4D-1: Properties of Logarithms

In this lesson we will develop some special properties of logarithms. Keep in mind that logarithms are exponents, so the properties of logarithms are closely related to the properties of exponents.

Fundamental Relationship: The fundamental connection between logarithms and exponentials is

$$\text{If} \quad b^x = y, \qquad \text{then} \quad \log_b y = x$$

This means that finding $\log_b y$ is equivalent to asking the question:

$$\text{"}What\ power\ of\ base\ b\ is\ needed\ to\ get\ the\ value\ y?\text{"}$$

So, ***the log is the power!***

Logarithmic Identities
Since the logarithm is the power of base b needed to produce x, we have the identities:

$$b^{\log_b x} = x, \qquad and \qquad \log_b b^x = x$$

On your calculator we have buttons that will evaluate $\log_{10} x$ and $\log_e x = \ln x$.

Properties of Logarithms

Exploration Use your calculator to approximate a solution to each of the following.

a) $\log(3 \cdot 5) =$

b) $\log 3 + \log 5 =$

c) $\log\left(\frac{3}{5}\right) =$

d) $\log 3 - \log 5 =$

e) $\log 3^5 =$

f) $5 \cdot \log 3 =$

We notice that there are some similar answers above, so let's explore more. Let

$$\log_b P = p, \qquad and \qquad \log_b Q = q.$$

Which implies

$$P = b^p, \qquad and \qquad Q = b^q$$

Let's find equivalent forms of these equations by changing them to exponential equations.

$y = \log_b(PQ)$

$y = \log_b \frac{P}{Q}$

$y = \log_b P^c$

Properties of Logarithms

Let $b, R,$ and S be positive real numbers with $b \neq 1$, and c a

Product Rule: $\log_b(PQ) = \log_b P + \log_b Q$

Quotient Rule: $\log_b \frac{P}{Q} = \log_b P - \log_b Q$

Power Rule: $\log_b P^c = c \log_b P$

Where $b, R,$ and S be positive real numbers with $b \neq 1$, and c is any real number.

Change of Base Formula for Logarithms

What if we have a logarithm that is not in base 10 or base e? Thanks to the power rule, we can change the base to something more familiar.

Consider this? Find a formula to expression that is equivalent to $\log_b x$ in terms of $\log_a x$

To begin, we define the logarithmic equation: $\log_b x = y$

Change the equation into exponential form: $b^y = x$

Now take the $\log_a$ of both sides: $\log_a(b^y) = \log_a(x)$

Finally use the Power Rule to solve for y: $y \log_a(b) = \log_a(x)$

$y = \frac{\log_a x}{\log_a b}$

Change of Base Formula:

To convert $\log_b x$ into another base, use the formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

In practice, we use these:

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

Example Use the properties of logarithms to find the exact solution, then use your calculator to find the approximate solutions.

a. $5^x = 2$

b. $10(4^x) = 25$

c. $e^x = \frac{e}{2}$