9D: Multiple-Angle Identities

Exercises

1. Use the appropriate sum or difference identity to prove the double-angle identity.

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\begin{aligned}\cos 2x &= \cos(x + x)\\ &= \cos x \cos x - \sin x \sin x\\ &= \cos^2 x - \sin^2 x\end{aligned}$$

2. Use the result from #1 to show the following:

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \cos^2 x - (1 - \cos^2 x)$$
$$= \cos^2 x - 1 + \cos^2 x$$
$$= 2\cos^2 x - 1$$

3. Use the result for #2 to derive a formula for $\cos^2 x$ in terms of $\cos 2x$

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x + 1 = 2\cos^2 x$$
$$\frac{\cos 2x + 1}{2} = \cos^2 x$$

4. Finally, use the result from #3 to derive a formula for $\cos\left(\frac{u}{2}\right)$ in terms of $\cos u$.

(Hint: begin by letting $x = \frac{u}{2}$.)

$$\cos^2\left(\frac{u}{2}\right) = \frac{\cos\left(2 \cdot \frac{u}{2}\right) + 1}{2}$$

$$\cos u = \pm\sqrt{\frac{\cos u + 1}{2}}$$

Find all the solutions to the equation in the interval $[0,2\pi)$

5. $\sin 2x = \cos x$

$$2\sin x\cos x = \cos x$$
$$2\sin x\cos x - \cos x = 0$$
$$\cos x\,(2\sin x - 1) = 0$$
$$\cos x = 0, \qquad or \quad 2\sin x - 1 = 0$$
$$x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}\right\}$$

6. $\cos 2x = \sin x$

$$1 - 2\sin^2 x = \sin x$$
$$0 = 2\sin^2 x + \sin x - 1$$
$$0 = (2\sin x - 1)(\sin x + 1)$$
$$0 = 2\sin x - 1, \qquad or \;\; 0 = \sin x + 1$$
$$\sin x = \frac{1}{2}, \qquad or \sin x = -1$$
$$\boldsymbol{x = \left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}}$$

7. $\cos^2 x + \cos x = \cos 2x$

$$\cos^2 x + \cos x = \cos 2x$$
$$\cos^2 x + \cos x = 2\cos^2 x - 1$$
$$0 = \cos^2 x - \cos x - 1$$

Consider the equation $0 = y^2 - y - 1$, where $y = \cos x$. Using the quadratic formula:

$$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$
$$\cos x = \frac{1+\sqrt{5}}{2} = 1.62 \;\Rightarrow\; \textit{no solution}$$
$$\cos x = \frac{1-\sqrt{5}}{2} = -.6180 \;\Rightarrow\; x = \cos^{-1}(-.6180) \approx \mathbf{128.17}°$$
$$\textit{There is another answer!} \ldots \;\; x = 360 - 128.17 = \mathbf{231.83}°$$

8. Write the expression as one involving only $\sin\theta$ and $\cos\theta$.

$$\sin 2\theta + \cos 3\theta$$

$$\sin 2x + \sin 3x = 2\sin x\cos x + \cos(x + 2x) = 2\sin x\cos x + \cos x\cos 2x - \sin x\sin 2x$$
$$= 2\sin x\cos x + \cos x\,(\cos^2 x - \sin^2 x) - \sin x\,(2\sin x\cos x)$$
$$= \mathbf{2\sin x\cos x + \cos^3 x - \sin^2 x}\cos \boldsymbol{x} \mathbf{- 2\sin^2 x\cos x}$$
$$= \mathbf{2\sin x\cos x + \cos^3 x - 3\sin^2 x}\cos \boldsymbol{x}$$

Prove the identity.

9. $\sin 4x = 2\sin 2x \cos 2x$

$$\begin{aligned}\sin(4x) &= \sin\big(2(2x)\big)\\ &= 2\sin(2x)\cos(2x)\end{aligned}$$

10. $2\csc 2x = \csc^2 x \tan x$

$$\begin{aligned}2\csc 2x &= \csc^2 x \tan x\\ \frac{2}{\sin 2x} &= \frac{1}{\sin^2 x}\cdot\frac{\sin x}{\cos x}\\ \frac{2}{2\sin x\cos x} &= \frac{1}{\sin x\cos x}\\ \frac{1}{\sin x\cos x} &= \frac{1}{\sin x\cos x}\end{aligned}$$

11. $\sin 3x = (\sin x)(4\cos^2 x - 1)$

$$\begin{aligned}&\sin(3x) = \sin(x + 2x)\\ &= \sin x\cos 2x + \cos x\sin 2x\\ &= \sin x\,(2\cos^2 x - 1) + \cos x\,(2\sin x\cos x)\\ &= \sin x\,\big((2\cos^2 x - 1) + 2\cos^2 x\big)\\ &= \sin x\,(4\cos^2 x - 1)\end{aligned}$$

Use half-angle identities to find an exact value without a calculator.

12. $\sin 15°$ (answer will be positive because the angle is in first quadrant.)

$$\sin 15 = \sin\left(\frac{30}{2}\right) = \sqrt{\frac{1-\cos 30}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \boldsymbol{\frac{\sqrt{2-\sqrt{3}}}{2}}$$

13. $\cos 75°$ (answer will be positive because the angle is in first quadrant.)

$$\cos 75° = \cos\left(\frac{150}{2}\right) = \sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \boldsymbol{\frac{\sqrt{2-\sqrt{3}}}{2}}$$

14. $\tan\frac{\pi}{8}$ (answer will be positive because the angle is in first quadrant.)

$$\tan\frac{\pi}{8} = \tan\left(\frac{\frac{\pi}{4}}{2}\right) = \frac{1-\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}-1}{1} = \boldsymbol{\sqrt{2}-1}$$