3B: Graphing Polynomial Functions

Polynomials

Definition – a **Polynomial** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0, \qquad a_n \neq 0$$

- Each monomial is called a ***term***
- The largest power n is called the ***degree***
- A polynomial with powers written in descending order is called ***standard form***
- The numbers $a_n, a_{n-1}, \ldots, a_0$ are called the ***coefficients*** of the polynomial
- The term a_nx^n is called the ***leading term***, a_n is the ***leading coefficient***, and a_0 is the ***constant term***.

Theorem: ***Polynomial Extrema and Zeros***

A polynomial function of degree n has at most $n - 1$ local extrema and at most n zeros.

End Behavior

Explore. Use your graphing calculator to fill in the end behavior for the functions below.

	$y = x^2$	$y = -x^2$	$y = x^3$	$y = -x^3$	$y = -2x^6 + 3x^5$	$y = -x^7 + 4x^4$
As $x \to \infty$, $y \to$						
As $x \to -\infty$, $y \to$						

Generalize: Explore more using your graphing calculator as needed to make generalizations.

Degree	***Leading Coefficient***	***As*** $x \to -\infty$	***As*** $x \to \infty$	***Picture***
Even	Positive	$y \to$	$y \to$	
Even	Negative	$y \to$	$y \to$	
Odd	Positive	$y \to$	$y \to$	
Odd	Negative	$y \to$	$y \to$	

Graphing and Transforming Basic Polynomials

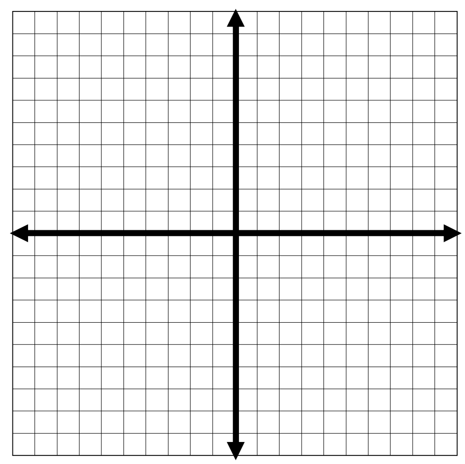
Transforming Polynomials

The following transformations will affect the graph of a polynomial $f(x)$:

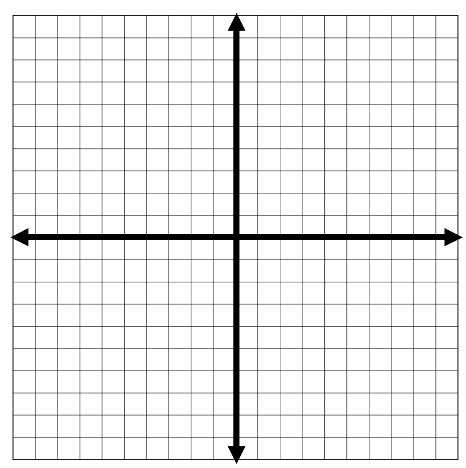
- $g(x) = f(x - h)$ will translate the graph of $f(x)$ horizontally h units.
- $g(x) = f(x) + k$ will translate the graph of $f(x)$ vertically k units.
- $g(x) = c \cdot f(x)$ will stretch the graph of $f(x)$ by a factor of c.
  If $c < 0$, the graph will be reflected across the y-axis.

Example Graph the following functions by hand and check with your graphing calculator, and simplify the right side to write it in standard form.

a) $g(x) = \frac{1}{3}(x - 1)^3 - 3$.
Hint: Use Transformations of $f(x) = x^3$ to graph



b) $h(x) = (x - 2)(x + 1)(x + 4)$.
Hint: Find zeros, then consider the degree of the polynomial.



Finding Zeros

Definition: The number k is a **zero** of a function f if and only if $f(k) = 0$.

Method: To find the zeros of a function, we set $f(x) = 0$ and solve for x by factoring. To factor we first look for common monomials, then factor out linear divisors $(x - k)$ using synthetic division, then use other methods such as grouping or quadratic methods.

Example: Find the zeros of $f(x) = (x - 4)(x + 3)(2x - 6)$.

When a function is factored as it is in the previous example, it's easy to find the zeros. However, the same function can be written as $f(x) = 2\,x^3 - 8\,x^2 - 18\,x + 72$, and it's more difficult to see the zeros. For polynomials in standard form, we can use our calculators to find the zeros.

Example: Use your calculator to find the zeros of $f(x) = x^3 - 6\,x^2 + 6$ to two decimal places.

Sometimes polynomials can be factored easily like the following example.

Example: Use factoring to find all the zeros of $f(x) = 2\,x^3 + 12\,x^2 + 18\,x$

Example: Use factoring to find all the zeros of $k(x) = x^4 - 7\,x^2 - 18$

Repeated zeros: If the factorization of a polynomial function includes a factor of $(x - k)^m$, then the zero k is said to have "multiplicity m".

- If a real zero c has odd multiplicity, the graph crosses the x axis at $(c, 0)$
- If a real zero c has even multiplicity, the graph does not cross the x axis at$(c, 0)$

Example: State the multiplicity of the zeros of $f(x) = 2\,x^3 + 12\,x^2 + 18\,x$.

Advanced Methods: Polynomial and Synthetic Division

When we cannot easily factor a polynomial function, then we must use more advanced techniques of polynomial and synthetic division.

Try These

Simplify using polynomial division.

1. $\frac{3x^4-5x^3+6x^2-5x+1}{x-1}$

2. $\frac{4x^6+8x^4+3x^2}{2x^2+1}$

Simplify completely using synthetic division.

3. $\frac{3x^4-5x^3+6x^2-5x+1}{x-1}$

4. $\frac{3x^4-3x^3-17x^2-x-6}{(x-3)(x+2)}$

So, how do we know what factors we should try to factor out with synthetic division?

Rational root theorem:

If we have a function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \qquad a_n \neq 0$$

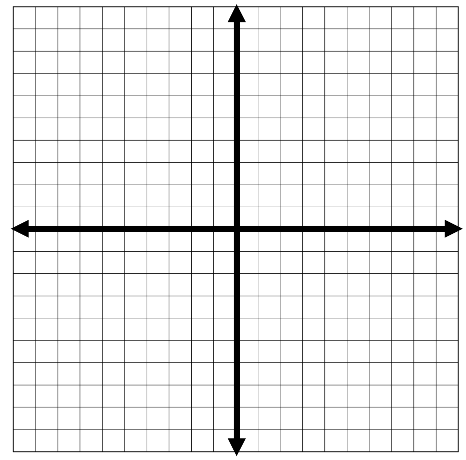
and $x = \frac{p}{q}$ is a rational zero of $f(x)$, where p and q are relatively prime, then p is a factor of a_0 and q is a factor of a_n .

In other words… only check for roots of the form $x = \frac{p}{q}$ that meet the conditions above.

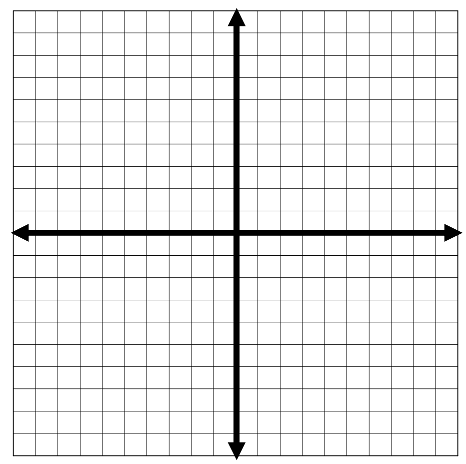
Try these:

Use the rational root theorem and synthetic division to completely factor and find all the *real* zeros of the functions below and graph

a) $f(x) = x^4 + 3\,x^3 - 6\,x - 4$



b) $g(x) = x^4 - 8\,x^2 - 5\,x + 6$



Equivalent statements:

1. The number k is a zero of the function f.
2. $x = k$ is a solution of $f(x) = 0$.
3. $(x - k)$ is a factor of $f(x)$.
4. k is an x-intercept of the graph of $f(x)$.

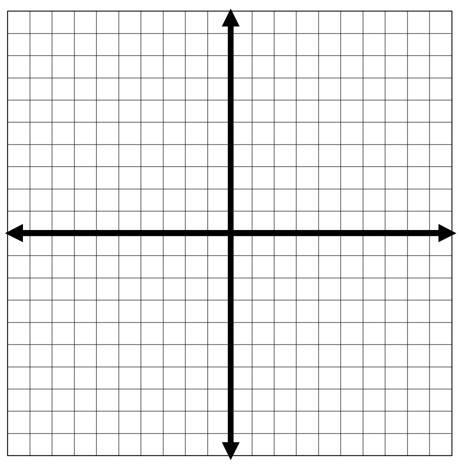
Assignment 3B-1

For each polynomial, write in standard form, state the degree, and describe the end behavior of the following polynomial functions without graphing. Then verify the end behavior with your graphing calculator and find all the zeros of the function

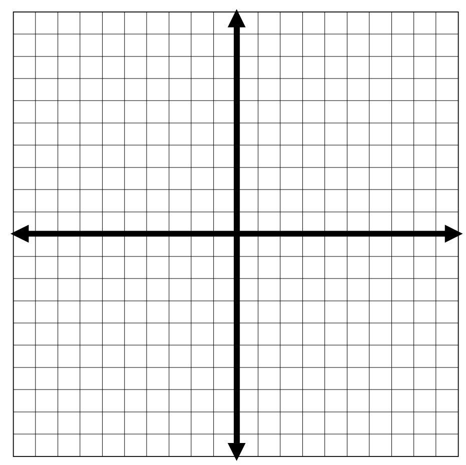
1. $y = 2x^3 - 2 + 3x^4 - 3x^2$
2. $y = -x^5 - 3x^6 - 4x^4 + 3x^5 + 10$
3. $y = 2 + x^4 - 10\,x^2 - 5\,x - 3\,x^3$
4. $y = 1 + 4x^3 - x^4 - 6x$

Use Transformations of $y = x^3$ to graph the following without your calculator. Then verify your graph using your calculator.

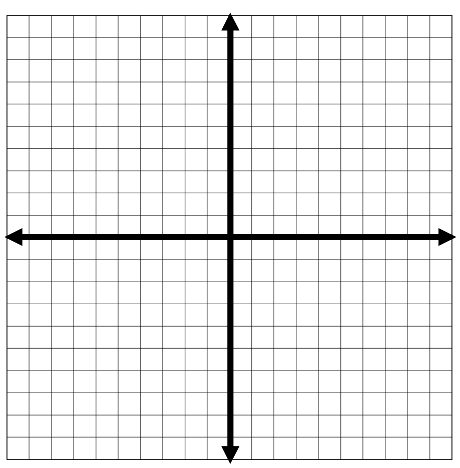
5. $y = \frac{1}{2}x^3 + 1$



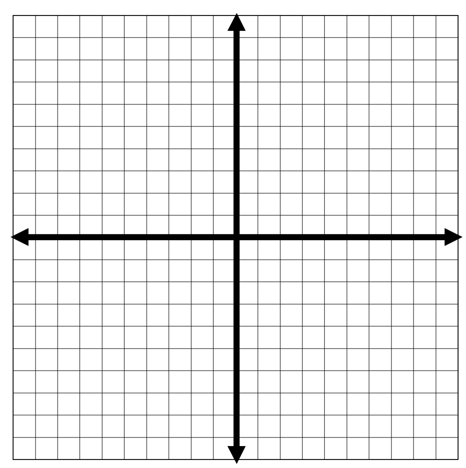
6. $y = (x + 3)^3 - 4$



7. $y = -(x - 1)^3 + 3$



8. $y = -2(x + 4)^3$



Use factoring to find the zeros of the functions. State the multiplicity of each zero.

9. $f(x) = x^3 - x^2 - 12x$

10. $g(x) = 3x^3 + 6x^2 + 3x$

11. $h(x) = x^4 - x^2 - 12$

12. $j(x) = 9x^2 + 12x + 4$