3D: Rational Functions and Asymptotes

Introduction to Rational Functions

Now that we have learned how to find critical values of polynomials, we will combine two polynomials to make a **rational function** in the form

$$f(x) = \frac{A(x)}{B(x)} = \frac{a_n\, x^n + \ldots}{b_m\, x^m + \ldots}$$

where $A(x)$ and $B(x)$ are polynomials. We will consider $A(x)$ as n^{th} degree and $B(x)$ as m^{th} degree where $m \geq 1$.

We already worked with some rational functions like

$$g(x) = x^{-1} = \frac{1}{x} \quad \text{and} \quad h(x) = x^{-2} = \frac{1}{x^2}$$

when we studied power functions. The graphs of these functions give are shown to the right.

All rational functions are discontinuous at some point. Many times we have discontinuous graphs like the ones here which get infinitely close to a line, but never touch the line. This line is called an **asymptote**.

For the functions $g(x) = \frac{1}{x}$ and $h(x) = \frac{1}{x^2}$, the graphs never touch the vertical axis where $x = 0$. The line $x = 0$ is a **vertical asymptote** because the domain of these functions does not include $x = 0$ to avoid dividing by zero. The two graphs also have a **horizontal asymptote** of $y = 0$ since they get infinitely close to the x-axis, but the graph never touches it.

g(x)=1/x

h(x)=1/(x^2)

Finding Asymptotes

The first step to understanding any rational function is to determine the asymptotes. There are two basic connections that are important to remember:

- **Domain** restrictions determine "**holes**" and **vertical asymptotes,** and
    1) Factor the numerator and denominator. If any factors cancel, the zero of this factor will be a **hole** (also called removable discontinuities) in the graph. (e.g. if $(x - 3)$ cancels, then $x = 3$ is a hole.)
    2) Determine domain restrictions by setting the denominator equal to 0 and solve. If z_i is a zero of the denominator, then $x = z_i$ is a vertical asymptote.
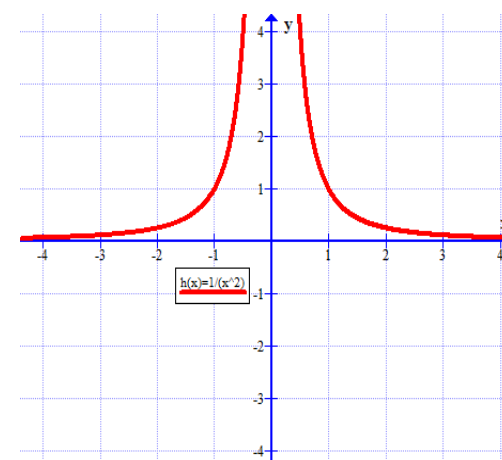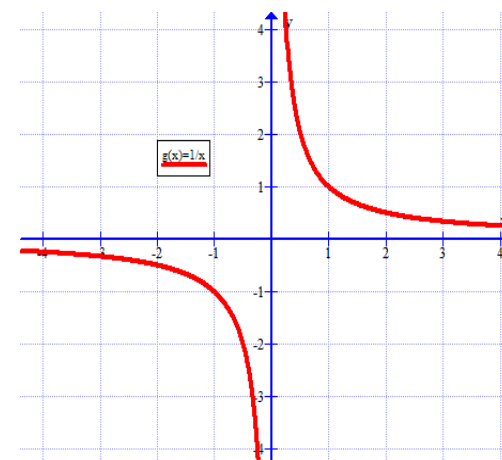
- **End Behavior (i.e. range at $\pm\infty$)** determines **horizontal asymptotes.**
  1) Use polynomial division to write the function as a *quotient function* with the remainder written as a fraction.
  2) Find the limit of y as $x \to \infty$ and $x \to -\infty$. *(Note, remainder fractions will go to zero when $x \to \pm\infty$.)*
     This limit is the equation of the horizontal (or slant) asymptote.

Example Consider the function $y = \dfrac{2x^2+6x}{x^2+2x-3}$

1. Write the function in factored form and find the domain restrictions for $f(x)$?

   a. Are there any removable discontinuities (i.e. holes)?

   b. Find the equation(s) of any vertical asymptotes.

2. Use polynomial division to write as sum of a polynomial and remainder fraction.

   a. For the value of each vertical asymptote, a, find the **Limits**
      as $x \to \pm\infty$, $y \to$
      as $x \to a^+$, $y \to$
      as $x \to a^-$, $y \to$

   b. Find the equation of any horizontal asymptotes using
      the infinite limits above.

3. Graph the function (test extra points if needed) and check it on your calculator.
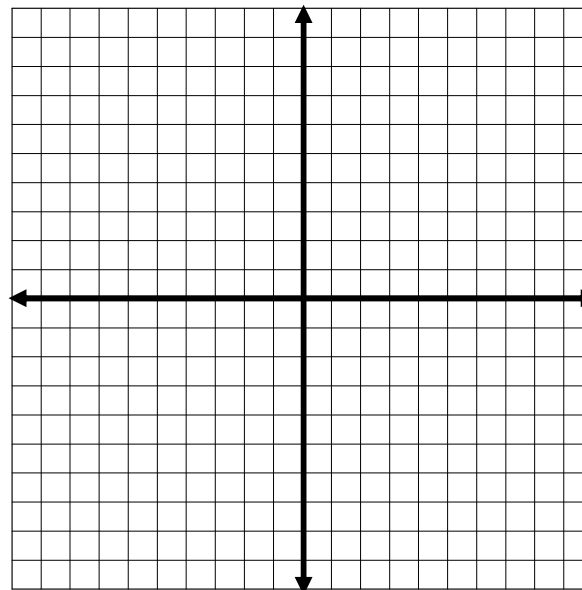
## Domain restrictions and vertical asymptotes

We have previously learned that we must carefully consider the denominator when determining the domain of a rational function. We will continue with this idea and use what we have learned about factoring to find these domain restrictions and other important characteristics of the graph of a rational function.

| Function	Factored Form	Domain Restrictions	Vertical Asymptotes (a) and Removable Discontinuities	Divide (write as sum of a polynomial and remainder fraction)	Limits as $x \to \pm\infty$, as $x \to a^+$ as $x \to a^-$	Horizontal Asymptotes
$a(x) = \dfrac{x+3}{x^2+x-6}$						

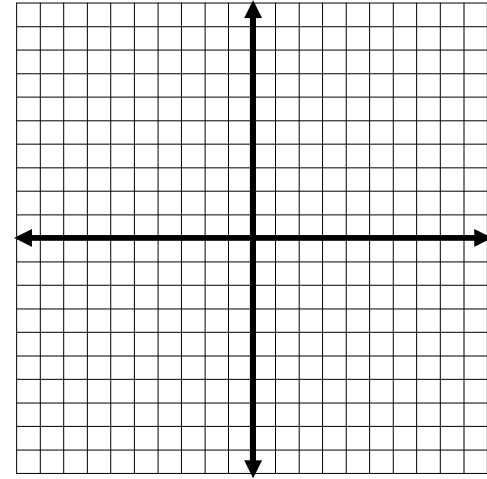| Function	Factored Form	Domain Restrictions	Vertical Asymptotes (a) and Removable Discontinuities	Divide (write as sum of a polynomial and remainder fraction)	Limits as $x \to \pm\infty$, as $x \to a^+$ as $x \to a^-$	Horizontal Asymptotes
$b(x) = \dfrac{2\,x^2 + x - 6}{x^2 + x - 2}$						
$c(x) = \dfrac{x^2 - x - 6}{x^2 + 2\,x - 3}$						
$d(x) = \dfrac{x^2 - 4}{x^2 + 1}$						
$y = \dfrac{x^3 + 4x^2 + 3x}{x + 1}$	$\dfrac{x(x+1)(x+3)}{x^3 + 4\,x^2 + 3\,x}$					

Graphs
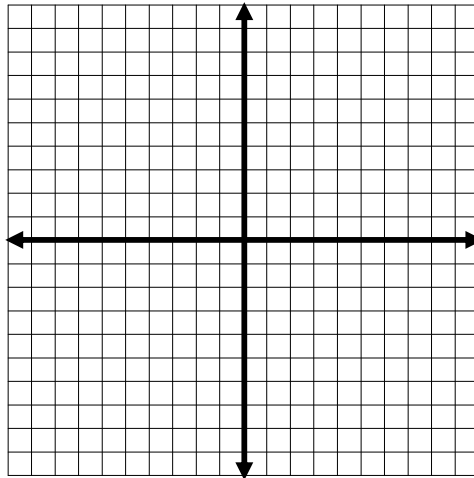
$$a(x) = \frac{x+3}{x^2+x-6}$$
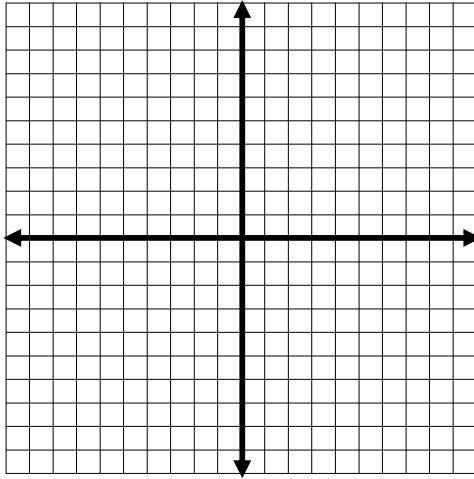
$$b(x) = \frac{2x^2+x-6}{x^2+x-2}$$
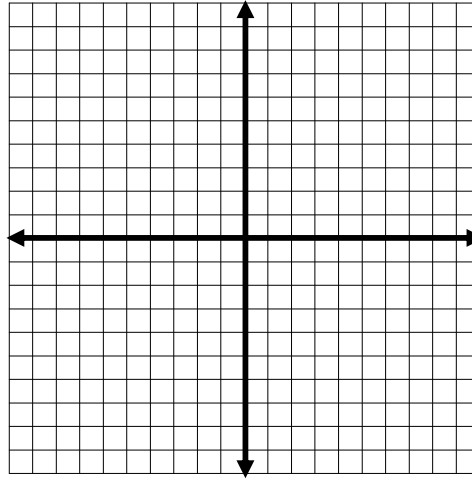
$$c(x) = \frac{x^2-x-6}{x^2+2x-3}$$

$$d(x) = \frac{x^2-4}{x^2+1}$$

$$e(x) = \frac{x^3+1}{x^2-1}$$