1B: Domain/Range and Intervals

When working with functions, we often need to describe sets of numbers that can be inputs and outputs. Two of the most important sets that we need to know when working with functions are

Domain: the set of all possible inputs for a function
Range: the set of all possible outputs for a function.

Two useful ways to do this are called **set builder notation** or **interval notation**. In this lesson, we will learn about when and how to use these different notations.

Number Families

For any mathematical situation, we need to know what types of numbers we can use.

Consider this: *Daily Temperatures*

Suppose we were to record the average temperature for a certain geographic location for each day of the year.

a) What are the two variables for this relation?

b) Which variable is the ***dependent*** variable?
Which variable is the ***independent*** variable?

c) Describe all the possible numbers for the independent variable (i.e. the domain):

d) Describe all the possible numbers for the dependent variable (i.e. the range):

To better describe sets of numbers, we need to define some special sets of numbers:

Number set	Notation	Description
Natural Numbers	$\mathbb{N}$	1, 2, 3, 4, 5, …
Whole Numbers	$\mathbb{W}$	0, 1, 2, 3, 4, 5, …
Integers	$\mathbb{Z}$	… -3, -2, -1, 0, 1, 2, 3, …
Rational Numbers	$\mathbb{Q}$	Can be written as a ratio of integers: $\left\{\frac{a}{b}\middle\| a, b, \in \mathbb{Z}\right\}$; Repeating or terminating decimals
Irrational Numbers	-	Non-repeating, non-terminating decimals e.g. $\sqrt{2}, -\sqrt{5},\ \pi,\ e,\ \phi\ (the\ golden\ ratio)$
Real Numbers	$\mathbb{R}$	The set of all numbers that are rational or irrational.

Back to Daily Temperatures: Consider the daily temperatures in the previous example.

a) The days in this example are the ***integers*** 1, 2, 3, …, 366

Instead of writing out all the possibilities, we can use **set builder notation** to describe the set:

$$D = \{d \mid 1 \leq d \leq 366\,, d \in \mathbb{Z}\}$$

where the symbol $\in$ mean "is in" or "is an element of", the symbol $\mid$ is read "such that", and $\mathbb{Z}$ is the integers. (We could have used the natural numbers $\mathbb{N}$.)

b) The temperatures could be any reasonable ***Real Numbers***.

Now, since theoretically the lowest possible temperature (which cannot be reached) is $-273.15°C$ and the highest possible temperature is approximately $10^{32}°C$, we could say

$$T = \{t \mid -273.15 < t \leq 10^{32}, t \in \mathbb{R}\}$$

We could also **graph** these numbers on a number line.

Since this set is a subset of the Real numbers, we can also write this in **interval notation** which shows the *lower boundary* and the *upper boundary* points or the set. We use a "hard bracket" [or] if a boundary is included and a "soft bracket" (or) if a boundary is not included. So, the temperature set would be

$$(-273.15, 10^{32}]$$

Bounded and Unbounded intervals

- If an interval has a maximum and a minimum number, it is called a **bounded interval.** For example:

$$\{x \mid 1 < x < 5, x \in \mathbb{R}\} \rightarrow (1,5)$$
$$\{x \mid 1 < x \leq 5, x \in \mathbb{R}\} \rightarrow (1,5]$$
$$\{x \mid 1 \leq x < 5, x \in \mathbb{R}\} \rightarrow [1,5)$$
$$\{x \mid 1 \leq x \leq 5, x \in \mathbb{R}\} \rightarrow [1,5]$$

If an interval does not have a maximum or it does not have minimum, it is called a **unbounded interval**. For example:

$$\{x \mid 1 < x\,,, x \in \mathbb{R}\} \rightarrow (1, \infty)$$
$$\{x \mid x < 1\,, x \in \mathbb{R}\} \rightarrow (-\infty, 1)$$
$$\{x \mid 1 \leq x\,, x \in \mathbb{R}\} \rightarrow [1, \infty)$$
$$\{x \mid x \leq 1\,, x \in \mathbb{R}\} \rightarrow (-\infty, 1]$$

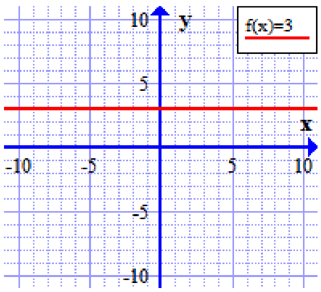
If we want to describe *All real numbers*, we can say $x \in \mathbb{R}$, or $(-\infty, \infty)$

Describing Domains and Ranges

Now we can use set builder notation and interval notation to describe the domain and range of functions. Since most functions involve Real numbers, interval notation will be the most useful.

<u>Try it:</u> Describe the domain and range for each of the following functions using interval notation

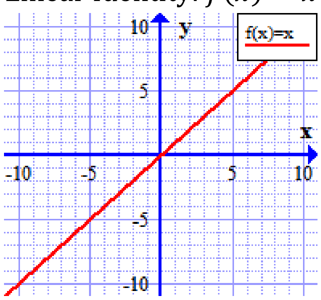
Linear Constant: $f(x) = 3$



Domain:

Range:

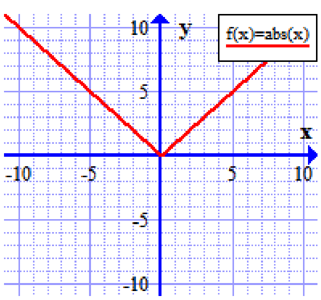
Linear Identity: $f(x) = x$



Domain:

Range:

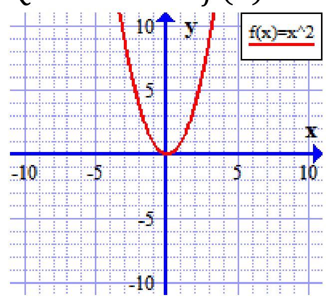
Absolute value: $f(x) = |x|$



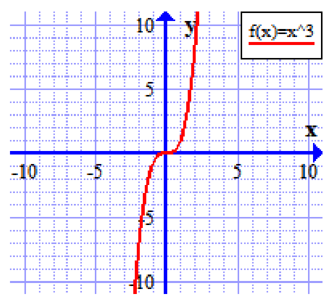
Domain:

Range:

Quadratic: $f(x) = x^2$



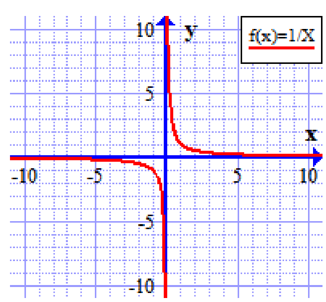
Cubic: $f(x) = x^3$



Domain:

Range:

Reciprocal: $f(x) = \frac{1}{x}$



Domain:

Range:

Domain from Equations. Remember, some functions have domain restrictions.

- Radical functions like $f(x) = \sqrt{x}$ must have a radicand (the thing under the radical) that is greater than or equal to zero.
- Rational functions like $g(x) = \frac{1}{x}$ with an x in the denominator cannot have a denominator of zero.

Example. Describe the domain of these functions in interval notation.

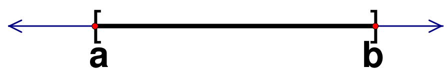
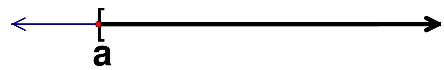
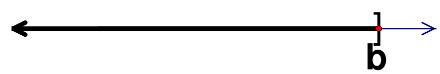
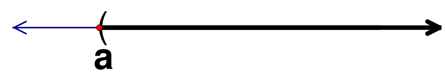
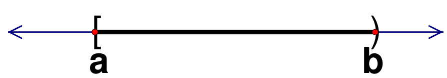
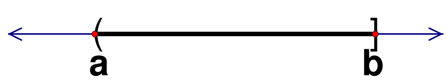
a) $f(x) = \sqrt{6 - x}$

b) $g(x) = \frac{1}{x-7}$

Interval Summary

Intervals can be classified in several ways. First, intervals involving infinity are “unbounded”, while intervals with two finite endpoints are called “bounded”. Secondly, if the finite endpoint(s) are included in the interval (shown by a hard-bracket or an *or equals to*), then the interval is “closed”. Otherwise, the interval is “open”.

The following is a table that summarizes the possible types of intervals shown in interval notation, as an inequality, and as a graph.

<table>
<tr><th></th><th>Bounded Interval</th><th colspan="2">Unbounded Interval</th></tr>
<tr><td rowspan="2">Closed</td><td rowspan="2">$[a, b]$
$a \leq x \leq b$
</td><td colspan="2">$[a, \infty)$
$x \geq a$
</td></tr>
<tr><td colspan="2">$(-\infty, b]$
$x \leq b$
</td></tr>
<tr><td rowspan="2">Open</td><td rowspan="2">(a, b)
$a < x < b$
</td><td colspan="2">(a, ∞)
$x > a$
</td></tr>
<tr><td colspan="2">$(-\infty, b)$
$x < b$
</td></tr>
<tr><td rowspan="2">Half-Open</td><td>$[a, b)$
$a \leq x < b$
</td><td rowspan="2">N/A</td><td rowspan="2">N/A</td></tr>
<tr><td>$(a, b]$
$a < x \leq b$
</td></tr>
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