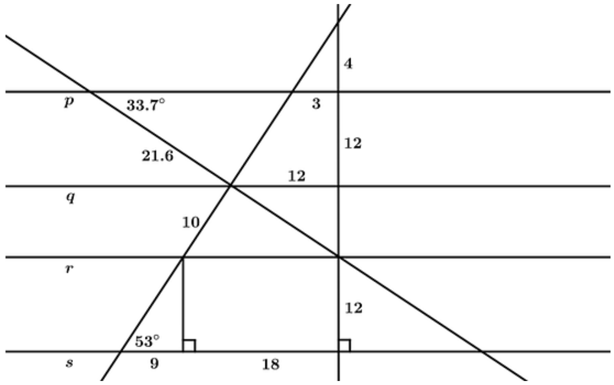
1. Find the measures of all missing sides and angles by using geometric reasoning, not rulers and protractors. If you think a measurement is impossible to find, identify what information you are missing.



2. Identify all similar triangles in the diagram. How do you know they are similar, by dilation or by AA similarity?



3. Identify at least three different quadrilaterals in the diagram.

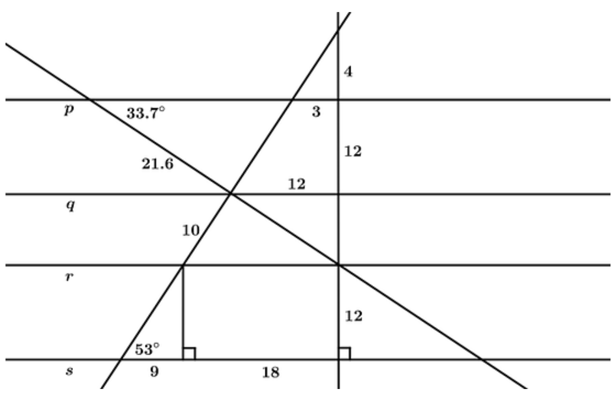


a. Find the sum of the interior angles for each quadrilateral.

b. Make a conjecture about the sum of the interior angles of a quadrilateral.

Conjecture:

4. Identify at least three different pentagons in the diagram. (Hint: The pentagons do not need to be convex.)



a. Find the sum of the interior angles for each pentagon.

b. Make a conjecture about the sum of the interior angles of a pentagon.

Conjecture:

5. Do you see a pattern in the sum of the angles of a polygon as the number of sides increases? If so, write a conjecture.

Ready for More?

How can you convince yourself that the pattern you have noticed for the sum of the interior angles of a triangle, a quadrilateral, a pentagon, and a hexagon holds for all n-gons? State and prove your conjecture.

Takeaways

Using the theorem that the sum of the interior angles of a triangle is 180, we were able to find a formula for the sum of the interior angles of any polygon: _______________

This diagram can be used to illustrate why this is true:

This theorem is true for both convex and concave polygons.

Finding missing sides and angles in a complex diagram requires that I look for structure in the diagram. Some of the geometric structures I used today included:

Vocabulary

- **concave and convex**
- diagonal

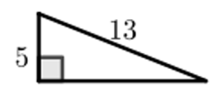
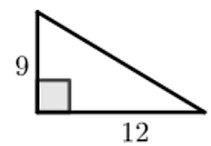
Bold terms are new in this lesson.

Lesson Summary

In this lesson, we drew upon a variety of theorems to support the computational work of finding missing sides and angles. To identify which theorems to use, we had to examine the available features of the diagram. For many measurements, multiple strategies could be used. We also used the diagram, along with our computed measurements, to develop and justify a conjecture for the sum of the interior angles of any polygon, similar to the theorem we proved previously about the sum of the interior angles in a triangle.

Retrieval

1.

Find the missing side lengths for the right triangles. Then determine if the triangles are similar or not.	 

Missing side in small triangle = _______ Missing side in large triangle = _______	Similar? _______

Solve each equation.

2. $3x + 7 = 5x - 5$

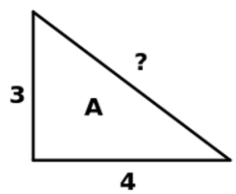
3. $5(x + 3) = 7(x + 1)$

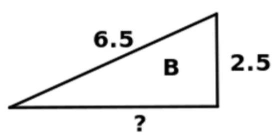
4. $\frac{12}{84} = \frac{3}{x+7}$

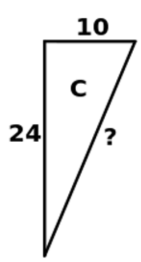
Unit 5, Lesson 5 – Ready, Set, Go

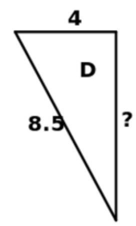
Ready

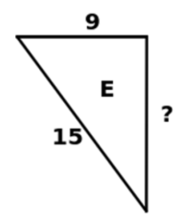
Find the missing side in each right triangle. Triangles are not drawn to scale.

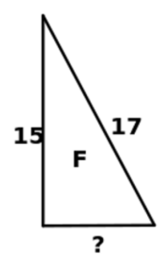
2. 

3. 

4. 

5. 

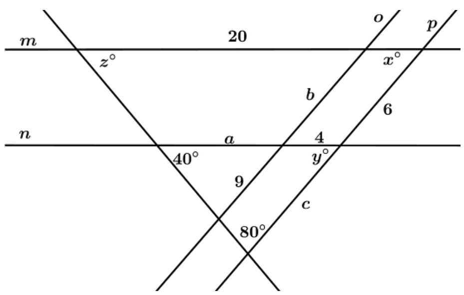
6. 

7. 

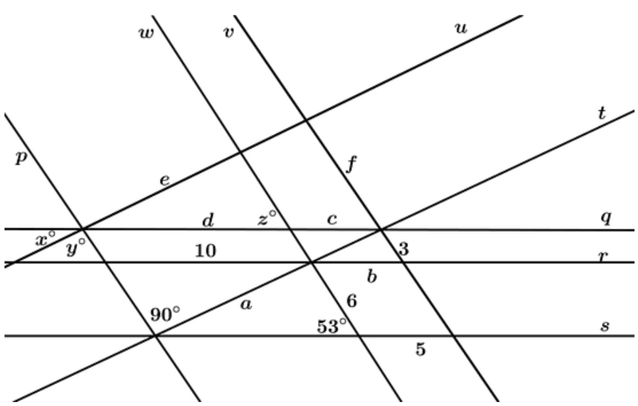
8. Based on ratios between side lengths, which of the previous right triangles are mathematically similar to each other? Name the triangles and write the proportion that demonstrates that they are similar.

Set

8. Line $m \parallel n$ and $\parallel p$, find the values of angles x, y, and z. Also, find the lengths of a, b, and c.



9. Line $q \parallel r \parallel s$ and $t \parallel u$ and $p \parallel w \parallel v$, find the values of angles x, y, and z. Also, find the lengths of a, b, c, d, e, and f.



Go

Solve each equation.

14. $3x - 5 = 2x + 7$

15. $\frac{5}{7} = \frac{x}{21}$

16. $\frac{3}{x} = \frac{18}{5x+2}$

17. $17 + 3(x - 5) = 2(x + 3)$

18. $\frac{x+5}{6} = \frac{3(x+2)}{9}$

19. $x + 2 + 3x - 8 = 90$

20. $\frac{5}{12} = \frac{x}{8}$

21. $\frac{4}{5} = \frac{x+2}{15}$

Unit 5, Lesson 6

Yard Work in Segments

Learning Focus

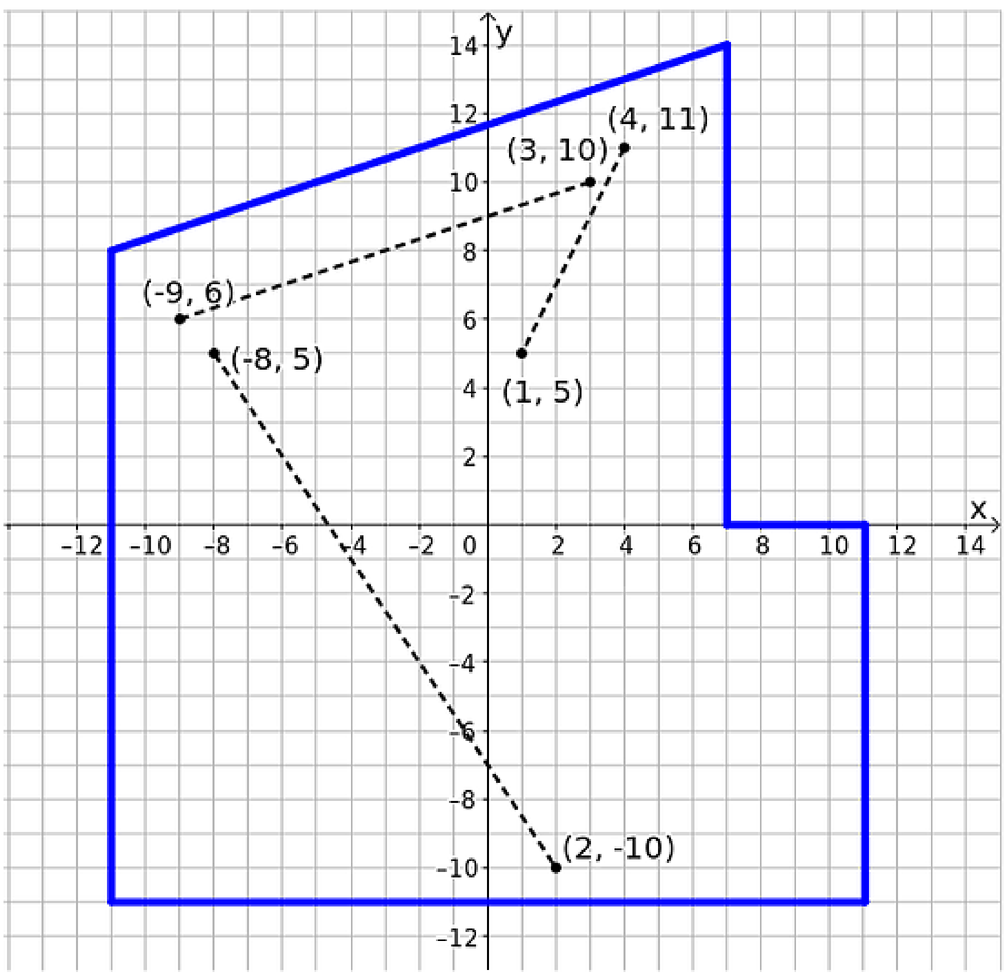
Locate the midpoint of a segment and a point that divides the segment in a given ratio.

How do I locate the midpoint of a segment given just the coordinates of its endpoints?

How do I divide a line segment drawn on a grid into proportional parts?

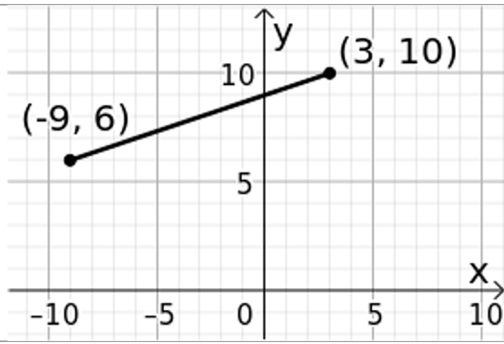
Open Up the Math: Launch, Explore, Discuss

Malik's family has purchased a new house with an unfinished yard. They drew the following map of the backyard:



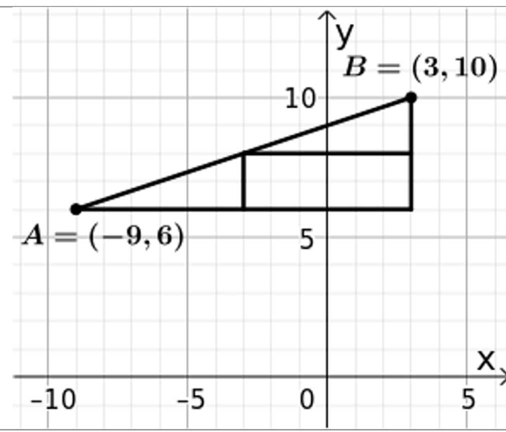
Malik and his family are using the map to set up gardens and patios for the yard. They plan to lay out the yard with stakes and strings (shown as points and dotted lines in the map) so they know where to plant grass, flowers, or vegetables. They want to begin with a vegetable garden that will be parallel to the fence shown at the top of the map.

1. They set the first stake at $(-9,6)$ and the stake at the end of the garden at $(3,10)$. They want to mark the middle of the garden with another stake. Where should the stake that is at the midpoint of the segment between the two end stakes be located? Using a diagram, describe your strategy for finding this point.



Midpoint:

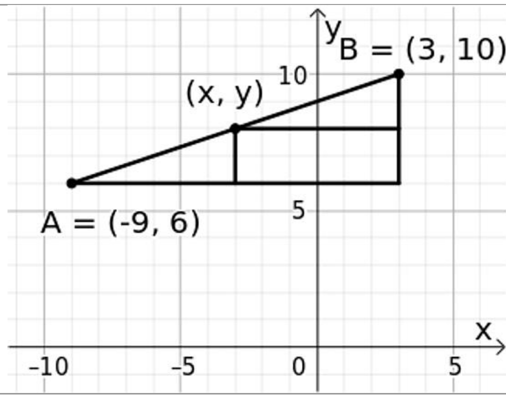
Malik figured out the midpoint by saying, "It makes sense to me that the midpoint is going to be halfway over and halfway up, so I drew a right triangle and cut the horizontal side in half and the vertical side in half like this:"
Malik continued, "That put me right at $(-3,8)$. The only thing that seems funny about that to me is that I know the base of the big triangle was 12 and the height of the triangle was 4, so I thought the midpoint might be $(6,2)$."



2. Explain to Malik why the logic that made him think the midpoint was $(6,2)$ is almost right, and how to extend his thinking to use the coordinates of the endpoints to get the midpoint of $(-3,6)$.

Malik's sister, Sapana, looked at his drawing and said, "Hey, I drew the same picture, but I noticed the two smaller triangles that were formed were congruent. Since I didn't know for sure what the midpoint was, I called it (x, y). Then I used that point to write an expression for the length of the sides of the small triangles. For instance, I figured that the base of the lower triangle was $x-(-9)$."

3. Label all of the other legs of the two smaller right triangles using Sapana's strategy.



Sapana continued, "Once I labeled the triangles, I wrote equations by making the bases equal and the heights equal."

4. Does Sapana's strategy work? Show why or why not.

5. Choose a strategy and use it to find the midpoint of the segment with endpoints $(-3,4)$ and $(2,9)$.

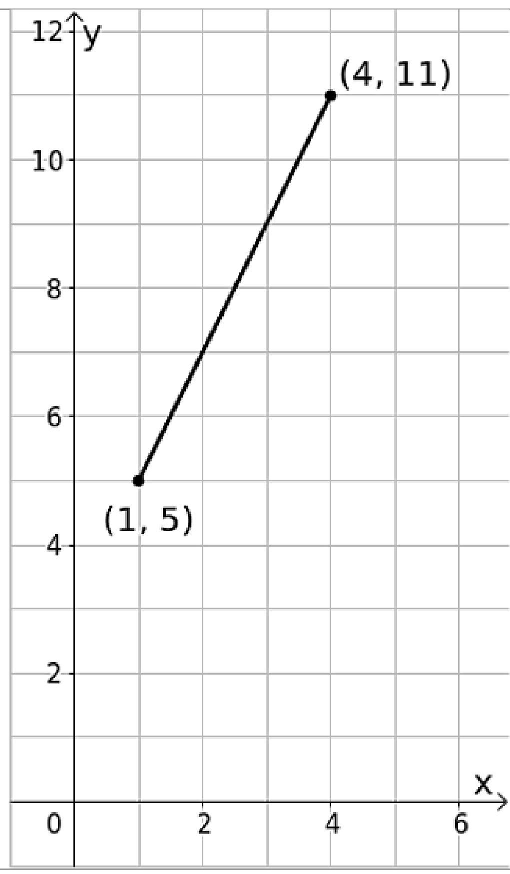
6.

Use either strategy to find the midpoint of the segment between (x_1, y_1) and (x_2, y_2).	

Pause and Reflect

The next area in the garden to be marked is for a flower garden. Malik's parents have the idea that part of the garden should contain a big rose bush and the rest of the garden will have smaller flowers like petunias. They want the section with the other flowers to be twice as long as the section with the rose bush. The stake on the endpoints of this garden will be at $(1,5)$ and $(4,11)$. Malik's dad says, "We'll need a stake that marks the end of the rose garden."

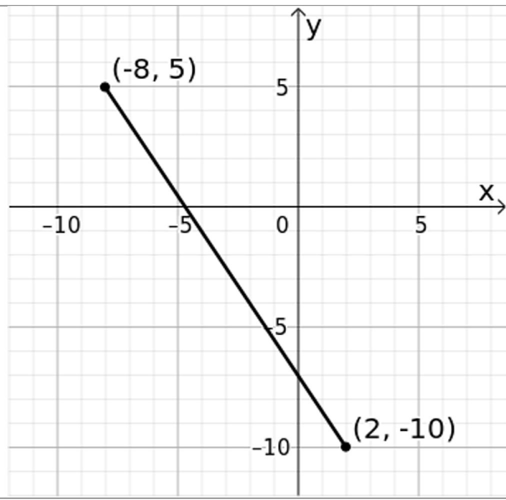
7. Help Malik and Sapana figure out where the stake will be located if the rose bush will be closer to the stake at $(1,5)$ than the stake at $(4,11)$.



The stake should be located at:

There's only one more set of stakes to put in. This time the endpoint stakes are at $(-8,5)$ and $(2,-10)$. Another stake needs to be placed that partitions this segment into two parts so that the ratio of the lengths is $2:3$.

10. Where must the stake be located if it is to be closer to the stake at $(-8,5)$ than to the stake at $(2,-10)$?



The stake should be located at:

Ready for More?

Generate a symbolic rule for locating the point that divides a line segment into two parts so that the ratio of the lengths is $m:n$, with the point closer to the left endpoint.

Takeaways

To find the midpoint of the segment whose endpoints are (x_1, y_1) and (x_2, y_2), I can use the midpoint rule:

To find the point that divides the segment whose endpoints are (x_1, y_1) and (x_2, y_2) into two parts in the ratio $m{:}n$, I can use a strategic method or remember the rule:

Strategy:

Rule: _______________

Here are my illustration and notes to explain why this rule or strategy works:

Vocabulary

- directed distance
- **midpoint**
- ratio

Bold terms are new in this lesson.

Lesson Summary

In this lesson, we examined strategies for dividing a line segment into two parts that fit a given ratio. One common application of this concept is to find the coordinates of the midpoint of a segment, given the coordinates of the endpoints.

Retrieval

Find the mean, or average, for each set of numbers.

1. $-5, 12, 32$

2. $3, 8, 14, 31$

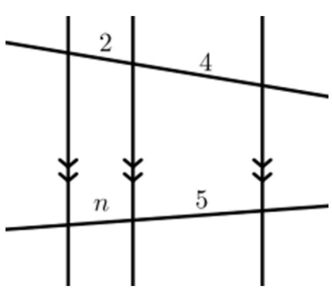
Find the value that is exactly halfway between the two given values.

3. $5, 13$

4. $2, 40$

5. Explain your process for finding the value halfway between two values in the last two problems.

6. Write a proportion and solve it for the missing value.



Unit 5, Lesson 6 – Ready, Set, Go

Ready

For each set of numbers, find the mean (average). Explain how the mean of the set compares to the values in the set.

3. $6, 12, 10, 8$

4. $2, 7, 12$

5. $-13, 21$

6. $3, -9, 15$

7. $43, 52$

8. $38, 64, 100$

Find the value that is exactly halfway between the two given values. Show your work.

8. $5, 13$

9. $26, 42$

10. $57, 77$

11. $-34, -22$

12. $-45, 3$

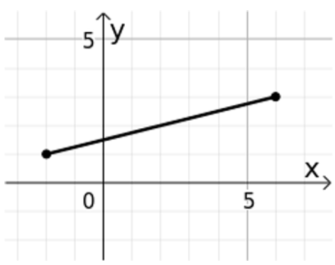
13. $-12, 18$

Set

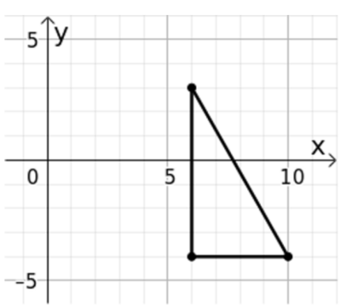
Find the coordinates of the midpoint of each line segment provided. If multiple line segments are given, then give the midpoints of all segments.

19.

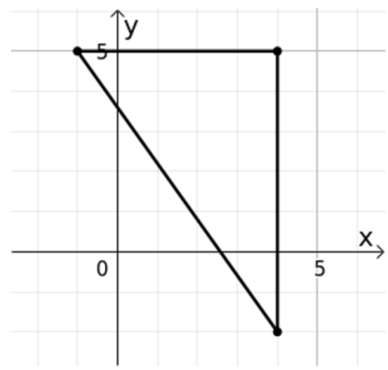
20.



21.



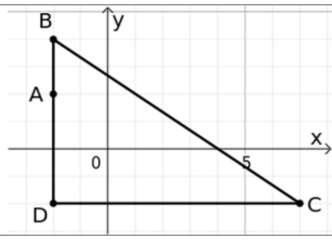
22.



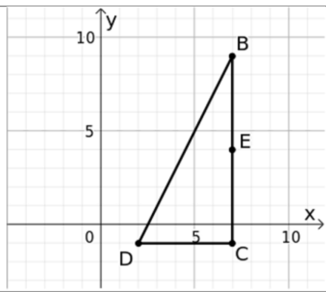
23. A line segment between $(2,3)$ and $(10,15)$.

24. A line segment between $(-2,7)$ and $(3,-8)$.

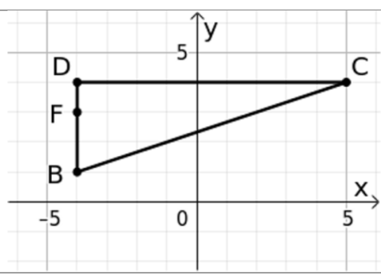
19.

	If a line is drawn parallel to $\overline{BC}$ and through point A, at what coordinate will the intersection of this parallel line be with $\overline{DC}$?

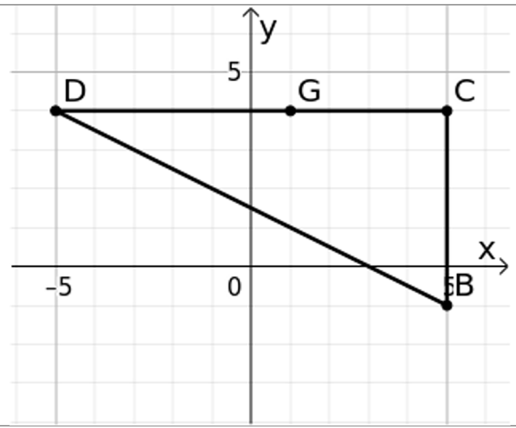
20.

	If a line is drawn parallel to $\overline{BD}$ and through point E, at what coordinate will the intersection of this parallel line be with $\overline{DC}$?

21.

	If a line is drawn parallel to $\overline{BC}$ and through point F, at what coordinate will the intersection of this parallel line be with $\overline{DC}$?

22.



If a line is drawn parallel to $\overline{BD}$ and through point G, at what coordinate will the intersection of this parallel line be with $\overline{BC}$?

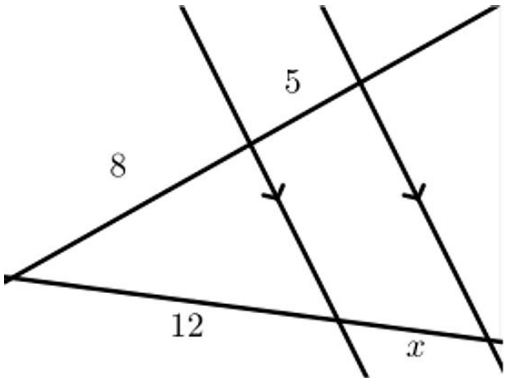
23. When a line is drawn parallel to one side of a triangle so that it intersects the other two sides of the triangle, how do the measures of the parts of the two intersected sides compare? Explain.

24. In problems 19–22, you were given right triangles. Could a determination of the coordinates be made if they were not right triangles? Why or why not?

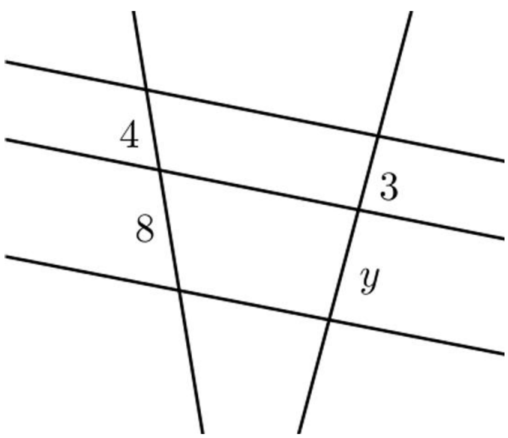
Go

Write a proportion for each of the diagrams below and solve for the missing value.

25.



26.

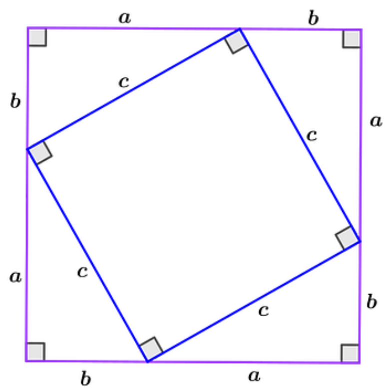


Unit 5, Lesson 7

Pythagoras by Proportions

Jump Start

In the diagram, the length of each side of the outer square is $a + b$, and the length of each side of the inner square is c. You are going to use this diagram to prove the Pythagorean theorem algebraically by showing that the area of the inner square, c^2, is also equal to $a^2 + b^2$.



1. Find an expression for the area of the outer square by expanding $(a + b)^2$.

2. Represent the area of one of the four triangles in terms of a and b.

3. Find the area enclosed by the four small triangles in terms of a and b.

4. Subtract the area of the four small triangles from the area of the larger square.

5. How does this prove the Pythagorean theorem?

Learning Focus

Prove the Pythagorean theorem algebraically.

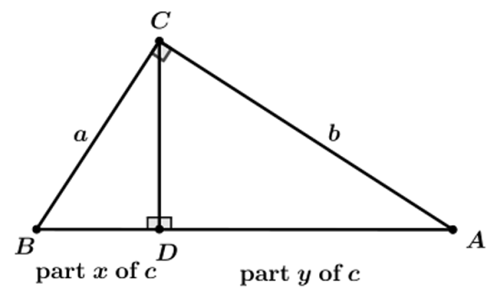
When and how do I use algebra in a geometric proof?

What does each proof of the Pythagorean theorem reveal?

Open Up the Math: Launch, Explore, Discuss

1.

There are many different proofs of the Pythagorean theorem. Here is one based on similar triangles.
Step 1: Cut a 4×6 index card along one of its diagonals to form two congruent right triangles.



Step 2: In each right triangle, draw an altitude from the right angle vertex to the hypotenuse. (Use the right angle in the other triangle to help you draw this altitude accurately.) Draw this altitude on both the front and back of the triangle.

Step 3: Label each triangle as shown in the diagram. Label the length of the altitude h. Flip each triangle over and label the matching sides and angles with the same names on the back as on the front.

Step 4: Cut one of the right triangles along the altitude to form two smaller right triangles

Step 5: Arrange the three triangles in a way that convinces you that all three right triangles are similar. You may need to reflect and/or rotate one or more triangles to form this arrangement.

Step 6: Write proportionality statements to represent relationships between the labeled sides of the triangles. (Note: Side c has been decomposed into segments labeled x and y. The sum of these two segments is c.)

Step 7: Solve one of your proportions for x and the other proportion for y. (If you have not written proportions that involve x and y, study your set of triangles until you can do so.)

Step 8: Work with the equations you wrote in step 7 until you can show algebraically that $a^2 + b^2 = c^2$. (Remember, $x + y = c$.)

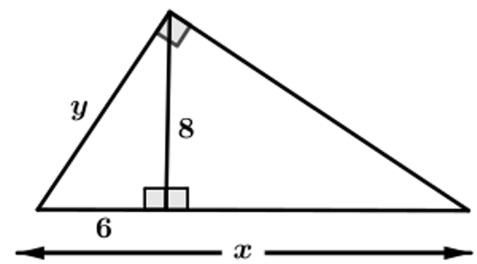
Ready for More?

Use your set of triangles to help you prove the following two theorems:

1. Right triangle altitude theorem 1: *If an altitude is drawn to the hypotenuse of a right triangle, the length of the altitude is the geometric mean between the lengths of the two segments formed on the hypotenuse.*

2. Right triangle altitude theorem 2: *If an altitude is drawn to the hypotenuse of a right triangle, the length of each leg of the right triangle is the geometric mean between the length of the hypotenuse and the length of the segment on the hypotenuse adjacent to the leg.*

3.

Use your set of triangles to help you find the values of x and y in the provided diagram.	

Takeaways

Here are some interesting things I noticed about right triangles today, by drawing the altitude of the triangle from the vertex at the right angle to the hypotenuse:

Vocabulary

- altitude
- **geometric mean**

Bold terms are new in this lesson.

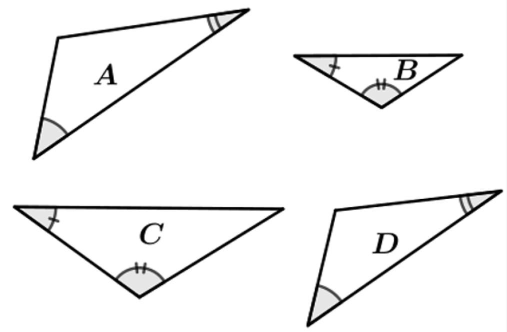
Lesson Summary

In today's lesson, we learned that drawing the altitude of a right triangle from the vertex at the right angle to the hypotenuse divides the right triangle into two smaller triangles that are similar to each other and to the original right triangle. We were able to prove the Pythagorean Theorem using proportionality statements about the three similar triangles.

Retrieval

1. What are the two ways to determine if two figures are similar?

2.

Which of the following are similar to each other? Why?	

3. An arithmetic sequence is represented in the table below. Find the missing values and write an explicit function rule for the sequence.

Term	1	2	3	4
Value	3			24

Explicit function rule: ________

4. A geometric sequence is represented in the table below. Find the missing values and write an explicit function rule for the sequence.

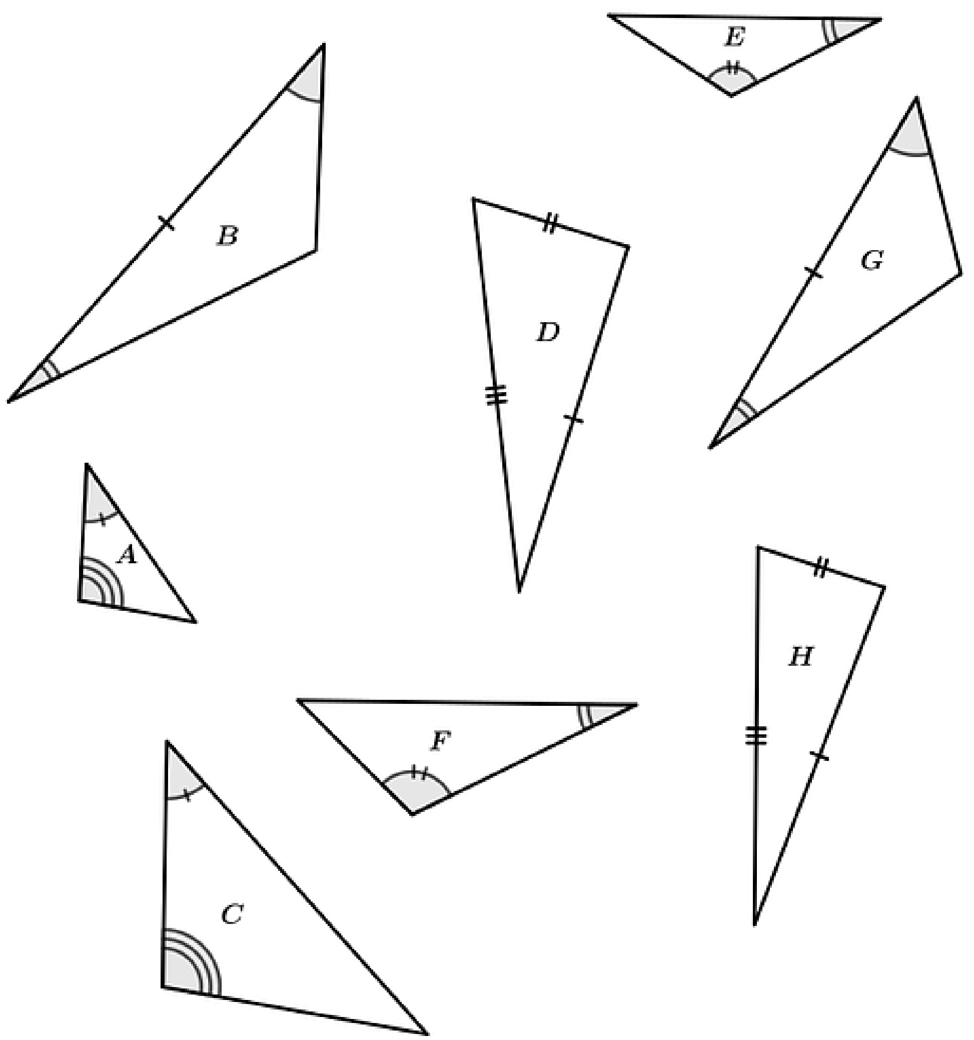
Term	1	2	3	4
Value	3			192

Explicit function rule: ________

Unit 5, Lesson 7 – Ready, Set, Go

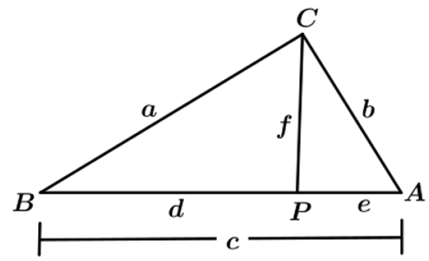
Ready

5. Determine which of the triangles below are similar and which are congruent. Justify your conclusions. Give your reasoning for the triangles you pick to be similar and congruent.



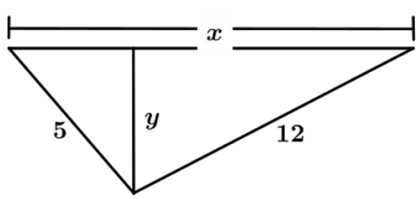
Set

3.

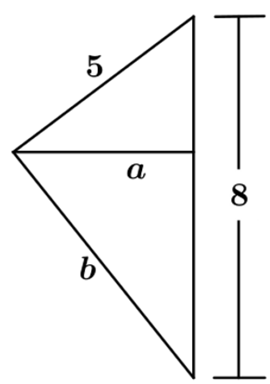
Using the right triangle with altitude drawn to the hypotenuse, prove the Pythagorean theorem, $a^2 + b^2 = c^2$.	

Find the missing values for each right triangle including the length of the altitude.

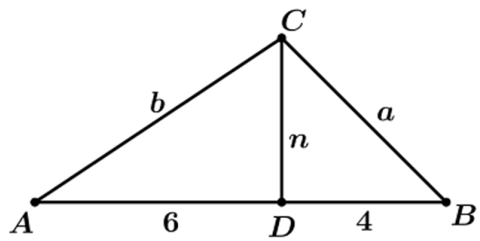
7.



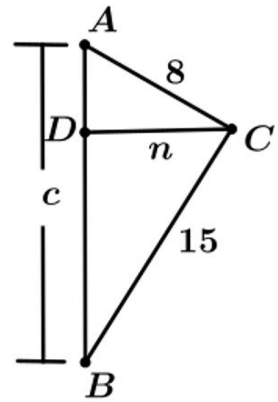
8.



9.

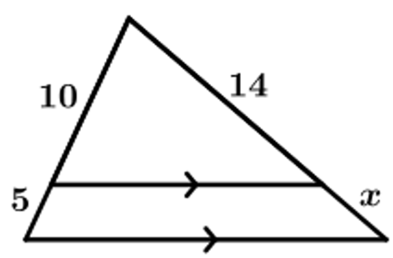


10.



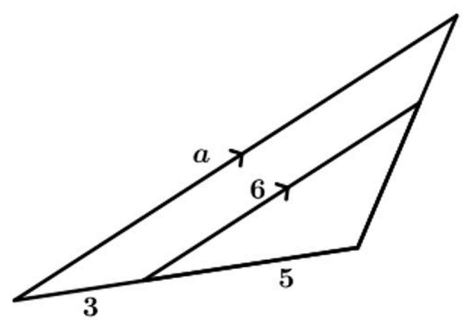
Go

In each problem, find the missing values using the similar triangles, parallel lines, and proportional relationships. Write a proportion and solve.



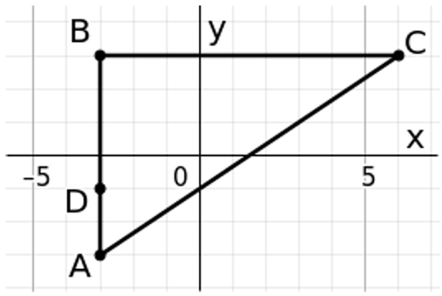
14.

15.



For questions 9–12, use the grids provided to determine the indicated segment measures and the proportions.

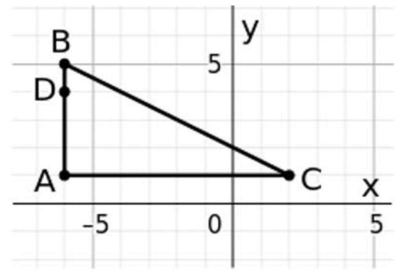
9.



a. When a line is drawn parallel to $\overline{AC}$ through point D, where will it intersect $\overline{BC}$?

b. What is the ratio of the lengths of the two segments created by point D?

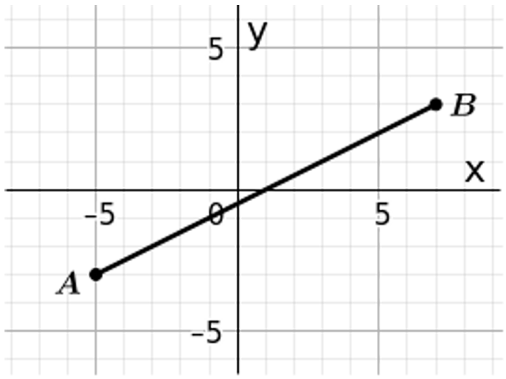
10.



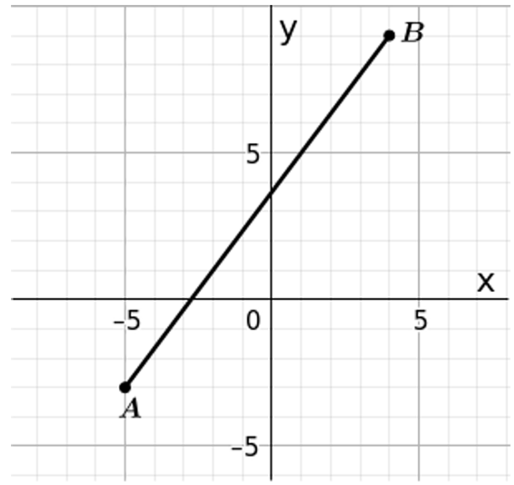
c. When a line is drawn parallel to $\overline{BC}$ through point D, where will it intersect $\overline{AC}$?

d. What is the ratio of the lengths of the two segments created by point D?

11. Place point C on the segment to split it into two segments with lengths that are a ratio of $1:2$.



12. Place point C on the segment to split it into two segments with lengths that are a ratio of $3:1$.



Analyze each table closely and determine the missing values based on the given information and values in the table. Create an explicit function rule for the sequence.

25. An Arithmetic Sequence

Explicit function rule: ________

Term	1	2	3	4
Value	7			22

26. A Geometric Sequence

Explicit function rule: ________

Term	1	2	3	4
Value	7			56

27. An Arithmetic Sequence

Explicit function rule: ________

Term	5	6	7	8
Value	10			43

28. A Geometric Sequence

Explicit function rule: ________

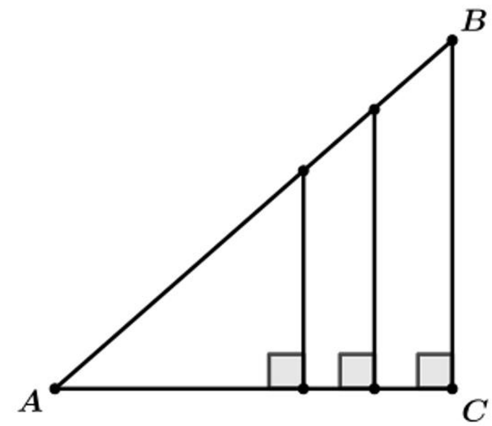
Term	7	8	9	10
Value	3			24

Unit 5, Lesson 8

Are Relationships Predictable?

Jump Start

In the previous task, one possible arrangement of the right triangles may have looked like this:



1. How does this arrangement of the triangles suggest that the three triangles are similar?

2. How did we justify that the triangles were similar?

3. If we had chosen to use a different-sized rectangle, such as a 3×5 card, or a 5×8 card, or an 8.5×11 inch sheet of paper, and cut along the diagonal to form the hypotenuse of the largest triangle, and then cut the second triangle formed by that same rectangle along the altitude drawn from the right angle to form two smaller triangles, would we have been able to arrange the three triangles in this same way? Why or why not?

Learning Focus

Investigate corresponding ratios of right triangles with the same acute angle.

What determines if two right triangles are similar?

Why are the ratios of sides in right triangles so special that they deserve a classification of their own (trigonometric ratios)?

Open Up the Math: Launch, Explore, Discuss

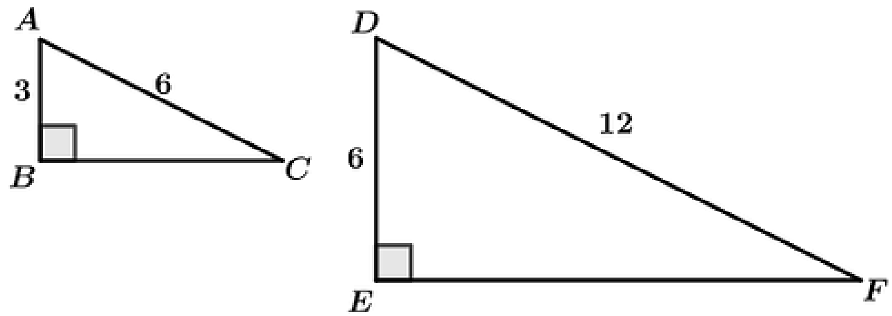
Draw a right triangle with an acute angle that measures $60°$.

Measure each side of your triangle as accurately as you can with a centimeter ruler.

Using the $60°$ angle as the angle of reference, list the measure for each of the following: Length of the adjacent side: _______ Length of the opposite side: _______ Length of the hypotenuse: _______	Create the following ratios using your measurements: $\frac{\text{opposite side}}{\text{hypotenuse}} =$ _______ $\frac{\text{adjacent side}}{\text{hypotenuse}} =$ _______ $\frac{\text{opposite side}}{\text{adjacent side}} =$ _______

1. Compare your ratios with others that had a triangle of a different size. What do you notice? Explain any connections you find to others' work.

2. In the right triangles provided, find the missing side length and then create the desired ratios based on the angle of reference (angle A and angle D).



List the ratios for $\triangle ABC$ using angle A as the angle of reference.	List the ratios for $\triangle DEF$ using angle D as the angle of reference.
$\frac{\text{opposite side}}{\text{hypotenuse}} =$	$\frac{\text{opposite side}}{\text{hypotenuse}} =$
$\frac{\text{adjacent side}}{\text{hypotenuse}} =$	$\frac{\text{adjacent side}}{\text{hypotenuse}} =$
$\frac{\text{opposite side}}{\text{adjacent side}} =$	$\frac{\text{opposite side}}{\text{adjacent side}} =$

3. What do you notice about the ratios from the two given triangles? How do these ratios compare to the ratios from the triangle you made for problem 1?

4. What can you infer about the angle measures of $\triangle ABC$ and $\triangle DEF$

5. Why do the relationships you have noticed occur?

6. Measure the angles in both $\triangle ABC$ and $\triangle DEF$. What can you conclude about the ratio of sides of a right triangle that have these specific angles?

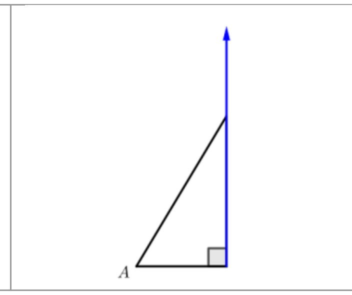
7. What might you conclude about the ratio of sides of a right triangle that has a $45°$ angle? If necessary, draw and experiment with examples of such a triangle.

The ratios we have worked with in this task are given special names.

- The **sine** is the ratio of the length of the side opposite an angle to the length of the hypotenuse.
- The **cosine** is the ratio of the length of the side adjacent to an angle to the length of the hypotenuse.
- The **tangent** is the ratio of the length of the side opposite an angle to the length of the side adjacent to the angle.

Ready for More?

Use the diagram to make observations about the sine, cosine and tangent ratios to answer the following questions:



a. As the reference angle A gets bigger, will the value of the sine ratio increase or decrease?

b. What happens to the value of the sine ratio when the reference angle A is close to $0°$?

c. What happens to the value of the sine ratio when the reference angle A is close to $90°$

d. As the reference angle A gets bigger, will the value of the cosine ratio increase or decrease?

e. What happens to the value of the cosine ratio when the reference angle A is close to $0°$?

f. What happens to the value of the cosine ratio when the reference angle A is close to $90°$?

g. As the reference angle A gets bigger, will the value of the tangent ratio increase or decrease?

h. What happens to the value of the tangent ratio when the reference angle A is close to $0°$?

i. What happens to the value of the tangent ratio when the reference angle A is close to $45°$?

j. What happens to the value of the tangent ratio when the reference angle A is close to $90°$?

Takeaways

The ratios of sides of a right triangle are called trigonometric ratios, and each specific ratio is given a name.

The ratio $\frac{\text{opposite side}}{\text{hypotenuse}}$ is called ________________.

The ratio $\frac{\text{adjacent side}}{\text{hypotenuse}}$ is called ________________.

The ratio $\frac{\text{opposite side}}{\text{adjacent side}}$ is called ________________.

The measure of the acute reference angle in a right triangle determines the value of these ratios, because ___________________________________

Before calculators, these values were recorded in tables for reference. For each, a table of trigonometric ratios would include these values:

$\sin(60°) =$ _ $\cos(60°) =$ _ $\tan(60°) =$ _

$\sin(30°) =$ _ $\cos(30°) =$ _ $\tan(30°) =$ _

$\sin(45°) =$ _ $\cos(45°) =$ _ $\tan(45°) =$ _

Adding Notation, Vocabulary, and Conventions

In a right triangle, we have given the name _______________ to the side opposite the right angle. It is always the longest side of the right triangle.

We will also give special names to the two legs of a right triangle, relative to one of the acute angles of the triangle.

In the diagram, $\overline{AC}$ is the side _______________ to $\angle A$ and $\overline{BC}$ is the side _______________ $\angle A$.

In the diagram, $\overline{AC}$ is the side _______________ $\angle B$ and $\overline{BC}$ is the side _______________ to $\angle B$.

I can quickly recognize how to name each of the following sides of a right triangle by…

Hypotenuse: _______________

Adjacent side: _______________

Opposite side: _______________

Vocabulary

- **adjacent**
- hypotenuse

- **opposite side in a triangle**
- **reference angle**
- **trigonometric ratios in right triangles: sine A, cosine A, tangent A**

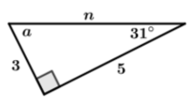
Bold terms are new in this lesson.

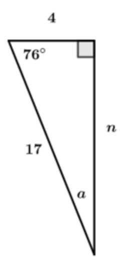
Lesson Summary

In this lesson, we learned about some special ratios, called trigonometric ratios, that occur in right triangles. If two right triangles have a pair of corresponding acute angles that are congruent, the right triangles will be similar. Therefore, corresponding ratios of the sides of these two right triangles will be equal. This observation is so useful when working with right triangles that have the same acute angle that values of these ratios were recorded in tables for each acute angle between 0° and 90°.

Retrieval

Find the missing angle measurement and side length in each right triangle.

1. 

2. 

Find the factored form and the intercepts for each quadratic function.

3. $g(x) = x^2 - 4x - 21$

   Factored form: _______

   y-intercept: _______

   x-intercepts: _______

4. $h(x) = x^2 + 8x + 12$

   Factored form: _______

   y-intercept: _______

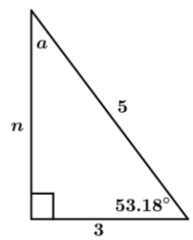
   x-intercepts: _______

Unit 5, Lesson 8 – Ready, Set, Go

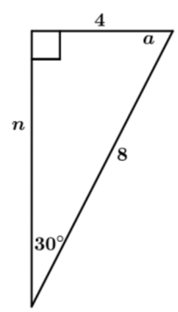
Ready

Find the missing side, n, and the missing angle, a.

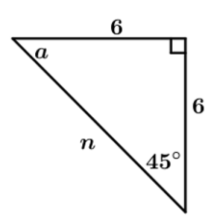
3.



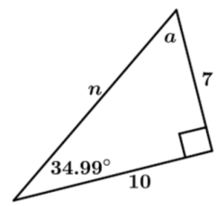
4.



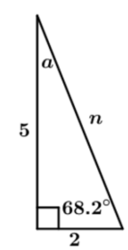
5.



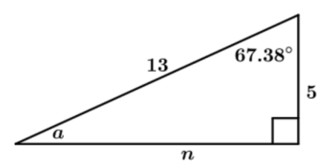
6.



7.



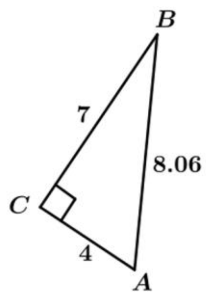
8.



Set

Write the indicated trigonometric ratios. If any sides of the triangle are missing, find them before determining the ratio.

16.

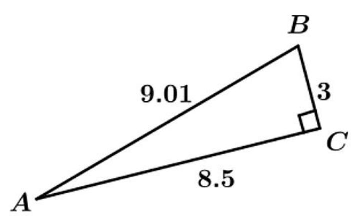


$\cos(A) =$ _______, $\cos(B) =$ _______

$\sin(A) =$ _______, $\sin(B) =$ _______

$\tan(A) =$ _______, $\tan(B) =$ _______

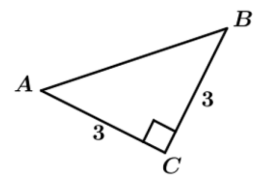
17.



$\cos(A) =$ _______, $\cos(B) =$ _______

$\sin(A) =$ _______, $\sin(B) =$ _______

$\tan(A) =$ _______, $\tan(B) =$ _______

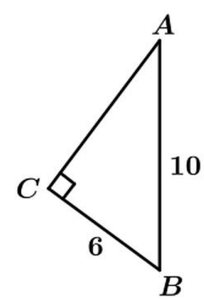


18.

$\cos(A) =$ _______, $\cos(B) =$ _______

$\sin(A) =$ _______, $\sin(B) =$ _______

$\tan(A) =$ _______, $\tan(B) =$ _______



19.

$\cos(A) =$ _______, $\cos(B) =$ _______

$\sin(A) =$ _______, $\sin(B) =$ _______

$\tan(A) =$ _______, $\tan(B) =$ _______

Go

15. $f(x) = x^2 + 9x + 20$

    factored form: _______

    x-intercepts: _______

    y-intercept: _______

16. $g(x) = x^2 + 2x - 15$

    factored form: _______

    x-intercepts: _______

    y-intercept: _______

17. $h(x) = x^2 - 49$

    factored form: _______

    x-intercepts: _______

    y-intercept: _______

18. $r(x) = x^2 - 13x + 30$

    factored form: _______

    x-intercepts: _______

    y-intercept: _______

19. $f(x) = x^2 + 20x + 100$

    factored form: _______

    x-intercepts: _______

    y-intercept: _______

20. $g(x) = x^2 - 8x - 48$

    factored form: _______

    factored form: _______

    x-intercepts: _______

    y-intercept: _______

21. $h(x) = x^2 + 16x + 64$

    factored form: _______

    x-intercepts: _______

    y-intercept: _______

22. $k(x) = x^2 - 36$

    factored form: _______

    x-intercepts: _______

    y-intercept: _______

23. $p(x) = x^2 - 2x - 24$

    factored form: _______

    x-intercepts: _______

    y-intercept: _______

Unit 5, Lesson 9

Relationships with Meaning

Learning Focus

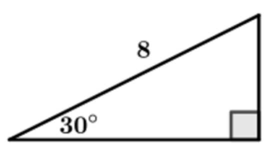
Examine properties of trigonometric expressions.

What observations can I make about the relationships between trigonometric ratios of the two different reference angles in a right triangle?

How do the properties of a right triangle influence algebraic statements involving trigonometric expressions?

Open Up the Math: Launch, Explore, Discuss

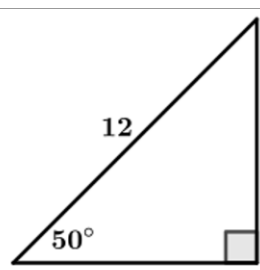
1. The sine ratio for a $30°$ angle is 0.5. We write this as $\sin(30) = 0.5$. Use this information to find the missing sides in the right triangle:



2.
    a. You can find the values of the trigonometric ratios on a scientific or graphing calculator.

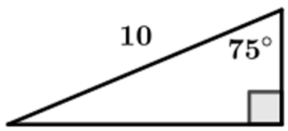
       Find the cosine and sine ratios for $50°$ on a calculator: $\cos(50) \approx$ _ , $\sin(50) \approx$ _ .

    b. Use this information to find both of the missing sides in the right triangle:

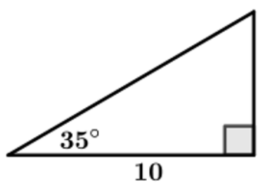


3. Use your calculator to find the information you need in order to find the desired side in the following right triangles. Record all of your work, including the trigonometric ratios you use, so someone else can follow it.

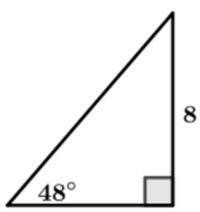
   a. Find the side adjacent to the 75° angle.



   b. Find the side opposite the 35° angle.

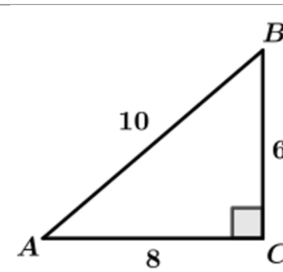


   c. Find the hypotenuse.



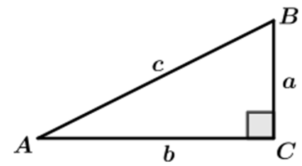
4.

Use the information from the given triangle to write the following trigonometric ratios:



$\sin(A) = \frac{\text{opposite}}{\text{hypotenuse}} =$ _ $\cos(A) = \frac{\text{adjacent}}{\text{hypotenuse}} =$ _ $\tan(A) = \frac{\text{opposite}}{\text{adjacent}} =$ _	$\sin(B) =$ _ $\cos(B) =$ _ $\tan(B) =$ _

5.

Use the information from the given triangle to write the following trigonometric ratios:	

$\sin(A) = \_$ $\cos(A) = \_$ $\tan(A) = \_$	$\sin(B) = \_$ $\cos(B) = \_$ $\tan(B) = \_$

6. Use the information from the previous two problems to write observations you notice about the relationships between trigonometric ratios of the two different reference angles in these right triangles.

7. Do you think these observations will always hold true? Why or why not?

The following is a list of conjectures made by students about right triangles and trigonometric relationships. For each, state whether you think the conjecture is true or false. Justify your answer.

8. $\cos(A) = \sin(A)$

9. $\tan(A) = \frac{\sin(A)}{\cos(A)}$

10. $\sin(A) = \cos(90° - A)$

11. $\cos(A) = \sin(B)$

12. $\cos(B) = \sin(90° - A)$

13. $\tan(A) = \frac{1}{\tan(B)}$

Note the following convention used to write $[\sin(A)]^2 = \sin^2(A)$.

20. $\sin^2(A) + \cos^2(A) = 1$

22. $\sin^2(A) = \sin(A^2)$

21. $1 - \sin^2(A) = \cos^2(A)$

Ready for More?

Given a right triangle with the following trigonometric ratio: $\sin(30°) = \frac{1}{2}$, find all of the trigonometric ratios for this triangle. How do you know these values are always going to be true when given this angle?

Takeaways

When given the measures of an acute angle and one side of a right triangle, we can find the other sides by:

- ______________________________
- ______________________________
- ______________________________
- ______________________________

An identity is an algebraic equation that is true for all replacements of the variable. When working with trigonometric ratios, some surprising results occur. The following trigonometric identities capture some of the observations we made today for the acute angles in a right triangle:

Vocabulary

- **complementary angles**

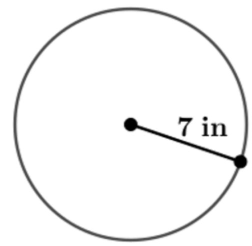
Bold terms are new in this lesson.

Lesson Summary

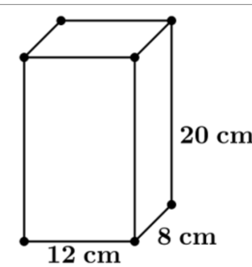
In this lesson, we examined some relationships between trigonometric ratios, such as a relationship between the sine and cosine of complementary angles. We were able to use the properties of a right triangle, including the Pythagorean theorem that describes a relationship between the lengths of the sides, to justify the observations we made today.

Retrieval

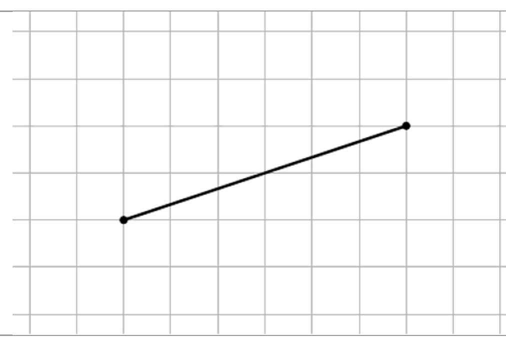
1.

Find the circumference and area of the circle.	

2.

Find the surface area and volume of the rectangular prism.	

3.

What is the difference between the slope of the line segment and the length of the line segment? Find both measurements.	

Unit 5, Lesson 10

Finding the Value of a Relationship

Learning Focus

Solve for the missing side and angle measures in a right triangle.

While we can find missing side lengths and angles in a right triangle if two sides or two angles are known using the Pythagorean theorem or the angle sum theorem for triangles, what information do we need to know to find missing measures in a right triangle now that trigonometric ratios are available as a computational tool?

How do we use trigonometry to make indirect measurements when the object can't be measured directly?

Open Up the Math: Launch, Explore, Discuss

Part 1: Pick a side

Andrea and Bonita are resting under their favorite tree before taking a nature walk up a hill. Both girls have been studying trigonometry in school, and now it seems like they see right triangles everywhere. For example, Andrea notices the length of the shadow of the tree they are sitting under and wonders if they can calculate the height of the tree just by measuring the length of its shadow.

Bonita thinks they also need to know the measure of an angle, so she checks an app on her phone and finds that the angle of elevation of the sun at the current location and time of day is 50°. In the meantime, Andrea has paced off the length of the tree's shadow and finds that it is 40 feet long.

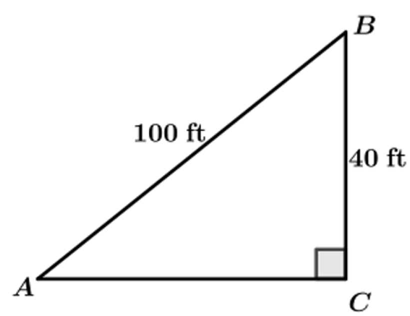
1. How might Andrea and Bonita use this information, along with their knowledge of trigonometric ratios, to calculate the height of the tree? (Andrea and Bonita know they can find the value of any trigonometric ratio they might need for any acute angle using a calculator.)

Part 2: What's your angle?

After their rest, Andrea and Bonita are going for a walk straight up the side of the hill. Andrea decided to stretch before heading up the hill while Bonita thought this would be a good time to get a head start. Once Bonita was 100 feet away from Andrea, she stopped to take a break and looked at her GPS device that told her that she had walked 100 feet and had already increased her elevation by 40 feet. With a bit of time to waste, Bonita wrote down the trigonometric ratios for $\angle A$ and for $\angle B$.

2. Name the trigonometric ratios for $\angle A$ and for $\angle B$ that involve the given sides.

When Andrea caught up, she said, "What about the unknown angle measures? When I was at the bottom and looked up to see you, I was thinking about the 'upward' angle measure from me to you. Based on your picture, this would be $\angle A$." Bonita wrote the trigonometric ratio $\sin A = \frac{40}{100} = \frac{2}{5}$ and asked, "So, how do we find angle A?"



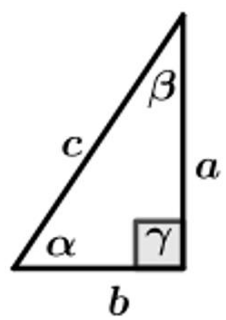
Together, the girls talked about how this was like thinking backward: instead of knowing an angle and using their calculators to find a trigonometric ratio like they did while working on the height of the tree problem, they now know the trigonometric ratio and need to find an unknown angle value. Bonita notices the $\sin^{-1}(\theta)$ button on her calculator and wonders if this might work like an "inverse trigonometric ratio" button, undoing the ratio to produce the angle. She decides to try it out and gets the following output on her calculator:

$$\sin A = \frac{2}{5} \sin^{-1}\left(\frac{2}{5}\right) = 23.578 \sin(23.578) = 0.4$$

3. How might this output convince Bonita that her assumption about the calculator was correct?

4. Use the trigonometric ratio you found for $\cos B$ to find the value of $\angle B$.

5. Find all unknown values for the given right triangle:



a. $\angle\alpha =$_______

b. $\angle\beta =$_______

c. $\angle\gamma = 90°$

d. $a = 12$ m

e. $b = 8$ m

f. $c =$_______

6. Bonita and Andrea started talking about all of the ways to find unknown values in right triangles and decided to make a list. What do you think should be on their list? Be specific and precise in your description. For example, "trigonometric ratios" is not specific enough. You may use the following sentence frame to assist with writing each item in your list:

When given _______________, you can find _______________ by ____.

Part 3: Angle of elevation and angle of depression

During their hike, Andrea mentioned that she looked up to see Bonita. In mathematics, when you look straight ahead, we say your line of sight is a horizontal line. From the horizontal, if you look up, the angle from the horizontal to your line of sight is called the angle of elevation. Likewise, if you are looking down, the angle from the horizontal to your line of sight is called the angle of depression.

7. After looking at this description, Andrea mentioned that her angle of elevation to see Bonita was about $23.5°$. They both agreed. Bonita then said her angle of depression to Andrea was about $66.5°$. Andrea agreed that Bonita was describing an angle of depression, but said Bonita's angle of depression was also $23.5°$. Who do you think is correct? Use drawings and words to justify your conclusion.

8. What conclusion can you make regarding the angle of depression and the angle of elevation? Why?

Ready for More?

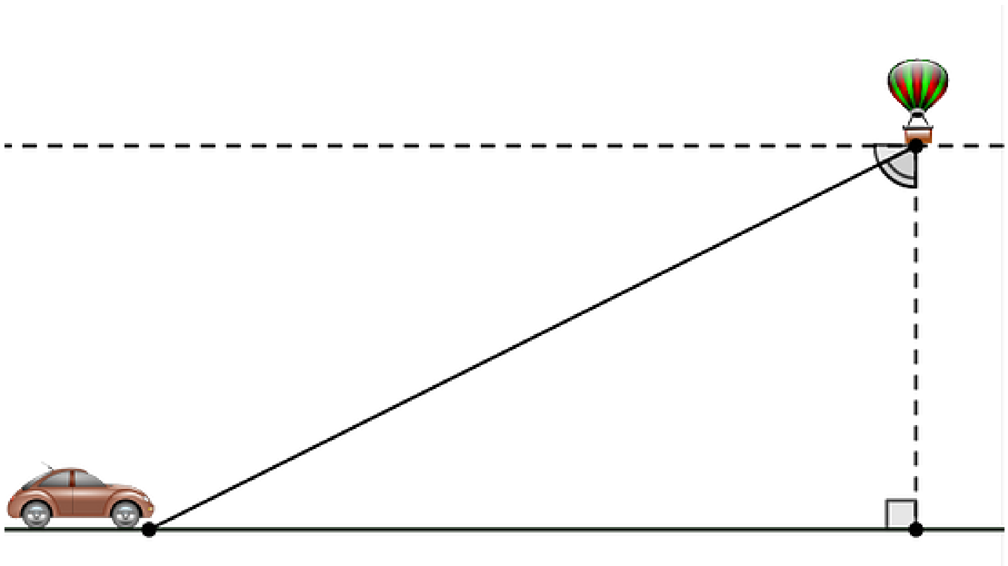
At night, as you walk away from a 10 ft high lamppost, your shadow extends farther and farther in front of you. Is there a position where you might stand so that your shadow is exactly as long as you are tall, since your height measures 5 ft? If you then walk twice as far away from the lamppost, will your shadow be twice as long? Use diagrams to help you think about this situation.

Takeaways

Make a list of all of the ways to find unknown values in right triangles. Be specific and precise in your description. For example, “trigonometric ratios” is not specific enough. You may use the following sentence frame to assist with writing each item in your list:

When given _______________, you can find _______________ by __________________________________.

Identify the angle of elevation and the angle of depression in the following diagram:



Vocabulary

- **angle of depression/angle of elevation**
- **inverse trigonometric ratio**

Bold terms are new in this lesson.

Lesson Summary

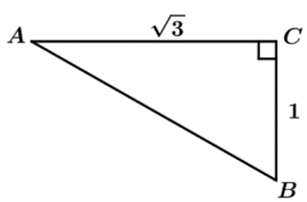
In this lesson, we extended our strategies for finding unknown sides and angles in a right triangle beyond using the Pythagorean theorem and the angle sum theorem for triangles, since sometimes we don’t have enough information in terms of side lengths or angle measures to use these theorems. We found that trigonometric ratios are useful in solving for unknown sides and that inverse trigonometric relationships are useful for finding unknown angles in a right triangle. Adding these tools allows us to find all of the missing sides and angles in a right triangle given two pieces of information: two sides of the triangle or one side and an angle.

Retrieval

1. Sketch a picture of this situation and label as much of the picture as possible.

   Tiana is standing in the bottom of a canyon with cliff walls that seem to reach the sky. She decided that she might try and measure one of them. So, Tiana measures a distance from the base of one of the cliffs out 200 ft and stands there. She then looks up to the top and finds that the angle from the ground to the top of the cliff is $68°$.

2. Use the right triangle to find the missing side length and angle measurements, as well as the desired ratios.



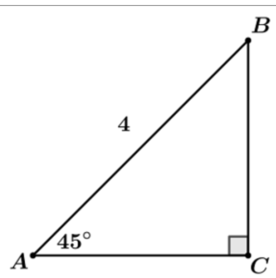
$\sin(A) =$ _______, $\sin(B) =$ _______

$\cos(A) =$ _______, $\cos(B) =$ _______

$\tan(A) =$ _______, $\tan(B) =$ _______

$m\angle A =$ _______, $m\angle B =$ _______

3. Use the given information in the right triangle to find the missing side lengths.

Unit 5, Lesson 10 – Ready, Set, Go

Ready

For each story presented, sketch a picture of the situation and label as much of the picture as possible.

4. Jill put a ladder up against the house to try to reach a light that is out and needs to be changed. She knows the ladder is 10 feet long, and the distance from the base of the house to the bottom of the ladder is 4 feet.

5. Francis is flying an airplane at an altitude of $3,000$ feet, when he begins a descent toward the ground. The plane has an angle of descent that is 15°.

6. Abby is standing at the top of a very tall skyscraper and looking through a telescope at the scenery all around her. The angle of decline on the telescope says 35°, and Abby knows she is 30 floors up and each floor is 15 feet tall.

Set

For problems 4–7, use the sketches you made in problems 1–3, or make new sketches, to help you find the missing values.

8. Jill put a ladder up against the house to try to reach a light that is out and needs to be changed. She knows the ladder is 10 feet long and the distance from the base of the house to the bottom of the ladder is 4 feet. How high does the ladder reach up on the side of the house? What is the angle of elevation formed by the ladder and the ground?

9. Francis is flying an airplane at an altitude of $3{,}000$ feet, when he begins a descent toward the ground. If the angle of descent of the plane is 15°, how far will the plane travel through the air before it is on the ground?

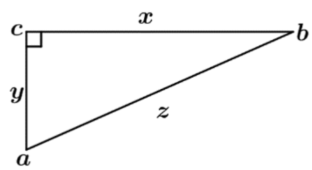
10. Abby is standing at the top of a very tall skyscraper and looking through a telescope at the scenery all around her. The angle of decline on the telescope says 35°, and Abby knows she is 30 floors up and each floor is 15 feet tall. How far from the base of the building is the object that Abby is looking at?

11. A 6-foot-tall person is standing 60 feet away from a skyscraper looking up at the top of the building and wondering how tall the skyscraper is. The angle of elevation for the person's line of sight is 70°. Determine the height of the building.

Go

Use the given right triangle to identify the trigonometric ratios and angles.

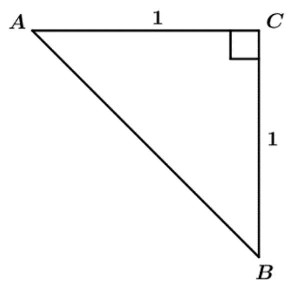
14.



$\sin(a) =$ _______, $\cos(a) =$ _______, $\tan(a) =$ _______

$\sin(b) =$ _______, $\cos(b) =$ _______, $\tan(b) =$ _______

15.

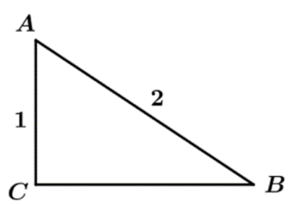


$\sin(A) =$ _______, $\cos(A) =$ _______, $\tan(A) =$ _______

$\sin(B) =$ _______, $\cos(B) =$ _______, $\tan(B) =$ _______

$m\angle A =$ _______, $m\angle B =$ _______

16.

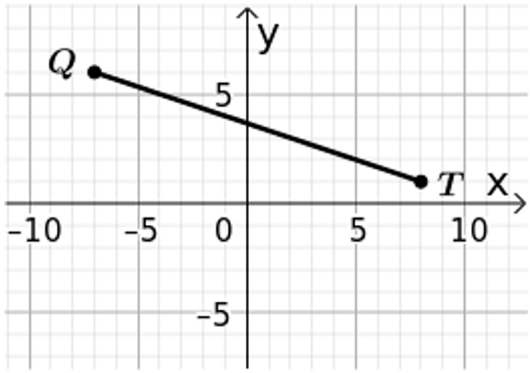


$\sin(A) =$ _______, $\cos(A) =$ _______, $\tan(A) =$ _______

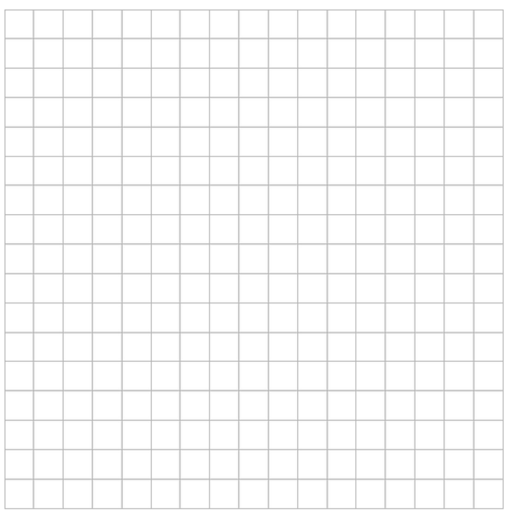
$\sin(B) =$ _______, $\cos(B) =$ _______, $\tan(B) =$ _______

$m\angle A =$ _______, $m\angle B =$ _______, $m\angle C = 90°$

17. Place a point on $\overline{QT}$ that splits the segment into two segments with lengths that have a ratio of $2:3$. Provide the coordinates of the point.



18. Draw a segment on the grid and place a point on the segment to split it into two segments with lengths that have a ratio of $2:5$.



Unit 5, Lesson 11

Using Trigonometric Relationships

Learning Focus

Solve application problems using trigonometry.

How do I apply trigonometric ratios to practical problems?

What are the essential elements of modeling a real-world context using a right triangle, even when only an imaginary right triangle exists?

Open Up the Math: Launch, Explore, Discuss

For each problem:

- Make a drawing to represent the situation
- Write an equation
- Solve (do not forget to include units of measure)

1. Carrie places a 10-foot ladder against a wall. If the ladder makes an angle of $65°$ with the level ground, how far up the wall is the top of the ladder?

2. A flagpole casts a shadow that is 15 feet long. The angle of elevation of the sun at this time is $40°$. How tall is the flagpole?

3. In Southern California, there is a six-mile section of Interstate 5 that decreases 2,500 feet in elevation as it descends Grapevine Hill in the Tejon Pass. What is the angle of descent?

Pause and Reflect

4. A hot air balloon is 600 feet above where it is planning to land and descending at a rate of 40 feet per minute. Sarah is driving over rough terrain at a speed of 10 miles per hour to meet the balloon when it lands. If the angle of elevation to the balloon is $20°$, how far away is Sarah from the place where the balloon will land? Who arrives at the landing spot first?

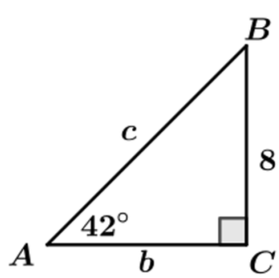
5. An airplane is descending as it approaches the airport. If the angle of depression from the plane to the ground is $7°$, and the plane is 6,000 feet above the ground, how far is the plane from the airport?

6. Michelle is 60 feet away from a building. The angle of elevation to the top of the building is $41°$. How tall is the building?

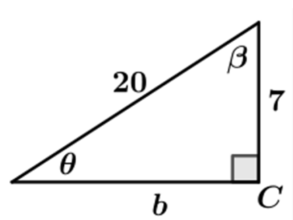
7. A ramp is used for loading equipment from a dock to a ship. The ramp is 10 feet long and the ship is 6 feet higher than the dock. What is the angle of elevation of the ramp?

For each right triangle, find all unknown side lengths and angle measures:

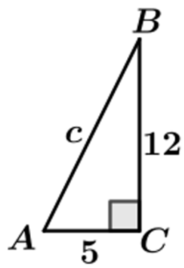
8.



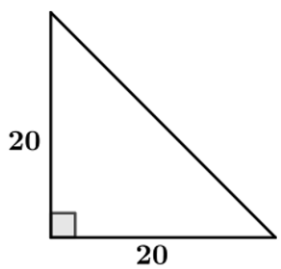
9.



10.



11.



12. Draw and find the missing angle measures of the right triangle whose sides measure 6, 8, and 10.

Determine the values of the two remaining trigonometric ratios when given one of the trigonometric ratios. Also find the measures of the acute angles of the triangle.

29. $\cos(\alpha) = \frac{3}{5}$

30. $\tan(\theta) = \frac{8}{3}$

31. $\sin(\beta) = \frac{4}{7}$

Ready for More?

Compare and contrast two different methods for answering problems like those in problems 13–15.

Illustrate both of these methods to find $\cos(A)$ and $\tan(A)$, given $\sin(A) = \frac{2}{3}$.

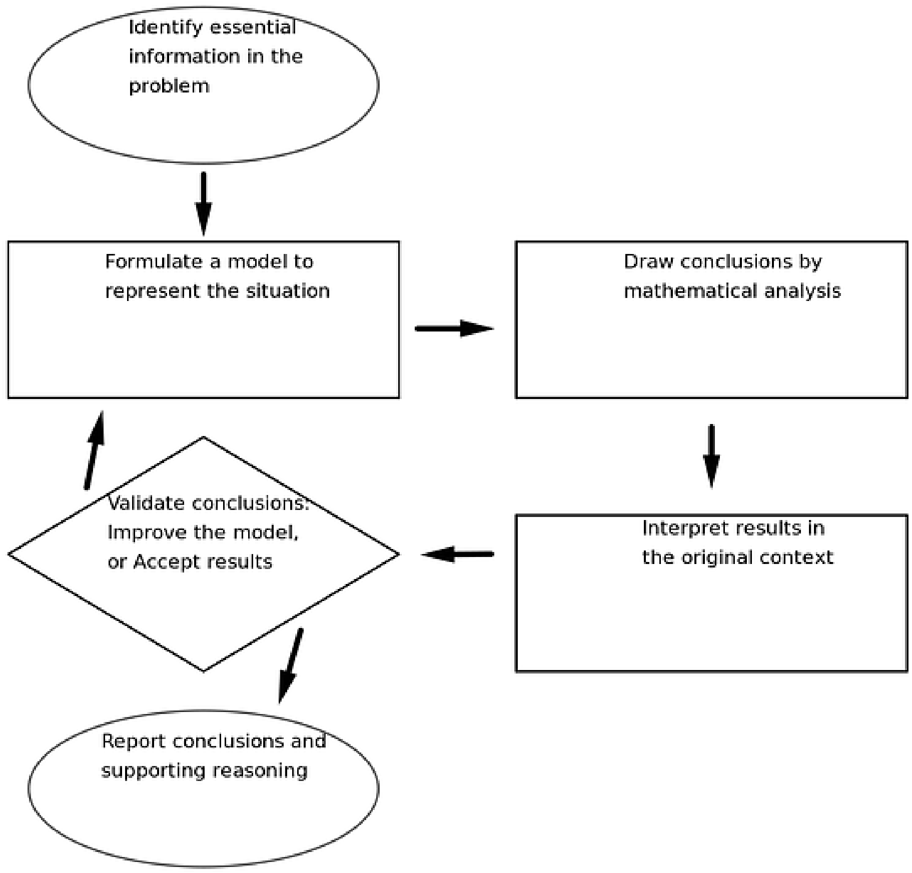
1. One method is to create a "reference triangle" from the information given in the trigonometric ratio.

Given: $\sin(A) = \frac{2}{3}$ $\cos(A) =$ $\tan(A) =$	

2. A second method is to use the trigonometric identities developed in this unit, such as: $\sin^2(A) + \cos^2(A) = 1$ or $\tan(A) = \frac{\sin(A)}{\cos(A)}$

Using trigonometric identities: $\cos(A) =$ $\tan(A) =$	

Takeaways



Vocabulary

- **model, mathematical**

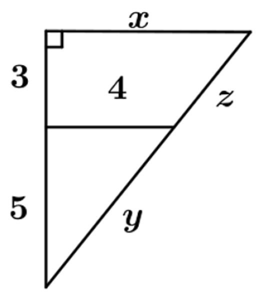
Bold terms are new in this lesson.

Lesson Summary

In this lesson, we learned about the modeling process and how to use right triangle trigonometry to model many different types of applications, even applications that didn't naturally include right triangles. A right triangle became a tool for representing a situation so we could draw upon trigonometric ratios and inverse trigonometric relationships to answer important problems in construction, aviation, transportation, and other contexts.

Retrieval

1. Find the missing values for the similar right triangles.



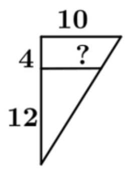
2. A doorstop, used as a wedge between the door and the floor to keep it open, is being designed with an angle of elevation of 11° and a height of 1 inch. How long should the bottom of the doorstop be? Sketch the situation and find the desired value.

Unit 5, Lesson 11 – Ready, Set, Go

Ready

Based on each set of similar triangles or parallel lines, create a proportion and solve it to find the missing values.

3.



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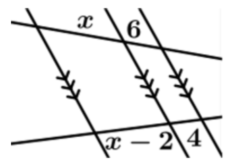

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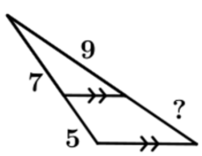

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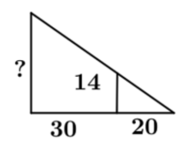

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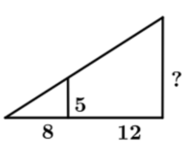
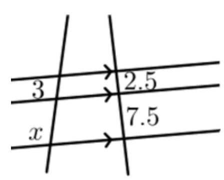
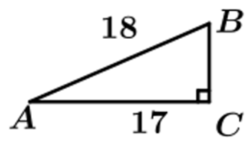


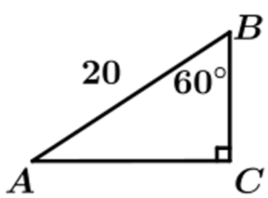
8.



Set

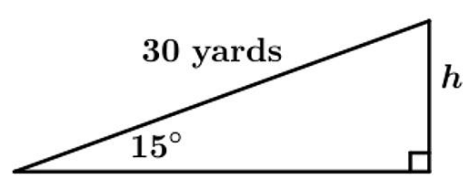
Solve each right triangle. Give any missing sides and missing angles.

9. 

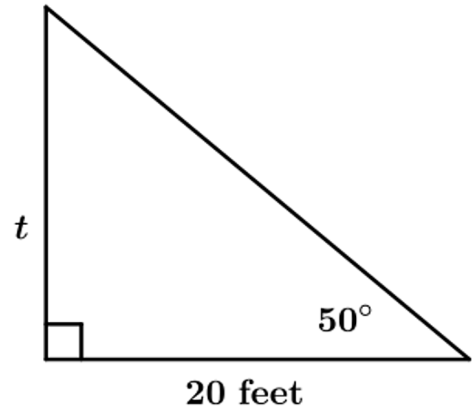
10. 

Use trigonometric ratios and the Pythagorean theorem to solve each problem.

13.

Jack is looking up the hill at Jill and wondering how much vertical increase there actually is from his position to her position on the hill. They know there is a 15° angle of incline, and Jack measures as he goes to meet Jill that he has gone 30 yards. Calculate the vertical distance.	

14.

Kim is trying to use the shadow of a large tree to measure the height of the tree. The length of the shadow has been measured as 20 feet and the angle of incline has been measured as $50°$. How tall is the tree?	

Model each of the situations with a right triangle and then solve the triangle to find the desired values.

24. Alex is standing on the top of a building looking out at a boat in the lake. The distance from the top of the building to the ground is 150 feet, and Alex is able to measure the angle from vertical to the line of sight as $70°$. How far from the base of the building is the boat?

Solution:	

25. The altitude of a plane is 6,000 feet and the angle of decent to the landing strip is $3°$. How much distance will the plane actually travel in its path through the air before it touches down?

Solution:	

Use the given trigonometric ratio, which is based on actual side length measures, to sketch a right triangle and solve the triangle.

32. $\sin(A) = \frac{1}{2}$

Solution:	Sketch:

33. $\cos(B) = \frac{3}{5}$

Solution:	Sketch:

34. $\tan(B) = \frac{6}{7}$

Solution:	Sketch:

35. $\sin(B) = \frac{7}{10}$

Solution:	Sketch:

Go

Sketch a drawing of the situation, then solve each problem.

17. Mark is building his son a pitcher's mound so he can practice for his upcoming baseball season in the backyard. Mark knows the league requires an incline of $12°$ and an elevation of 8 inches in height. How long will the front of the pitcher's mound need to be?

Solution:	Sketch:

18. Susan is designing a wheelchair ramp. Wheelchair ramps require a slope that is no more than 1 inch of rise for every 12 inches of ramp length. Susan wants to determine how much horizontal distance a ramp of 6 feet in length will span. She also wants to know the degree of incline from the base of the ramp to the ground.

Solution:	Sketch:

19. Michael is designing a house with a roof pitch of 5. Roof pitch is the number of inches a roof will rise for every 12 inches of run. What is the angle that will need to be used in building the trusses and supports for the roof? At the peak of the roof, what angle will there be when the front and the back of the roof come together?

Solution:	Sketch: