Name: Date: Period:

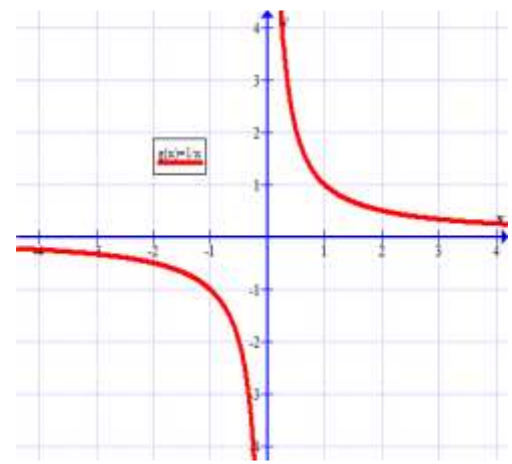
3B: Graphing Basic Rational Functions

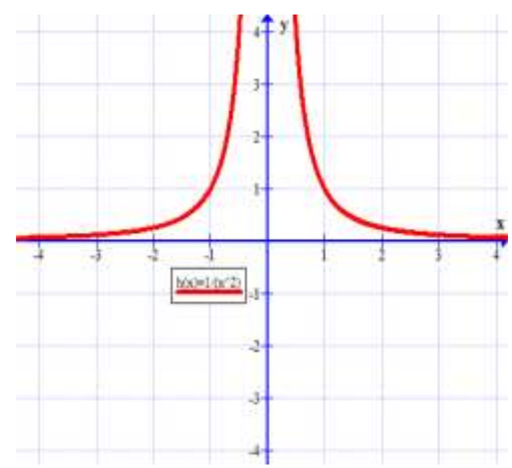
Introduction to Rational Functions

Now that we have learned how to find critical values of polynomials, we will combine two polynomials to make a **rational function** in the form

$$f(x) = \frac{A(x)}{B(x)} = \frac{a_n\, x^n + \ldots}{b_m\, x^m + \ldots}$$

where $A(x)$ and $B(x)$ are polynomials. We will consider $A(x)$ as n^{th} degree and $B(x)$ as m^{th} degree where $m \geq 1$.





We already worked with some rational functions like

$$g(x) = x^{-1} = \frac{1}{x} \quad \text{and} \quad h(x) = x^{-2} = \frac{1}{x^2}$$

when we studied power functions. The graphs of these functions give are shown to the right.

All rational functions are discontinuous at some point. Many times we have discontinuous graphs like the ones here which get infinitely close to a line, but never touch the line. This line is called an **asymptote.**

For the functions $g(x) = \frac{1}{x}$ and $h(x) = \frac{1}{x^2}$, the graphs never touch the vertical axis where $x = 0$. The line $x = 0$ is a **vertical asymptote** because the domain of these functions does not include $x = 0$ to avoid dividing by zero. The two graphs also have a **horizontal asymptote** of $y = 0$ since they get infinitely close to the *x*-axis, but the graph never touches it.

Finding Asymptotes

The first step to understanding any rational function is to determine the asymptotes. There are two basic connections that are important to remember:

- **Vertical Asymptotes: Domain restrictions** determine "**holes**" and **vertical asymptotes,** and
  1) Factor the numerator and denominator. If any factors cancel, the zero of this factor will be a **hole** (also called removable discontinuities) in the graph. (e.g. if $(x - 3)$ cancels, then $x = 3$ is a hole.)
  2) Determine domain restrictions by setting the denominator equal to 0 and solve.
     If z_i is a zero of the denominator, then $x = z_i$ is a vertical asymptote.

- **Horizontal Asymptotes: End Behavior (i.e. range at $\pm\infty$)** determines **horizontal asymptotes.**
  1) Method 1: Use polynomial division to write the function as a *quotient function* with the remainder written as a fraction.
     Method 2: Divide all terms of numerator and denominator by the *greatest power of x in the denominator*.
  2) Find the limit of y as $x \to \infty$ and $x \to -\infty$. *(Note, remainder fractions will go to zero when $x \to \pm\infty$.)*
     This limit is the equation of the horizontal (or slant) asymptote.

Summary of horizontal/slant asymptote behavior for a rational function of the form $f(x) = \frac{a_m x^m + \cdots}{a_n x^n + \cdots}$	
Degree relationship of numerator & denominator	**Horizontal or slant asymptote behavior.**
$m < n$	$y = 0$
$m = n$	$y = \frac{a_m}{a_n}$
$m > n$	Slant asymptote determined by the limit as $x \to \pm\infty$

Other important points for graphing

Before graphing a rational function, find the asymptotes and holes.

- x-intercepts: these come from the zeros of the numerator. If the numerator is 0, then the whole fraction is 0!
- y-intercept: Set $x = 0$ to find this!
- Test points: to determine if the curve is above or below an asymptote, plug in a few x values into the *factored version*.

Assignment

Find the key asymptotes and intercepts for each function, then graph them on the last page.

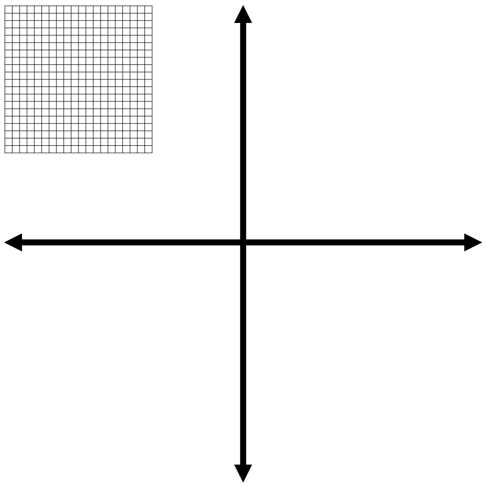
Function	**Factored Form**	**Domain Restrictions** (from Denom.) **and zeros** (from Numerator)	**Vertical Asymptotes (a)** and Removable Discontinuities	**Divide** by greatest power of x in denominator	**Limits** as $x \to \pm\infty$	**Horizontal Asymptotes** from $\lim_{x\to\infty} f(x)$
$a(x) = \frac{x+3}{x^2+x-6}$						

Function	Factored Form	**Domain Restrictions** (from Denom.) **and zeros** (from Numerator)	**Vertical Asymptotes** ($\boldsymbol{a}$) and Removable Discontinuities	**Divide** by greatest power of x in denominator	**Limits** as $x \to \pm\infty$	**Horizontal Asymptotes** from $\lim_{x\to\infty} f(x)$
$b(x) = \frac{3}{4x - 12}$						
$c(x) = \frac{x^2 + x}{x^3 - x}$						
$d(x) = \frac{3x}{x + 1}$						

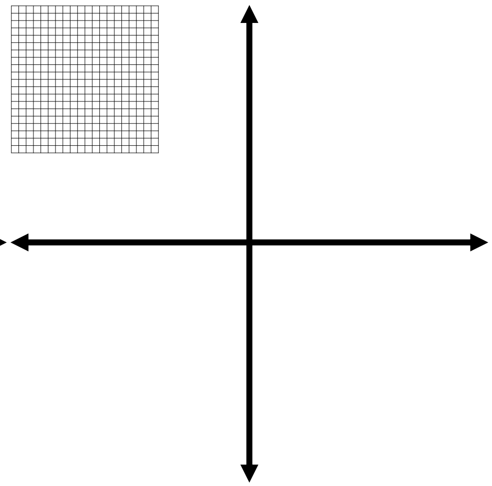
$e(x) = \dfrac{x^2 - 2x - 3}{x^2 + 4x + 3}$						
$f(x) = \dfrac{2\,x^2 + x - 6}{x^2 + x - 2}$						
$g(x) = \dfrac{x^2 - 4}{x^2 + 1}$						

Graphs

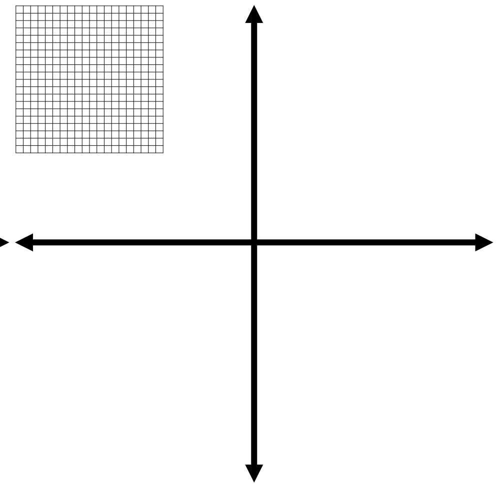
$a(x) = \frac{x+3}{x^2+x-6}$



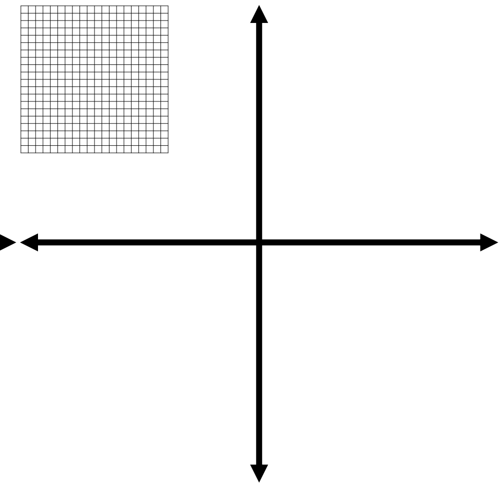
$b(x) = \frac{3}{4x-12}$



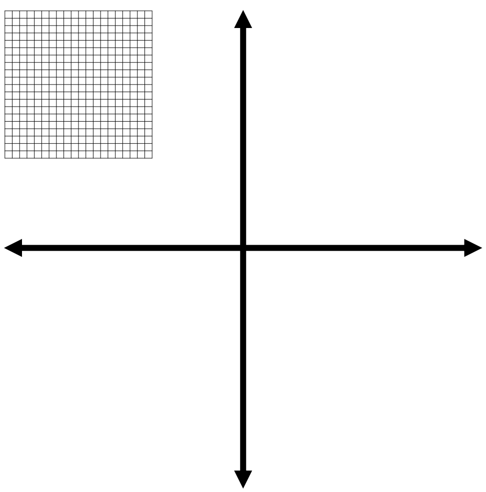
$c(x) = \frac{x^2+x}{x^3-x}$



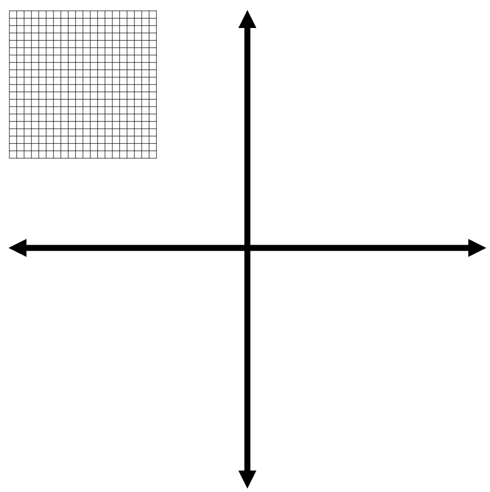
$d(x) = \frac{3x}{x+1}$



$e(x) = \frac{x^2-2x-3}{x^2+4x+3}$



$f(x) = \frac{2x^2+x-6}{x^2+x-2}$



$g(x) = \frac{x^2-4}{x^2+1}$