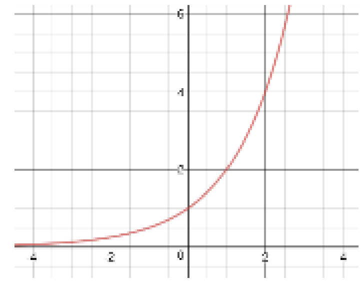
Pre-Calculus

Unit 4 Toolkit – Exponential and Logarithmic functions

This toolkit is a summary of some of the key topics you will need to master in this unit.

4A: Graphing Exponential Functions

Learning Target: I can graph and describe transformations for exponential functions.



An **exponential function** is of the form $f(x) = a(b^x)$

Where a is nonzero, b is positive, and $b \neq 1$. The constant a is the ***initial value*** and b is the ***base***.

If $b > 1$, the graph is **exponential growth** (increasing).
If $0 < b < 1$, the graph is **exponential decay** (decreasing).

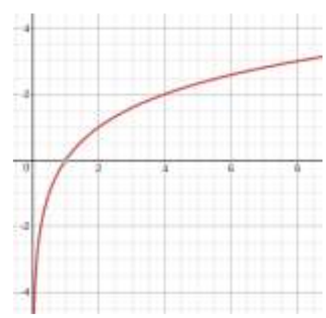
Transformations of Exponential Functions:
The values of a, b, c and d affect the graph of $\boldsymbol{y = a(b^{x+c}) + d}$ as follows:
- a: Vertical stretch/shrink. If a is negative, it is reflected across the x-axis
- b: Rate of change.
  - If $b > 1$, the graph is **exponential growth** (increasing).
  - If $0 < b < 1$, the graph is **exponential decay** (decreasing).
- c: Horizontal shift.
  - *if c>0, graph moves left; if c<0, graph moves right.*
- d: Vertical shift.
  - *if d>0, graph moves up; If d<0, graph moves down.*

Horizontal Flip: *If a function is written as $y = b^{-x}$, this is a horizontal reflection across the y-axis because*

$$y = b^{-x} = (b^{-1})^x = \left(\frac{1}{b}\right)^x$$

4B: Graphing Logarithmic Functions

Learning Target: I can graph and describe transformations for logarithmic functions.



Key Connection: If $x > 0$ and $0 < b \neq 1$, then these are equivalent:

$$\boldsymbol{y = \log_b(x) \;\leftrightarrow\; b^y = x.}$$

Transformations of Logarithmic Functions:
The values of a, b, c and d affect the graph of $\boldsymbol{y = a\log\big(b(x+c)\big) + d}$ as follows:
- a: Vertical stretch/shrink. If a is negative, it is reflected across the x-axis
- b: Horizontal stretch/shrink. If b is negative, the graph is reflected across the y-axis.
- c: Horizontal shift.
  - *if c>0, graph moves left; if c<0, graph moves right.*
- d: Vertical shift.
  - *if d>0, graph moves up; If d<0, graph moves down.*

Finding Domain of a logarithmic function: The domain of $y = \log_b x$ is $\{x | x > 0,\ x \in \mathbb{R}\}$

The domain of $f(x) = a\log(px + q) + d$, set $px + q > 0$ and solve

4C: Solving Problems with Exponentials Equations

Learning Target: I can solve problems involving exponential functions.

When solving exponential equations, we have 2 methods:

Method 1: Equivalent Exponentials

Write each side of the equation as an equivalent exponential, then set the exponents equal to each other.

Example Solve $3^{2x+3} = 81$

$$3^{2x+3} = 3^4$$
$$2x + 3 = 4$$
$$x = {}^{1}\!/_{2}$$

Method 2: Using Logarithms to Solve Exponential Equations

Fundamental Relationship: The fundamental connection between logarithms and exponentials is If $b^x = y$, then $\log_b y = x$
Change of Base Formula: To convert $\log_b x$ into another base, use the formula $\log_b x = \dfrac{\log_a x}{\log_a b}$ More specifically $\log_b x = \dfrac{\log x}{\log b} = \dfrac{\ln x}{\ln b}$

Example Solve $6^{x+3} = 14$

$$x + 3 = \log_6 14$$
$$x + 3 = \frac{\ln 14}{\ln 6}$$
$$x = \frac{\ln 14}{\ln 6} - 3 \approx -1.53$$

4D: Solving Problems with Logarithmic Equations

Learning Target: I can solve problems involving logarithmic functions with properties.

Properties of Logarithms

Let $b, R,$ and S be positive real numbers with $b \neq 1$, and c a

Product Rule: $\log_b(PQ) = \log_b P + \log_b Q$

Quotient Rule: $\log_b \frac{P}{Q} = \log_b P - \log_b Q$

Power Rule: $\log_b P^c = c \log_b P$

Where $b, R,$ and S be positive real numbers with $b \neq 1$, and c is any real number.

To solve a logarithmic equation:

1. Method 1:
   a. Simplify one side to a single logarithm and the other side as a constant,
   b. Convert to exponential form, and
   c. Solve.
2. Method 2:
   a. Write both sides as single logarithms with the same base.
   b. Remove the logs and set the arguments (the stuff in the log) equal.
   c. Solve.

Example Solve with Method 1. $\log_2 x + \log_2 2x = 4$

$$\log_2(x \cdot 2x) = 4$$
$$2x^2 = 2^4$$
$$2x^2 = 16$$
$$x^2 = 8$$
$$x = \pm 2\sqrt{2}$$
$$However, by\ the\ domain, x > 0$$
$$Solution: x = 2\sqrt{2}$$

Example Solve with Method 2: $\log 2x + \log(x+1) = \log(2x^2+1)$

$$\log\big(2x(x+1)\big) = \log(2x^2+1)$$
$$2x(x+1) = 2x^2+1$$
$$2x^2+2x = 2x^2+1$$
$$2x = 1$$
$$x = \frac{1}{2}$$