3D: Rational Functions and Asymptotes

A rational function is one of the form

$$f(x) = \frac{A(x)}{B(x)} = \frac{a_n\, x^n + \ldots}{b_m\, x^m + \ldots}$$

where $A(x)$ and $B(x)$ are polynomials.
We will consider $A(x)$ as n^{th} degree and $B(x)$ as m^{th} degree.

Domain restrictions and vertical asymptotes

We have previously learned that we must carefully consider the denominator when determining the domain of a rational function. We will continue with this idea and use what we have learned about factoring to find these domain restrictions and other important characteristics of the graph of a rational function.

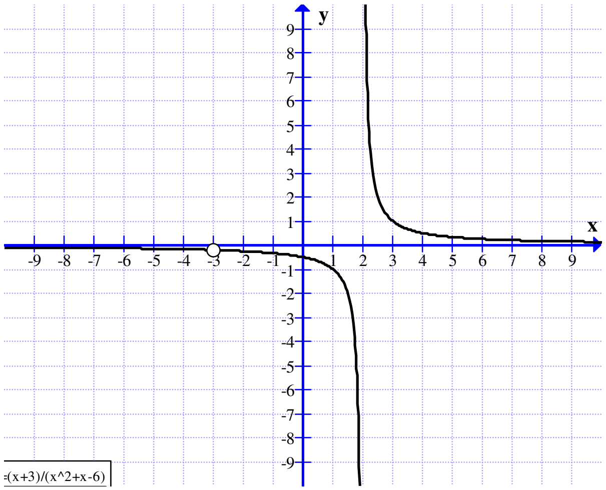
Function	**Factored Form** and **Simplified Form**	**Domain Restrictions**	**Vertical Asymptotes** $(\boldsymbol{a})$ and Removable Discontinuities	**Divide** (write as sum of a polynomial and remainder fraction)	**Limits** as $x \to \pm\infty$, as $x \to a^+$ as $x \to a^-$	**Horizontal Asymptotes**
$a(x) = \dfrac{x+3}{x^2+x-6}$	$= \dfrac{x+3}{(x+3)(x-2)}$ $= \dfrac{1}{x-2}$	$x \neq -3$ $x \neq 2$	Vert. Asym.: $x = 2$ Hole: $x = -3$	Cannot be Divided Because numerator degree is less than denominator degree	*Since num. degree < denom. Degree:* As $x \to -\infty$, $y \to 0$ As $x \to \infty$ $y \to 0$ *Consider Vert. asymptote:* As $x \to 2^+$ $y \to \infty$ As $x \to 2^-$ $y \to -\infty$	$y = 0$ Since infinite limits go to 0 *(No Zeros)*

Function	Factored Form	Domain Restrictions	Vertical Asymptotes ($\boldsymbol{a}$) and Removable Discontinuities	Divide (write as sum of a polynomial and remainder fraction)	Limits as $x \to \pm\infty$, as $x \to a^+$ as$x \to a^-$	Horizontal Asymptotes
$b(x) = \dfrac{2\,x^2 + x - 6}{x^2 + x - 2}$	$= \dfrac{(x+2)(2x-3)}{(x+2)(x-1)}$ $= \dfrac{(2x-3)}{(x-1)}$	$x \neq -2$ $x \neq 1$	Vert. Asym.: $x = 1$ Hole: $x = -2$	Use simplified version and long division: $\dfrac{(2x-3)}{(x-1)}$ $= 2 + \dfrac{-1}{x-1}$ (Use this to find limits and horizontal asymptote)	As $x \to -\infty$, $y \to 2$ As $x \to \infty$ $y \to 2$ *<u>Consider Vert. asymptote:</u>* As $x \to 1^-$ $y \to \infty$ As $x \to 1^+$ $y \to -\infty$	From Limits: $y = 2$ *(by numerator: zero at* $x = \frac{3}{2}$*)*
$c(x) = \dfrac{x^2 - x - 6}{x^2 + 2\,x - 3}$	$= \dfrac{(x+2)(x-3)}{(x+3)(x-1)}$ *Can't be simplified*	$x \neq -3$ $x \neq 1$	Vert. Asym.: $x = -3$ and $x = 1$ *No holes because no factors cancel*	Use simplified version and long division: $\dfrac{x^2 - x - 6}{x^2 + 2\,x - 3} =$ $1 + \dfrac{-3x-3}{x^2 + 2x - 3}$ (Use this to find limits and horizontal asymptote)	As $x \to -\infty$, $y \to 1$ As $x \to \infty$ $y \to 1$ *<u>Consider Vert. asymptote:</u>* As $x \to -3^-$ $y \to -\infty$ As $x \to -3^+$ $y \to \infty$ As $x \to 1^-$ $y \to \infty$ As $x \to 1^+$ $y \to -\infty$	From Limits: $y = 1$ *(by numerator: zero at* $x = -2{,}3$*)*

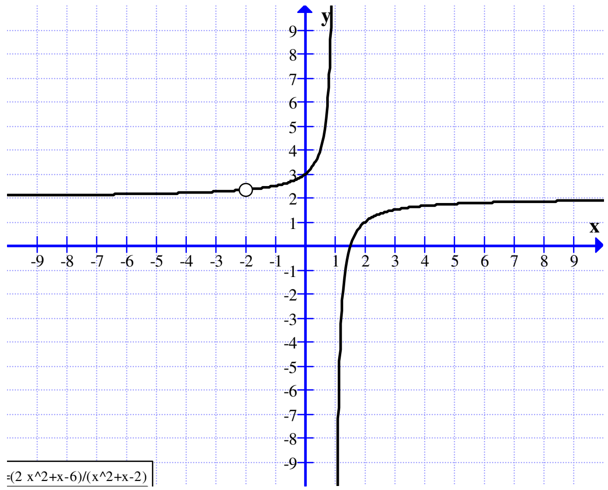
Function	**Factored Form**	**Domain Restrictions**	**Vertical Asymptotes ($\boldsymbol{a}$)** and Removable Discontinuities	**Divide** (write as sum of a polynomial and remainder fraction)	**Limits** as $x \to \pm\infty$, as $x \to a^+$ as$x \to a^-$	**Horizontal Asymptotes**
$d(x) = \frac{x^2 - 4}{x^2 + 1}$	$\frac{(x+2)(x-2)}{x^2+1}$	*No Domain Restrictions*	*None because there are no domain restrictions*	Use simplified version and long division: $1 + \frac{-5}{x^2+1}$ (Use this to find limits and horizontal asymptote)	As $x \to -\infty$, $y \to 1$ As $x \to \infty$ $y \to 1$ *No vertical asymptotes, so there's no other limits to consider*	From Limits: $y = 1$ *(by numerator: zero at* $x = \pm 2$*; and y-intercept at* $(0, -4)$*)*
$y = \frac{x^3 + 4x^2 + 3x}{x + 1}$	$y = \frac{x(x+1)(x+3)}{\mathrm{x}+1}$	$x \neq 1$	Hole: $x = 1$	It simplifies to $x^2 + 3x$, so there is no long division needed.	As $x \to -\infty$, $y \to \infty$ As $x \to \infty$ $y \to \infty$	No Horizontal Asymptote.

Graphs

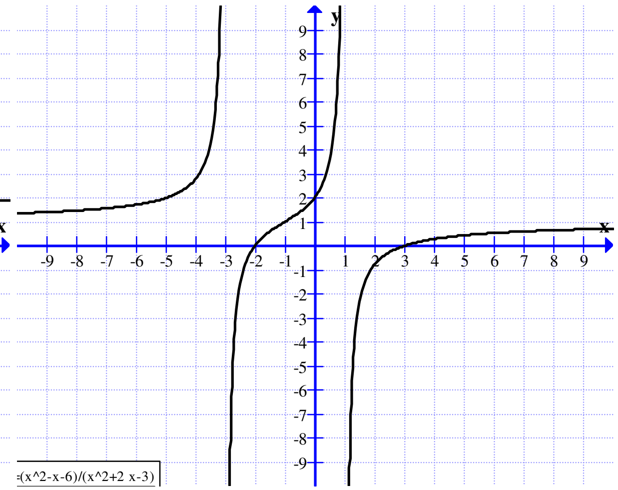
$a(x) = \frac{x+3}{x^2+x-6}$



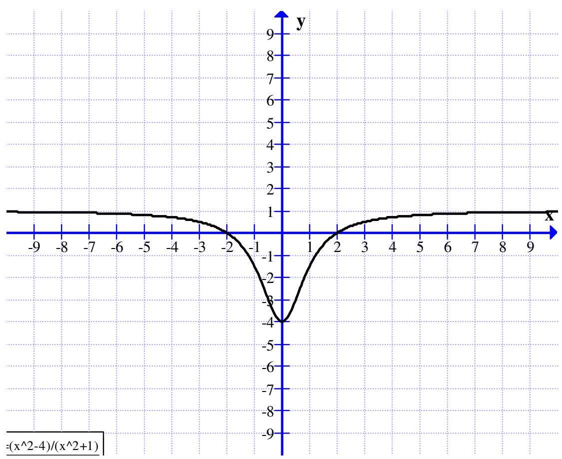
$b(x) = \frac{2x^2+x-6}{x^2+x-2}$



$c(x) = \frac{x^2-x-6}{x^2+2x-3}$



$d(x) = \frac{x^2-4}{x^2+1}$



$y = \frac{x^3+4x^2+3x}{x+1}$