Pre-Calculus

Name: SOLUTIONS Date: Period:

2B: Vertex Form & Completing the Square

A Useful Form

By observing transformations of the graph of the quadratic equation $y = x^2$ we have learned how to move the parabola vertically or horizontally, flip it vertically, and modify its rate of change (i.e. its vertical stretch). These transformations can all be summarized in the general form, $f(x) = a(x-h)^2 + k$, which we call vertex form. Below is a description of the affects of the parameters a, h, and k.

Vertex form of a quadratic function: $\boldsymbol{f(x) = a(x-h)^2 + k}$

> $\boldsymbol{a}$**:** This parameter changes the shape of the parabola by increasing the rate of change for large values of $|a|$ and decreasing the rate of change for small values of $|a|$. If $a > 0$ the parabola opens up, If $a < 0$, the parabola opens down.
>
> $\boldsymbol{h}$**:** This parameter moves the parabola right if $h > 0$ and left if $h < 0$.
> (Careful, when $h = 5$, we have $y = (x-5)^2$, and when $h = -5$, we have $y = (x+5)^2$)
>
> $\boldsymbol{k}$**:** This parameter moves the parabola up if $k > 0$ and down if $k < 0$.
>
> **Vertex:** The vertex of the parabola is (h, k)

Try These:

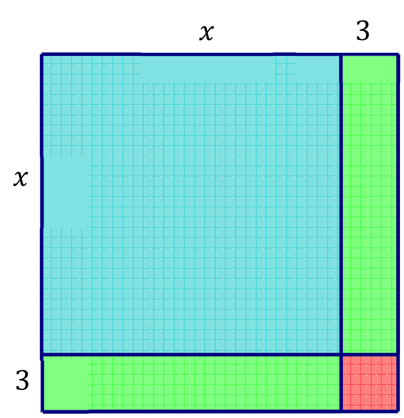
1. Find the equation of the parabola that is twice as steep as $y = x^2$ and is moved left 5 units and down 14 units.

2. Describe how the graph of $y = -(x-18)^2 + 9$ differs from the graph of $y = x^2$, and find the coordinates of its vertex.

Completing the square to find the vertex

What if a quadratic function is not in vertex form?

This is a good question.
For this we need to know how to "Complete the Square"



Consider the geometric model shown to the right. We want to describe the area of this square in multiple ways.

Try this: Find an algebraic expression to describe the total area of this square.

Can you find a different expression for the total area of the large square?

<u>Two different expressions:</u>
Using the length of the side, we get the factored form:

$$(x+3)^2,$$

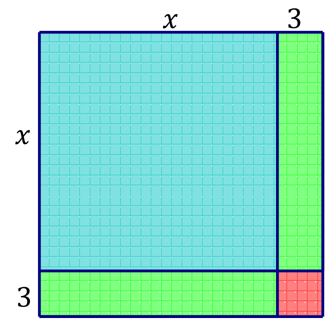
Using the areas of the four rectangles we get the expanded form:

$$x^2+3x+3x+9\ .$$

From which we get the final simplified version

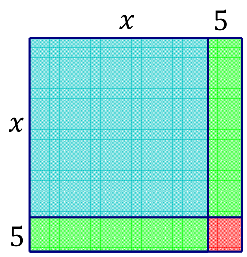
$$x^2+3x+3x+9\ =x^2+6x+9.$$

This gives us a nice visual proof that $(x+3)^2=x^2+6x+9$.

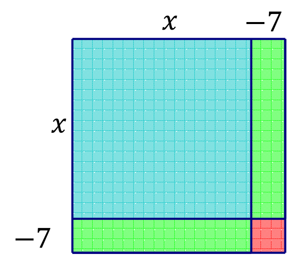


<u>Try these:</u>
For each square, write the area represented in factored, expanded, and simplified form.

a.



b.

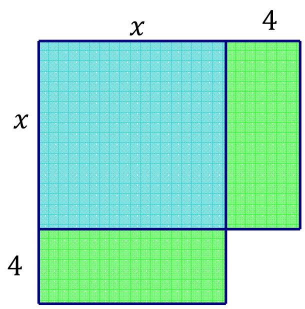


<u>Reverse it.</u>
Now, draw a square that represents these expressions.

a. $x^2+24x+144$

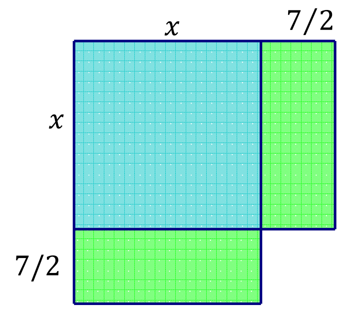
b. $x^2+5x+\frac{25}{4}$

<u>Missing squares...</u>
These "squares", are missing something. As they are, we define the area as the square of a binomial (i.e. factored form.) For each shape,
a) write an expanded expression for the area.
b) Then decide the area of the corner square what needs to be added to make a complete square.



$Current\ Area =$

$"Missing"\ area =$



$Current\ Area =$

$"Missing"\ area =$

Solution: The first shape above represents $x^2 + 8x$ which needs $4^2 = 16$ to complete the square. The second represents $x^2 + 7x$, which needs $\left(\frac{7}{2}\right)^2 = \frac{49}{4}$ to complete the square.

To generalize this:
To complete the square for $\boldsymbol{x^2 + bx}$, add $\boldsymbol{\left(\frac{b}{2}\right)^2}$.

How does this help us?

Here's how we can complete the square to change a quadratic function into vertex form.

Example:

Write the function $y = 2x^2 + 20x + 6$ in vertex form.

Solution:

$$y = 2x^2 + 20x + 6$$
$$y - 6 = 2x^2 + 20x$$
$$\frac{y-6}{2} = x^2 + 10x$$

Complete the square by adding $\left(\frac{10}{2}\right)^2 = 25$ to each side:

$$\frac{y-6}{2} + 25 = x^2 + 10x + 25$$

Factor the right side

$$\frac{y-6}{2} + 25 = (x+5)^2$$

Finally, re-solve for y

$$\frac{y-6}{2} = (x+5)^2 - 25$$
$$y - 6 = 2(x+5)^2 - 50$$
$$y = 2(x+5)^2 - 44$$

So, the vertex is at $(-5, -44)$ and it is 2 times "steeper" than $y = x^2$

Assignment

For each of the following, determine the amount that needs to be added to complete the square. Then use this value to complete the square and factor the expression. (Use fractions when necessary)

1. $x^2 + 22x$

   $\boldsymbol{x^2 + 22x + 121 = (x + 11)^2}$

2. $x^2 + 5x$

   $\boldsymbol{x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2}$

Complete the square to rewrite the following equations in vertex form. Then find the coordinates of the vertex.

3. $y = x^2 + 12x$

   $\boldsymbol{y + 36 = x^2 + 12x + 36}$

   $\boldsymbol{y + 36 = (x + 6)^2}$

   $\boldsymbol{y = (x + 6)^2 - 36}$

   $\boldsymbol{Vertex: (-6, -36)}$

4. $y = 4x^2 - 12x$

   $\boldsymbol{\frac{y}{4} = x^2 - 3x}$

   $\boldsymbol{\frac{y}{4} + \frac{9}{4} = x^2 - 3x + \frac{9}{4}}$

   $\boldsymbol{\frac{y}{4} + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2}$

   $\boldsymbol{\frac{y}{4} = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}}$

   $\boldsymbol{y = 4\left(x - \frac{3}{2}\right)^2 - 9}$

   $\boldsymbol{Vertex: \left(\frac{3}{2}, -9\right)}$

5. $y = 5x^2 + 30x$

   $\boldsymbol{\frac{y}{5} = x^2 + 6x}$

   $\boldsymbol{\frac{y}{5} + 9 = x^2 + 6x + 9}$

   $\boldsymbol{\frac{y}{5} + 9 = (x + 3)^2}$

   $\boldsymbol{y = 5(x + 3)^2 - 45}$

   $\boldsymbol{Veretex: (-3, -45)}$

6. $y = x^2 + 5x - 3$

   $\boldsymbol{y + 3 = x^2 + 5x}$

   $\boldsymbol{y + 3 + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2}$

   $\boldsymbol{y = \left(x + \frac{5}{2}\right)^2 - \frac{37}{4}}$

   $\boldsymbol{Vertex: \left(-\frac{5}{2}, -\frac{37}{4}\right)}$

7. $y = -2x^2 + 3x - 5$

   $\boldsymbol{\frac{y + 5}{-2} = x^2 - \frac{3}{2}x}$

   $\boldsymbol{\frac{y + 5}{-2} + \frac{9}{16} = \left(x - \frac{3}{4}\right)^2}$

   $\boldsymbol{y + 5 = -2\left(x - \frac{3}{4}\right)^2 + \frac{9}{8}}$

   $\boldsymbol{y = -2\left(x - \frac{3}{4}\right)^2 - \frac{31}{8}}$

   $\boldsymbol{Vertex: \left(\frac{3}{4} - \frac{31}{8}\right)}$

8. $y = 5x^2 + x - 22$

   $\frac{y + 22}{5} = x^2 + \frac{1}{5}x$

   $\frac{y + 22}{5} + \frac{1}{100} = x^2 + \frac{1}{5}x + \frac{1}{100}$

   $\frac{y + 22}{5} + \frac{1}{100} = \left(x + \frac{1}{10}\right)^2$

   $\frac{y + 22}{5} = \left(x + \frac{1}{10}\right)^2 - \frac{1}{100}$

   $y + 22 = 5\left(x + \frac{1}{10}\right)^2 - \frac{1}{20}$

   $y = 5\left(x + \frac{1}{10}\right)^2 - 22\frac{1}{20}$

   $Vertex: \left(-\frac{1}{10}, -22\frac{1}{20}\right)$

9. Suppose we take the parabola $y = x^2$ and we move it 5 units up, 6 units left, and flip it upside down. Find the standard form equation of the new parabola.

$$y = -(x+6)^2 + 5$$
$$y = -(x^2 + 12x + 36) + 5$$
$$y = -x^2 - 12x - 36 + 5$$
$$y = -x^2 - 12x - 31$$