Pre-Calculus

Unit 3 Toolkit – Rational Functions

This toolkit is a summary of some of the key topics you will need to master in this unit.

3A: Rational Expressions and Equations.

Learning Target: I can simplify and solve rational equations.

Multiplying Rational Expressions using Cross-Canceling

When multiplying rational expressions, remember to:

- "Cross-cancel" first by finding factors in the numerators and denominators that are the same.
- Divide out these factors because they make 1. Always leave something behind, even if it's only a 1.
- Multiply the remaining factors to get your final fraction.

Adding Rational Expressions

When adding or subtracting rational expressions, remember to:

- Begin by defining the common denominator in factored form.
- Multiply each term of the expression by a special form of 1 to change them all to a common denominator.
- Combine into one fraction and simplify both the numerator and common denominator.

Solving Rational Equations

A ***rational equation*** has expressions in the denominator(s) that involve powers or functions of x.
* * When solving rational equations, the goal is to ***eliminate (or clear) the denominators*** by multiplying both sides of the equation by the ***common denominator.***

> The key to solving rational equations is a few simple steps:
> 1. Find the ***least common denominator*** for all terms, then
> 2. ***Multiply both sides*** of the equation by the common denominator, then
> 3. ***Solve*** the remaining equation using standard techniques, then
> 4. ***Check for extraneous solutions*** by substituting answers back into original equation. *Specifically, make sure that there are no conflicts with the domain restrictions.*

Example: Solve.

$$\frac{1}{x+3}+\frac{2}{x}=\frac{3}{x^2+3x}$$

$$\frac{x}{x+3}+\frac{2}{x}=\frac{6}{x(x+3)}$$

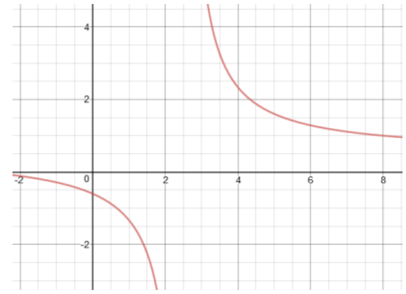
$$x(x+3)\left(\frac{x}{x+3}+\frac{2}{x}\right)=\left(\frac{6}{x(x+3)}\right)\cdot x(x+3)$$

$$\frac{x\cdot x(x+3)}{x+3}+\frac{2\cdot x(x+3)}{x}=\frac{6\cdot x(x+3)}{x(x+3)}$$

$$x\cdot x+2(x+3)=6\rightarrow x^2+2x+6=6\rightarrow x=0\ \ or\ \ x=-2$$

Since $x\neq 0$ for the second term on left-hand side $\frac{2}{x}$, we must eliminate this solution. ***Solution set:*** $\boldsymbol{x=-2}$

- *Factor all denominators.*
- *Multiply all unique denominator factors.* $x(x+3)$
- *Make sure you distribute to each term.*
- *Cancel factors that you can, then solve.*
- *Check solutions with denominators to avoid domain restrictions.*



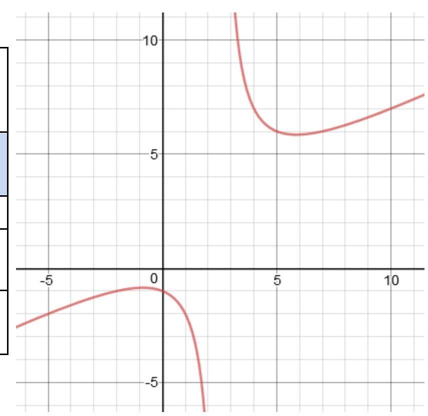
3B: Graphing Basic Rational Functions.

Learning Target: I can graph rational functions and identify their asymptotes.

Steps for finding key features of the graph of a rational function $(x) = \frac{A(x)}{B(x)}$.

If the function has several terms, combine into one fraction using a common denominator.
Always begin by **Factoring** the numerator and denominator

1. **Domain restrictions** → these come from the zeros of the denominator.
2. **Simplify** the expression by canceling common factors.
3. **Vertical Asymptotes** and **Holes** (a.k.a. removable discontinuities).
   a. Vertical Asymptotes come from factors that *do not cancel.*
   A vertical asymptote is of the form $x = a$ and is *graphed as a vertical dotted line.*
   b. Holes come from zeros of factors that *do cancel.* Find the exact position of a hole by using the *simplified function* to evaluate $f(z)$ for the zero z that cancels out.
   After graphing the basic function, draw in the holes as an open dot at this point.
4. **Horizontal asymptotes** are determined by the end behavior.
   a. *Method 1:* Use polynomial division to write the function as a *quotient function* with the remainder written as a fraction.
   Method 2: Divide all terms of numerator and denominator by the *greatest power of x in the denominator*.
   b. Find the limit of y as $x \to \infty$ and $x \to -\infty$. *(Note, remainder fractions will go to zero when $x \to \pm\infty$.)* This limit is the equation of the horizontal (or slant) asymptote.
5. **Intercepts** and extra points.
   a. x-intercepts come from the zeros of the numerator of the *simplified* function.
   b. y-intercepts come from evaluating $f(0)$.



Shortcut for horizontal/slant asymptote behavior for a rational function of the form $f(x) = \frac{a_m x^m + \cdots}{a_n x^n + \cdots}$

Degree relationship of numerator & denominator	**Horizontal or slant asymptote behavior.**
$m < n$	$y = 0$
$m = n$	$y = \frac{a_m}{a_n}$
$m > n$	Slant asymptote determined by the limit as $x \to \pm\infty$

3C: Graphing Advanced Rational Functions.

Learning Target: I can graph advanced rational functions including functions with slant asymptotes and multiple vertical asymptotes.

Begin graphing all rational functions using the steps in section 3B. Then remember these exceptions:

- **Multiple Vertical Asymptotes**: Remember that the *horizontal asymptote* only affects end behavior. The graph between multiple vertical asymptotes can cross this horizontal line.
  - Determine the shape of the graph between vertical asymptotes by testing points and considering the limit from the left and right of an asymptote.
- **Slant Asymptote:** Whenever the degree of the numerator is greater than the degree of the denominator, we get a slant asymptote. Graph this slanted line (or curve) and do all other steps the same.