10B.1: Dot Products

We have seen how to multiply a scalar and a vector and add two vectors. Now we will see how multiplication of two vectors gives us an important product called the ***dot product*** of two vectors. The dot product is the simplest case of a larger class of operators called ***inner products*** that play a key role in advanced mathematics. The dot product will be used to find the angle between two vectors and to find important components of physical situations like force and work.

The Dot Product

The standard definition of the Euclidean dot product is

Definition: The **dot product** (a.k.a. *inner product*) of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is $$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Try These: Find the dot product of the following vectors.

a) $\langle 3, -4 \rangle \cdot \langle -2, 5 \rangle$

b) $(2\mathbf{i} - 3\mathbf{j}) \cdot (5\mathbf{i} + 4\mathbf{j})$

Now that we have the definition, we will investigate some useful properties of dot products.

Properties of Dot Products

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, c be a scalar, and $\mathbf{0} = \langle 0,0 \rangle$ be the zero vector.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
3. $\mathbf{0} \cdot \mathbf{u} = 0$
4. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
   $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
5. $(c\,\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

Try These Proofs:

Prove the properties above by letting $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and c be a scalar.

Property 1:

$\mathbf{u} \cdot \mathbf{v} =$

Property 2:

$\mathbf{u} \cdot \mathbf{u} =$

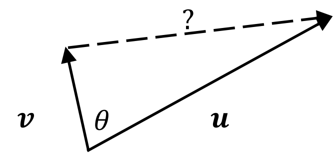
Try This: *Use Property 2 to find the length of the vector* $\mathbf{u} = \langle 5, -12 \rangle$.

Aside: *It can also be useful to think of the dot product as the product of two vectors in matrix form as below.*

Alternate Matrix-form Definition: When thinking of vectors in matrix form, we have
$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2.$$

Finding the Angle Between Vectors

Consider this:



a) Describe the dotted vector in the diagram to the right in terms of **u** and **v.**

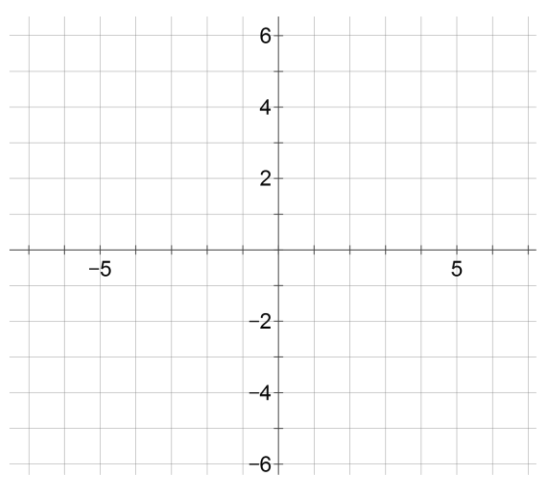
b) Use the law of cosines to write an equation relating the sides and the angle θ. Then solve this equation for $\cos\theta$.

Angle Between Two Vectors:
If θ is the angle between two nonzero vectors **u** and **v**, then
$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}, \qquad \textit{and} \qquad \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

Try These: *Graph the vectors and find the angle between the two given vectors*

a) $\mathbf{u} = \langle -3, -1 \rangle, \mathbf{v} = \langle -4, 1 \rangle$

b) $\mathbf{w} = \langle 5, -2 \rangle,\ \mathbf{r} = \langle 2, 5 \rangle$



c) *What is the geometric relationship between* $\mathbf{w}$ *and* $\mathbf{r}$ *in (b) above?*

d) *Use the formula,* $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$*, to explain what has to be true to make* $\theta = 90$.

Definition of Orthogonal Vectors: The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal *if and only if* $\mathbf{u}\cdot\mathbf{v} = 0$

Note: "Orthogonal" is another way of saying "Perpendicular". However, it is a broader term since the zero vector is orthogonal to all vectors, but not perpendicular to them. Furthermore, if a vector has more than 3 dimensions (like $\langle 1,1,1,1\rangle$*), the term "perpendicular" does not have a physical meaning.*

Try These:

a) Find the value of x that makes the vectors orthogonal: $\mathbf{u} = \langle x, 4\rangle$, and $\mathbf{v} = \langle 3,7\rangle$

b) Find $\mathbf{u}\cdot\mathbf{v}$ satisfying the given conditions where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\theta = 150°, |\mathbf{u}| = 3, |\mathbf{v}| = 8$$

Assignment 10B.1-Dot Products

Let $\mathbf{u} = \langle 3,6 \rangle$, $\mathbf{v} = \langle -4,3 \rangle$, and $\mathbf{w} = \langle 3,4 \rangle$

1. Find the following dot products
   a. $\mathbf{u} \cdot \mathbf{v}$

   b. $\mathbf{u} \cdot \mathbf{w}$

   c. $\mathbf{v} \cdot \mathbf{w}$

2. Find the angle between each pair of vectors:
   a. $\mathbf{u}, and\ \mathbf{v}$

   b. $\mathbf{u}, and\ \mathbf{w}$

   c. $\mathbf{v}, and\ \mathbf{w}$

3. Which vectors pairs above are orthogonal? Explain.

4. Let $\mathbf{r} = \langle x, -2 \rangle$. Find the value of x that will make $\mathbf{u}$ and $\mathbf{r}$ orthogonal.

5. Let $\mathbf{s} = \langle -5, y \rangle$. Find the value of y that will make $\mathbf{v}$ and $\mathbf{s}$ orthogonal.

6. Find $\mathbf{u} \cdot \mathbf{v}$ satisfying the given conditions where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\theta = 100°, |\mathbf{u}| = 6, |\mathbf{v}| = 9$$