3A: Solving Quadratic Equations

When solving quadratic equations, we have several methods that we can use depending on the terms of the quadratic expression. Below is a summary of these methods with an example.

Standard Form of a quadratic equation: $\boldsymbol{ax^2 + bx + c = 0}$ ***for integers*** $\boldsymbol{a, b, c}$.

Method 1: Square Root Method

$$ax^2 + c = 0$$

Example: Solve $4x^2 - 27 = 0$

$4x^2 - 27 = 0$

$4x^2 = 27$

$x^2 = \frac{27}{4}$ Isolate the x^2

$x = \pm\sqrt{\frac{27}{4}}$ Square Root

$x = \pm\frac{3\sqrt{3}}{2}$ Simplify

Method 2: Factoring

The factoring method employs the *zero-product rule* which states:

"If $pq = 0$, then either $p = 0$, or $q = 0$."

When $\boldsymbol{a = 1}$***:*** $x^2 + bx + c = 0$

Strategy: Find two numbers m and n such that

$$m + n = b, \quad \textit{and} \quad m \cdot n = 0$$

then we write

$$x^2 + bx + c = (x + m)(x + n) = 0$$

and use the zero product rule.

Example: *Solve* $x^2 - 2x - 7 = 8$

$x^2 - 2x - 7 = 8$

$x^2 - 2x - 15 = 0$ Make one side equal to 0

$(x - 5)(x + 3) = 0$ Factor the quadratic expression

$x - 5 = 0, \quad or \quad x + 3 = 0$ Set each factor equal to zero.

$x = 5, \quad or \quad x = -3$ Solve each sub-equation to get solutions.

When $\neq 1$ ***, use the "ac" method:*** $ax^2 + bx + c = 0$

Example: *Solve* $2x^2 + x - 15 = 0$

$2x^2 + x - 15 = 0$	First make sure that the equation is equal to 0.
	Compute the value of ac $ac = 2(-15) = -30$
$2x^2 + 6x - 5x - 15 = 0$	Find two numbers that have a product of $ac = -30$ and a sum of $b = 1$. They are -5 and 6. Split the bx term into a sum with these coefficients.
$2x(x+3) - 5(x+3) = 0$ $(2x-5)(x+3) = 0$	Factor the expression by *grouping*.
$2x - 5 = 0, \quad or \;\; x + 3 = 0$	Set both factors equal to zero.
$x = \frac{5}{2}, \quad or \quad x = -3$	Solve the sub-equations.

Square Trinomials: $a^2x^2 + 2abx + b^2 = c$

Example: *Solve* $9x^2 + 30x + 25 = 40$

$9x^2 + 30x + 25 = 40$	First check to see if the left side is a quadratic trinomial in the form $a^2x^2 + 2abx + b^2$ *... This one is!*
$(3x+5)^2 = 40$	Factor the expression.
$3x + 5 = \pm\sqrt{40}$	Square root (don't forget the $\pm$ roots)
$3x = \pm\sqrt{40} - 5$	Isolate the x variable to solve.
$x = \frac{\pm 2\sqrt{10} - 5}{3}$	

Method 3: Completing the Square

Use for any quadratic equation in standard from $ax^2 + bx + c = 0$.

Example: Solve $3x^2 + 24x - 21 = 0$

$3x^2 + 24x - 21 = 0$	
$3x^2 + 24x = 21$	Isolate constant term c
$x^2 + 8x = 7$	Divide by $a = 3$
$x^2 + 8x + 16 = 23$	Add $(b/2)^2$ to both sides, where b is the coefficient on x
$(x+4)^2 = 23$	Factor the perfect square trinomial
$x + 4 = \pm\sqrt{23}$	Square root both sides
$x = \pm\sqrt{23} - 4$	Subtract 4 to solve.

Method 4: The Quadratic Formula

Use for any quadratic in standard form $ax^2 + bx + c = 0$.

The solutions to $ax^2 + bx + c = 0$ can be found using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example: Solve $2x^2 + 4x - 5 = 6$

$$2x^2 + 4x - 5 = 6$$

$$2x^2 + 4x - 11 = 0$$ Set equation equal to 0

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(2)(-11)}}{2(2)}.$$ Use coefficients $a = 2, b = 4, c = -11$ for the formula.

$$x = \frac{-4 \pm \sqrt{16 + 88}}{4}$$

$$x = \frac{-4 \pm \sqrt{104}}{4}$$

Simplify *carefuly!*

$$x = \frac{-4 \pm 2\sqrt{26}}{4}$$

$$x = \frac{-2 \pm \sqrt{26}}{2}$$

Complex Solutions

In the set of real numbers the square root of all real numbers is a real number. However, the square root of a negative real number is undefined. This is because when we square any real number, we get a non-negative number.

So, we have to define a new type of number called an ***imaginary number***. So, we define

$$\sqrt{-1} = i$$

to be our imaginary unit. An *imaginary number* is any multiple of i.

For example:

Write the $\sqrt{-4}$ and $\sqrt{-8}$ using the imaginary unit i.

$$\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2i$$

$$\sqrt{-8} = \sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{2} = 2 \cdot i \cdot \sqrt{2} = 2i\sqrt{2}$$

A ***complex number*** has two parts: a real part and an imaginary part of the form

$$\underbrace{a}_{real\ part} + \underbrace{bi}_{imag.\ part}$$

When solving quadratic equations, we sometimes end up with no real solutions but we have imaginary solutions.

Example. Use the square root method to solve $4x^2 + 20 = 10$

$$4x^2 + 20 = 10$$

$$x^2 = \frac{-10}{4}$$

$$x = \sqrt{\frac{-10}{4}} = \frac{\sqrt{-10}}{2} = \frac{\sqrt{10}}{2} \cdot \sqrt{-1} = \frac{\sqrt{10}}{2}i$$

Pre-Calculus Name: Date: Period:

Assignment 3A

Solve using the square root method

1. $3(x+3)^2 - 5 = 31$

2. $2x^2 + 9 = 5$

3. $(x-4)^2 = -9$

Solve by Factoring

1. $x^2 + 15x - 34 = 0$

2. $x^2 - 12x = -35$

3. $4x^2 + 12x + 9 = 0$

4. $2\,x^2 + 5\,x + 3 = 0$

5. Try this: Factor the left side *before* setting equal to 0.
   $9\,x^2 - 24\,x + 16 = -5$

Solve by completing the square

6. $2x^2 + 8x - 5 = 7$

7. $x^2 + 6x = -25$

8. $4x^2 + 8x + 20 = 0$

Solve using the quadratic formula

9. $4x^2 - 3x - 5 = 2$

10. $5x^2 - 4x = 5$

11. $3x^2 + x + 1 = 0$

12. $-5x^2 + 6x + 3 = 5$