4C: Solving Exponential Equations

Definition

An **exponential function** is of the form

$$f(x) = a(b^x)$$

Where a is nonzero, b is positive, and $b \neq 1$. The constant a is the *initial value* and b is the *base*.

Exploring Exponential Growth and Decay

Some quantities grow exponentially as the rice in the previous example, while others decrease exponentially.

Legend has it…

There once was a peasant in India who invented the game of chess. The ruler of the land was so pleased the game, that he wanted to reward the peasant for his creation. So, he asked the man what he would like for his reward. After thinking for a few minutes he said, "I don't need much. How about if you take my 8×8 chess board and put 1 grain of rice on the first square, 2 grains on the second square, 4 on the third, and continuing doubling the grains of rice on each square

1) Write an exponential function to model the number of grains of rice g that would be placed on square x.

2) How many grains of rice were needed to put on the last square?

3) If a grain of rice has a volume of $1 grain = .03\ cm^3$, what is the volume of the rice that would be on the last square?

4) The world production of rice is approximately 700,000 liters per year. How does the amount of rice *on the last square* compare to the amount of rice produced in a year? $(1\ liter = 1000ml = 1000cm^3)$

5) *Extra Challenge:* How many liters of rice would there be on the entire board?

Example: Decaying Elements
Consider the radioactive isotope promethium-128 that has a half-life of one second. *Half-life* is the amount of time it takes an element to decrease to half of its original quantity. Suppose that we measure $64\mu g$ (micrograms) of promethium-128.

a. Find the mass of Promethium-128 at the beginning of each of the first five seconds.

Seconds (sec.)	0	1	2	3	4	5
Mass (μg)						

b. Write an equation to describe the mass of the isotope as a function of time (in seconds).

c. Suppose that after further experiments it is discovered that the amebas population doubled every 1.5 hours. Write a new function to describe the ameba's population as a function of time (in hours) if we start with 3 organisms.

Exponential Growth and Decay: The exponential function $y = a(b^x)$ is either a
- *Exponential Growth* equation if $b > 1$ (here we call b the "growth factor"), or
- *Exponential Decay* equation if $0 < b < 1$ (here we call b the "decay factor").

Solving Basic Exponential Equations

So, how do we solve equations that have variables in the exponent? We have two methods that can be useful.

Method 1: Equivalent Exponentials

One method is to write each side of the equation as an equivalent exponential.

Example: Solve

a. $3^x = 81$

b. $5(2^x) - 30 = 130$

Method 2: Using Logarithms to Solve Exponential Equations

Another method is to use the inverse of an exponential function, which we call a *logarithm*. When we find a logarithm, we are answering the question "what is the power of base b that gives us the desired value?"

Fundamental Relationship: The fundamental connection between logarithms and exponentials is

$$\text{If} \quad b^x = y, \qquad \text{then} \quad \log_b y = x$$

This means that finding $\log_b y$ is equivalent to asking the question:

$$\text{"}What\ power\ of\ base\ b\ is\ needed\ to\ get\ the\ value\ y?\text{"}$$

So, ***the log is the power!***

Logarithmic Identities

Since the logarithm is the power of base b needed to produce x, we have the identities:

$$b^{\log_b x} = x, \qquad and \qquad \log_b b^x = x$$

On your calculator we have buttons that will evaluate $\log_{10} x$ and $\log_e x = \ln x$. Whenever you have an exponential equation of base 10 or base e (sometimes called the natural occurring number that is approximately 2.7182818285).

Example. Solve the following exponential equations by changing to a logarithmic equation and using your calculator.

a) $10^x = 110$

b) $10^x = 0.005$

c) $3(10^x) = 213$

d) $10^{4x} = 0.5$

e) $e^x = 23$

f) $2e^{x+2} = 24$

Change of Base Formula:

To convert $\log_b x$ into another base, use the formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Example Use the change-of-base formula to evaluate the following to 3 decimal places. Check your answer by evaluating the power.

a) $\log_2 5 =$

b) $\log_5 130 =$

c) $\log_5 2 =$

d) $\log_{0.5} 4 =$

Example Use the change of base formula to solve. $5^{x+3} = 20$