Pre-Calculus Name: Date: Period:

# 3B: Graphing Basic Rational Functions

## Introduction to Rational Functions

Now that we have learned how to find critical values of polynomials, we will combine two polynomials to make a **rational function** in the form

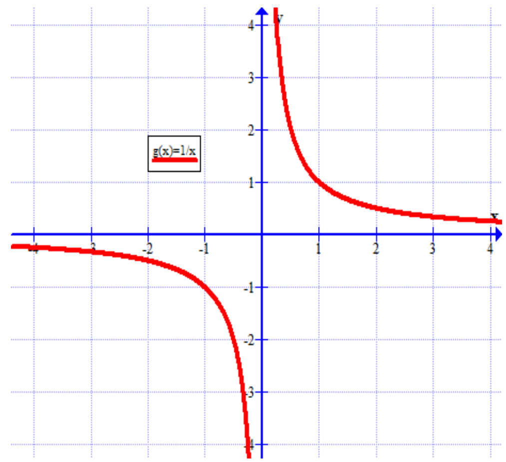
$$f(x) = \frac{A(x)}{B(x)} = \frac{a_n\, x^n + \ldots}{b_m\, x^m + \ldots}$$

where $A(x)$ and $B(x)$ are polynomials. We will consider $A(x)$ as $n^{th}$ degree and $B(x)$ as $m^{th}$ degree where $m \geq 1$.

We already worked with some rational functions like

$$g(x) = x^{-1} = \frac{1}{x} \quad \text{and} \quad h(x) = x^{-2} = \frac{1}{x^2}$$

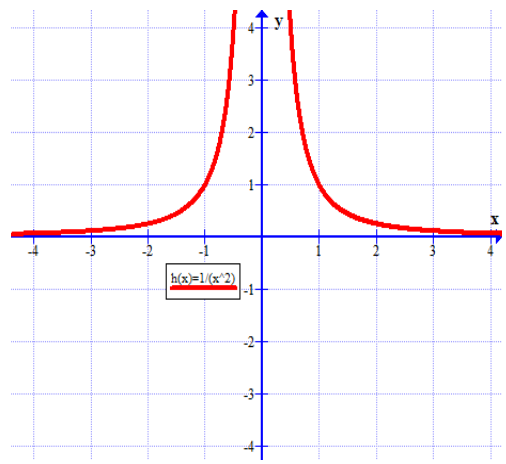
when we studied power functions. The graphs of these functions give are shown to the right.




*All rational functions are discontinuous at some point.* Many times we have discontinuous graphs like the ones here which get infinitely close to a line, but never touch the line. This line is called an **asymptote**.

For the functions $g(x) = \frac{1}{x}$ and $h(x) = \frac{1}{x^2}$, the graphs never touch the vertical axis where $x = 0$. The line $x = 0$ is a **vertical asymptote** because the domain of these functions does not include $x = 0$ to avoid dividing by zero. The two graphs also have a **horizontal asymptote** of $y = 0$ since they get infinitely close to the $x$-axis, but the graph never touches it.

## Finding Asymptotes




The first step to understanding any rational function is to determine the asymptotes. There are two basic connections that are important to remember:

- **Domain** restrictions determine "**holes**" and **vertical asymptotes,** and
  1) Factor the numerator and denominator. If any factors cancel, the zero of this factor will be a **hole** (also called removable discontinuities) in the graph. (e.g. if $(x - 3)$ cancels, then $x = 3$ is a hole.)
  2) Determine domain restrictions by setting the denominator equal to 0 and solve. If $z_i$ is a zero of the denominator, then $x = z_i$ is a vertical asymptote.

- **End Behavior (i.e. range at** $\pm\infty$**)** determines **horizontal asymptotes.**
  1) Use polynomial division to write the function as a *quotient function* with the remainder written as a fraction.
  2) Find the limit of $y$ as $x \to \infty$ and $x \to -\infty$. *(Note, remainder fractions will go to zero when* $x \to \pm\infty$*.)*
     This limit is the equation of the horizontal (or slant) asymptote.

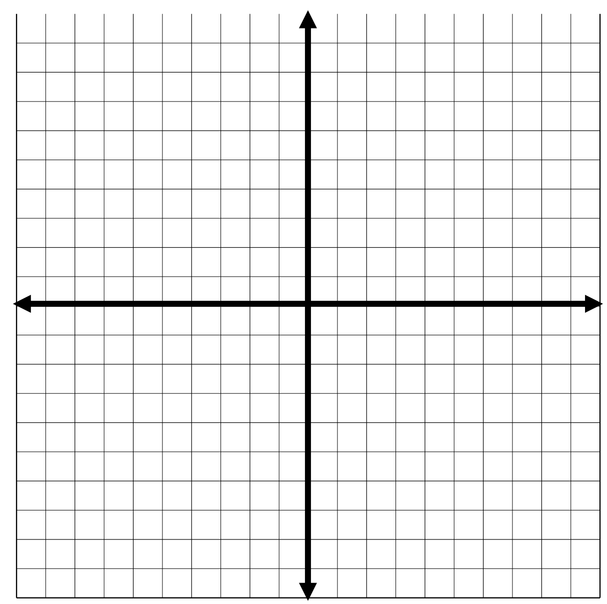
*Example* Consider the function $y = \frac{3x^2+3x}{x^2-2x-3}$

1. Write the function in factored form and find the domain restrictions for $f(x)$?

   a. Are there any removable discontinuities (i.e. holes)?

   b. Find the equation(s) of any vertical asymptotes.

2. Use polynomial division to write as sum of a polynomial and remainder fraction.



   a. For the value of each vertical asymptote, $a$, find the **Limits**
      as $x \to \pm\infty$ , $y \to$
      as $x \to a^+$, $y \to$
      as $x \to a^-$, $y \to$

   b. Find the equation of any horizontal asymptotes using
      the infinite limits above.

3. Graph the function (test extra points if needed) and check it on your calculator.

## Domain restrictions and vertical asymptotes

We have previously learned that we must carefully consider the denominator when determining the domain of a rational function. We will continue with this idea and use what we have learned about factoring to find these domain restrictions and other important characteristics of the graph of a rational function.

| **Function** | **Factored Form** | **Domain Restrictions** | **Vertical Asymptotes** ($\boldsymbol{a}$) and Removable Discontinuities | **Divide** (write as sum of a polynomial and remainder fraction) | **Limits** as $x \to \pm\infty$ , as $x \to a^+$ as $x \to a^-$ | **Horizontal Asymptotes** |
|---|---|---|---|---|---|---|
| $a(x) = \dfrac{x+3}{x^2+x-6}$ | | | | | | |

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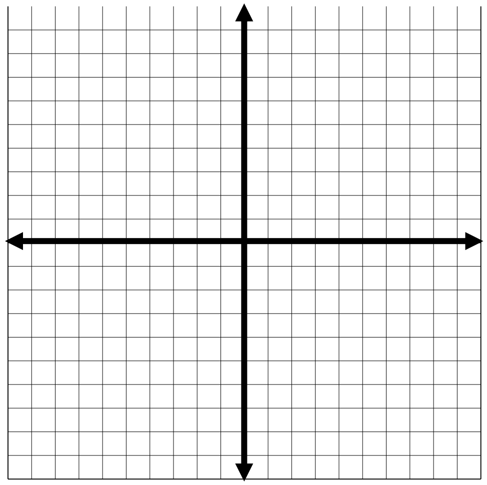
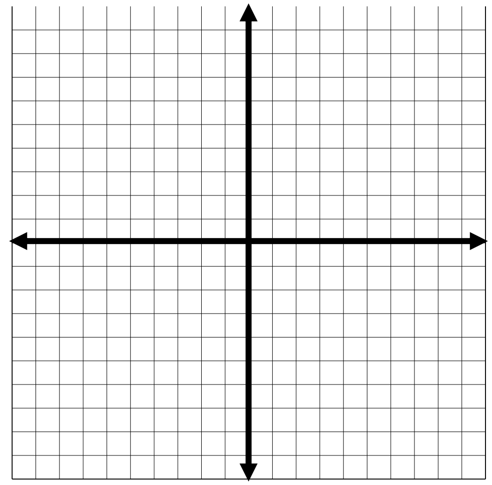
|---|---|---|---|---|---|---|
| $b(x) = \dfrac{2\,x^2 + x - 6}{x^2 + x - 2}$ | | | | | | |
| $d(x) = \dfrac{x^2 - 4}{x^2 + 1}$ | | | | | | |
| $e(x) = \dfrac{x^3 + 4x^2 + 3x}{x + 1}$ | | | | | | |

## Graphs

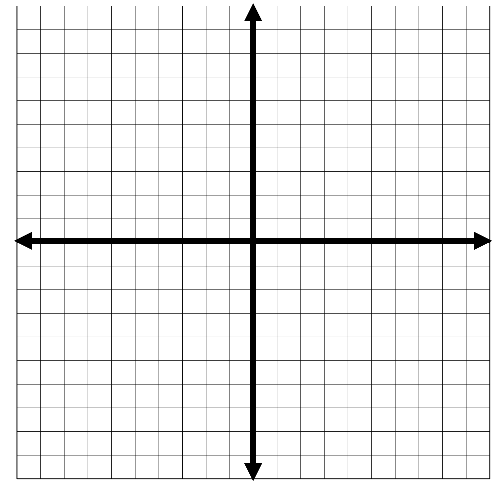
$a(x) = \frac{x+3}{x^2+x-6}$



$b(x) = \frac{2\,x^2+x-6}{x^2+x-2}$



$d(x) = \frac{x^2-4}{x^2+1}$



$$e(x) = \frac{x^3 + 4x^2 + 3x}{x + 1}$$