9C: Angle Sum & Difference Formulas

We have learned about several different families of trigonometric identities that allow us to rewrite expressions that involve trigonometric functions of one unknown angle. We now want to consider how torewrite a function that involves the sum or difference of two angles. Specifically, we want to find ways to simplify the functions

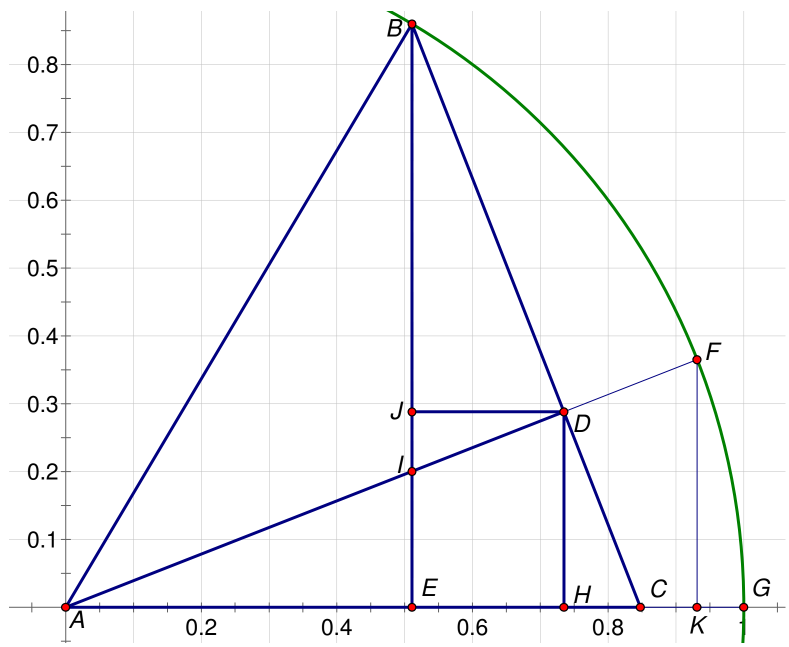
$$\sin(u+v), \qquad \sin(u-v), \qquad \cos(u+v), \qquad \cos(u-v)$$

by rewriting them as functions of the angle measures u and v.

Angle Sum Puzzle

The six pieces that you were given to build a triangle give us the triangle to the right. This is a triangle constructed in the unit circle, as shown to the right, such that $m\angle CAB = (u+v)$.

To construct the triangle, we first draw radii $\overline{AG}$, $\overline{AF}$, and $\overline{AB}$ to make adjacent angles of u and v. We then will construct $\overline{BE}$ perpendicular to the x-axis, and $\overline{BC}$ perpendicular to $\overline{AF}$. Finally, we construct $\overline{JD}$ and $\overline{DH}$ perpendicular to $\overline{BE}$ and the x-axis.

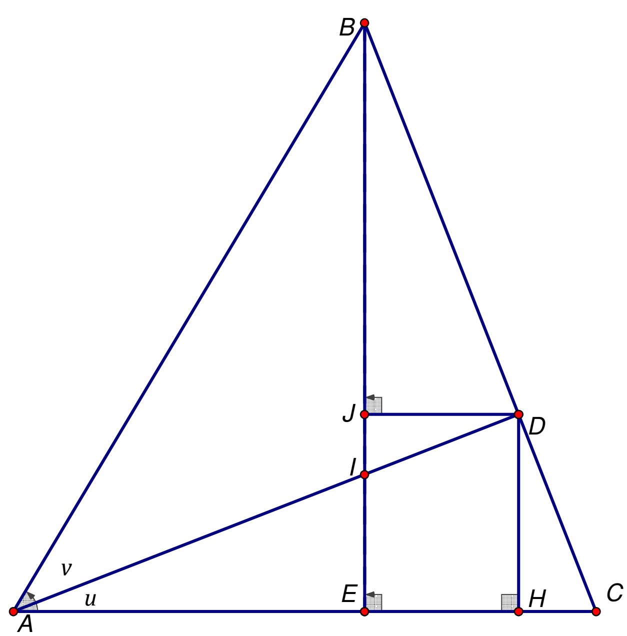


Exploring the Triangle

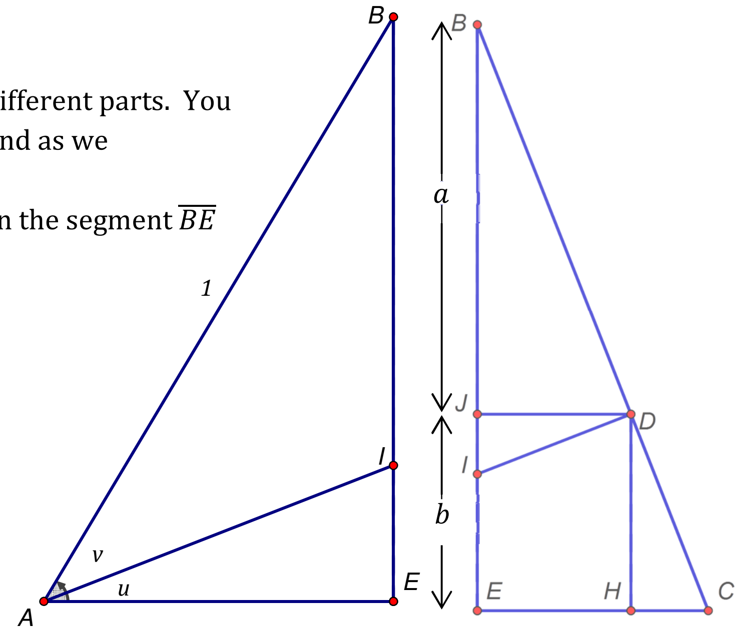
Now, let's explore the triangle to see what important discoveries we can make. As we investigate different parts of the triangle, use your puzzle pieces to break ΔABC into different parts to help you see what's going on.

1. Begin by using your puzzle pieces to find all of the angles that have the measure u. Mark these angles on the figure to the right.

Extra Challenge: *Can you prove that* $m\angle DBJ = u$*?*



2. Now let's analyze the triangle by breaking it into different parts. You may find it useful to move your puzzle pieces around as we investigate these smaller triangles.
First, split the original triangle into two parts down the segment $\overline{BE}$ as shown to the right.



Consider ΔABE.

a. Find $m\angle A =$____________________

b. Using ΔABE, find the following using trigonometric ratios of the angle $(u+v)$. *Note that* $AB = 1$.

$BE =$

$AE =$

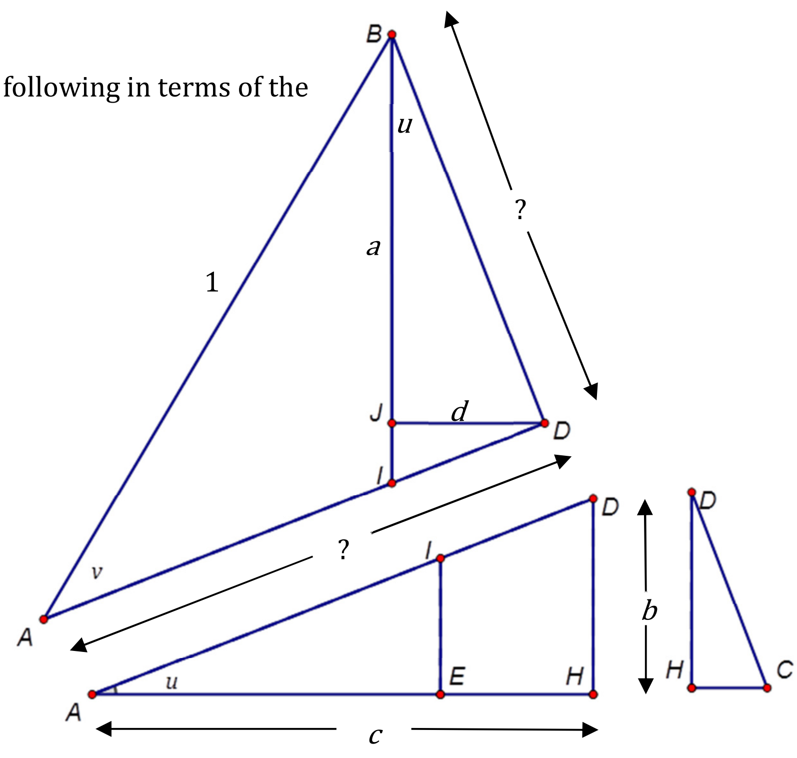
3. Now, separate the original triangle into 3 smaller triangles as shown below. Notice that these are all right triangles.

a. Use *trigonometric ratios* to find the following in terms of the angle measure v in ΔABD:

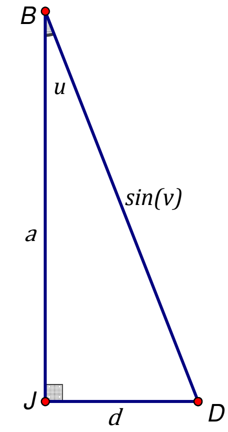
$BD =$

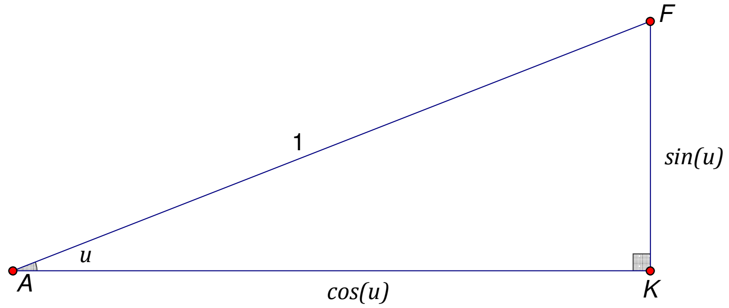
$AD =$

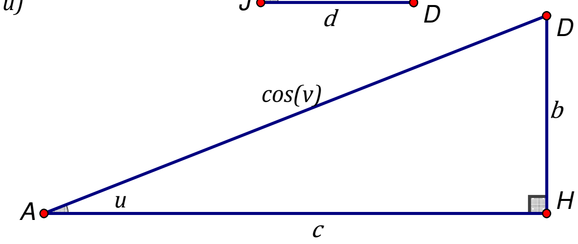
Label these on the diagram.



4. **Now we consider the lengths labeled $a, b, c,$ and d for the following.** To find these values, we will compare ΔAFK (in top figure on pg. 1) with ΔADH and ΔBDJ.







Observe that all of these triangles are *similar triangles* because of their angles.

a. What is the scale factor that shrinks ΔAFK to ΔBDJ? ___________

b. Use this scale factor to find an expression for a and d in ΔBDJ:

$a =$ $\qquad\qquad$ $d =$

c. What is the scale factor that shrinks ΔAFK to ΔADH? __________

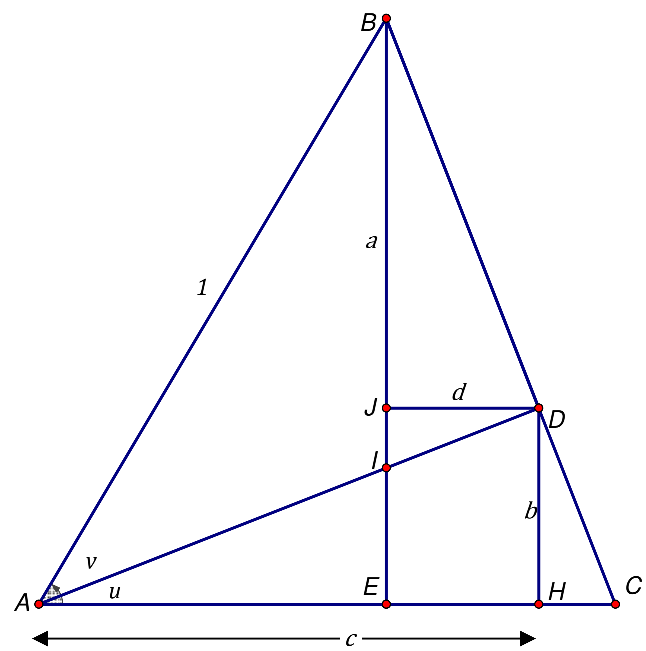
d. Use this scale factor to find an expression for b and c in ΔADH:

$b =$ $\qquad\qquad$ $c =$

5. Now, let's put it all together!
Looking back at the original triangle, we see that

$$BE = a + b$$

Use what we found in steps #2 and #4 above to substitute for BE, a, and b and get an equation *in terms of trig. ratios*.



6. In the original triangle, we also see that

$$AE = c - d$$

Use what we found in steps #2 and #3 above to substitute for AE, c, and d and get an equation *in terms of trig. ratios.*

Angle Sum Formulas

We now have two angle sum formulas!

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

Try These

1. Use the angle sum formulas to find the exact answer for each of the following. Try rewriting the angle as the sum of two familiar angles (like $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, etc.$ or their decimal equivalent.)

   a) $\sin(75°) = \sin(\qquad + \qquad) =$

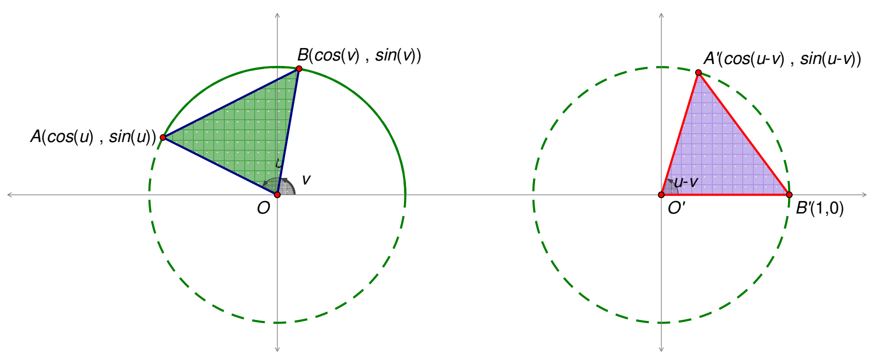
   b) $\cos\frac{7\pi}{12} = \cos(\qquad + \qquad) =$

Exploring Difference Formulas

Sine Difference identity: We now need to find a difference identity for $\sin(u-v)$. Use the sum identity to transform $\sin(u-v)$ into an expression with only the sines and cosines of u and v.

$$\sin(u-v) = \sin\big(u+(-v)\big)$$

Cosine Difference identity



We can find the cosine difference identity the same way, but let's look at a slightly different way to find a formula for $\cos(u-v)$.

In the figures above, the two shaded triangles are congruent, but the second triangle is moved into standard position. We now see that $AB = A'B'$. We will define $\theta = (u-v)$, an using the distance formula, we get the equation below. Simplify it to see what we end up with.

$$\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2} = \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2}$$

Sine and Cosine Sum and Difference Identities

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$
$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

*Note: Notice that the sign in the RHS of the **cosine** identity are opposite of the sign of the LHS. However, the sign is the same for both RHS and LHS of the sine angle-sum identity.*

Try These

a) Use a sum or difference identity to verify that $\cos 0 = 1$

a) Write $\cos\frac{\pi}{5}\cos\frac{\pi}{3} - \sin\frac{\pi}{5}\sin\frac{\pi}{3}$ as a single sine or cosine expression

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b) Prove that $\sin(u+v) - \sin(u-v) = 2\cos u \sin v$

Assignment 9C: Angle Sum Formulas

Use a sum or difference identity to find the exact value.

1. $\sin 15°$

2. $\cos 75°$

3. $\cos \frac{7\pi}{12}$

4. $\tan 15\,°$
   (Hint: remember $\tan x = \frac{\sin x}{\cos x}$*)*

Write the expression as the sine or cosine of a single angle.

5. $\sin 35° \cos 15° - \cos 35° \sin 15°$

6. $\cos \frac{\pi}{5} \cos \frac{\pi}{10} - \sin \frac{\pi}{5} \sin \frac{\pi}{10}$

7. $\sin 3x \cos x + \cos 3x \sin x$

8. Use a sum identity to verify that $\sin \frac{\pi}{2} = 1$

9. Use a difference identity to verify that $\sin\left(\frac{\pi}{2} - u\right) = \cos u$