Pre-Calculus Name: Date: Period:

Unit 2 Toolkit – Polynomial Functions

This toolkit is a summary of some of the key topics you will need to master in this unit.

2A: Quadratic Equations with Complex Solutions

When solving a quadratic equation, we have several methods that we can use. Choosing the best method will improve accuracy and efficiency of your solving. Here are some quick reminders of the different methods and when to use them:

- **Method 1: Square Root Method**
  Use when the equation in the form $ax^2 + c = 0$ or $a(x + p)^2 = q$. Solve for the square, then square root. ***Don't for get to use*** $\pm$ ***when you square root!***

- **Method 2: Factoring**
  The factoring method employs the zero-product rule which states:
  "If pq = 0, then either p = 0, or q = 0."

  a. ***When*** $\boldsymbol{a = 1}$***:*** $x^2 + bx + c = 0$
  Strategy: Find two numbers m and n such that
  $$m + n = b, \quad and \quad m \cdot n = 0$$
  then we write
  $$x^2 + bx + c = (x + m)(x + n) = 0$$
  and use the zero product rule.

  b. ***When*** $\neq$ **1** ***, use the "ac" method:*** $ax^2 + bx + c = 0$
  1. First make sure that the equation is equal to 0.
  2. Compute the value of ac
  3. Find two numbers that have a product of ac and a sum of b = 1.
  4. Split the bx term into a sum with these coefficients.
  5. Factor the expression by grouping.

  c. ***Square Trinomials:*** $a^2x^2 + 2abx + b^2 = c$
  First check to see if the left side is a quadratic trinomial in the form $a^2x^2 + 2abx + b^2$*… This one is!* Remember, square trinomials come from
  $$(a + b)^2 = a^2 + 2ab + b^2 \text{ and } (a - b)^2 = a^2 - 2ab + b^2$$

- **Method 3: Completing the Square**
  Use for any quadratic equation in standard from $ax^2 + bx + c = 0$.
  1. Isolate constant term c
  2. Divide by a
  3. Add $(b/2)^2$ to both sides, where b is the coefficient on x
  4. Factor the perfect square trinomial
  5. Finish with the square root method

- **Method 4: The Quadratic Formula**
  Use for any quadratic in standard form $ax^2 + bx + c = 0$. Set the equation equal to 0 first!
  The solutions to $ax^2 + bx + c = 0$ can be found using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2B: Completing the Square and the Vertex Formula

Vertex form of a quadratic function: $\boldsymbol{f(x) = a(x-h)^2 + k}$

> $\boldsymbol{a}$: This parameter changes the shape of the parabola by increasing the rate of change for large values of $|a|$ and decreasing the rate of change for small values of $|a|$. If $a > 0$ the parabola opens up, If $a < 0$, the parabola opens down.
>
> $\boldsymbol{h}$: This parameter moves the parabola right if $h > 0$ and left if $h < 0$.
> (Careful, when $h = 5$, we have $y = (x-5)^2$, and when $h = -5$, we have $y = (x+5)^2$)
>
> $\boldsymbol{k}$: This parameter moves the parabola up if $k > 0$ and down if $k < 0$.
>
> **Vertex:** The vertex of the parabola is (h, k)

- <u>**Key point:**</u> To complete the square for $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.

<u>*Example:*</u> Write the function $y = 2x^2 + 20x + 6$ in vertex form.

Solution: Our first goal is to get an expression of the form $x^2 + bx$.

$$y = 2x^2 + 20x + 6$$
$$y = 2(x^2 + 10x) + 6$$

Complete the square by adding $\left(\frac{10}{2}\right)^2 = 25$ and subtracting 25 at the same time to maintain equality.

$$y = 2(x^2 + 10x + 25 - 25) + 6$$
$$y = 2(x^2 + 10x + 25) - 50 + 6$$

Now factor the parenthesis

$$y = 2(x+5)^2 - 44$$

So, the vertex is at $(-5, -44)$ and it is 2 times "steeper" than $y = x^2$. The axis of symmetry is $x = -5$.

Standard Form

The standard form of a quadratic function is $f(x) = ax^2 + bx + c$ if we complete the square (try it for yourself), then we can get

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

This shows us that the x coordinate of the vertex is $x = -\frac{b}{2a}$, which is also the axis of symmetry.

> The **axis of symmetry** of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$

2C: Graphing Polynomial Functions and Modeling

A **Polynomial** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \qquad a_n \neq 0$$

- Each monomial is called a ***term***
- The largest power n is called the ***degree***
- A polynomial with powers written in descending order is called ***standard form***
- The numbers $a_n, a_{n-1}, \dots, a_0$ are called the ***coefficients*** of the polynomial
- The term $a_n x^n$ is called the ***leading term***, a_n is the ***leading coefficient***, and a_0 is the ***constant term***.
- If a polynomial has only one term, it is called a ***monomial.***

Theorem: *Polynomial Extrema and Zeros*
A polynomial function of degree n has at most $n-1$ local extrema and at most n zeros.

The following transformations will affect the graph of a polynomial $f(x)$:

- $g(x) = f(x-h)$ will translate the graph of $f(x)$ horizontally h units.
- $g(x) = f(x) + k$ will translate the graph of $f(x)$ vertically k units.
- $g(x) = c \cdot f(x)$ will stretch the graph of $f(x)$ by a factor of c.
  If $c < 0$, the graph will be reflected across the y-axis.

Equivalent statements:

1. The number k is a zero of the function f.
2. $x = k$ is a solution of $f(x) = 0$.
3. $(x-k)$ is a factor of $f(x)$.
4. k is an x-intercept of the graph of $f(x)$.

End (Long run) behavior of polynomials

We can know what the graph of a polynomial can look like just by knowing its degree and its leading coefficient as shown below.

Degree	***Leading Coefficient***	**As $x \to -\infty$**	**As $x \to \infty$**
Even	Positive	$y \to \infty$	$y \to \infty$
Even	Negative	$y \to -\infty$	$y \to -\infty$
Odd	Positive	$y \to -\infty$	$y \to \infty$
Odd	Negative	$y \to \infty$	$y \to -\infty$

Finding Zeros

Definition: The number k is a **zero** of a function f if and only if $f(k) = 0$.

Method: To find the zeros of a function, we set $f(x) = 0$ and solve for x by factoring. To factor we first look for common monomials, then factor out linear divisors $(x - k)$ using synthetic division, then use other methods such as grouping or quadratic methods.

Polynomial Division

When performing polynomial division, the *zero value* (the number that makes a divisor zero) is what goes in the box.

Example

$$\frac{3x^4 + 2x^2 + 20x + 12}{x + 2}$$

Don't forget to put a 0 for the x^2 term!

-2	3	0	2	20	12
		-6	12	-28	-16
	3	-6	14	8	Remainder=-4

$$\frac{3x^4 + 2x^2 + 20x + 12}{x + 2} = 3x^3 - 3x^2 + 14x + 8 - \frac{4}{x + 2}$$

2D: Polynomial Equations and the Fundamental Theorem of Algebra

Fundamental Theorem of Algebra: A polynomial function of degree $n > 0$ has exactly n complex zeros. (Some of these may be repeated zeros.)

Linear Factorization Theorem

Every polynomial function of degree $n > 0$ can be factored into the form

$$f(x) = a(x - z_1)(x - z_2) \cdots (x - z_n)$$

where $z_1, z_2, \dots, z_n$ are the complex zeros of $f(x)$. Keep in mind that if there are repeated zeros, some z_i values may be the same.

****Important Note: Complex zeros always come in conjugate pairs!**

i.e. If $a + b$ i is a zero of a polynomial, then $a - b$ i is also a zero.

Key Connections: The following statements about a polynomial function are all equivalent for any complex number k:

1. $(x - k)$ is a factor of $f(x)$.
2. k is a zero of the function f.
3. $f(k) = 0$, which implies that $x = k$ is a root or x-intercept of the graph of $f(x)$.

Finding all complex zeros of a polynomial:

1. Use the rational root theorem or a graphing calculator to find a possible real zero z_1.
2. Use synthetic division to divide the polynomial by z_1.
   If we get a remainder of 0, then z is a zero of the function and this gives us a factorization
   $$f(x) = (x - z_1)(q_1 x^{n-1} + q_2 x^{n-2} + \cdots q_{n-1})$$
   where $q_1, q_2, \dots, q_{n-1}$ are the coefficients of the quotient that result from the synthetic division.
3. If the quotient is a quadratic, use the techniques we know to solve $0 = q_1 x^2 + q_2 x + q_3$. If you can factor the quotient, do this.
4. If the quotient in (2) is not a quadratic and we cannot factor it other ways, then repeat steps (1) and (2) until one of these is true.