4B-1: Exponential Equations and Log Properties

Exponential Equations of Base 10 and *e*

Fundamental Relationship: The fundamental connection between logarithms and exponentials is

If $b^x = y$, then $\log_b y = x$

This means that finding $\log_b y$ is equivalent to asking the question:

"*What power of base b is needed to get the value y?*"

So, ***the log is the power!***

Logarithmic Identities

Since the logarithm is the power of base b needed to produce x, we have the itentities:

$$b^{\log_b x} = x, \quad and \quad \log_b b^x = x$$

Example. Solve the following exponential equations using the logarithmic identities above. Use substitution to check your answer.

a) $10^x = 110$

$x = \log(110) \approx 2.041$

b) $10^x = 0.005$

$x = -log(200) \approx -2.301$

c) $3(10^x) = 213$

$x = log(71) \approx 1.851$

d) $10^{4x} = 0.5$

$x = \frac{log(.5)}{4} \approx -0.075$

e) $e^x = 23$

$x = ln(23) \approx 3.135$

f) $2e^{x+2} = 24$

$x = ln(12) - 2 \approx 0.485$

Properties of Logarithms

Exploration Use your calculator to approximate a solution to each of the following.

a) $\log(3 \cdot 5) =$

b) $\log 3 + \log 5 =$

c) $\log\left(\frac{3}{5}\right) =$

d) $\log 3 - \log 5 =$

e) $\log 3^5 =$

f) $5 \cdot \log 3 =$

We notice that there are some similar answers above, so let's explore more. Let

$$\log_b P = p, \qquad and \quad \log_b Q = q.$$

Which implies

$$P = b^p, \qquad and \quad Q = b^q$$

Let's find equivalent forms of these equations by changing them to exponential equations.

$\log_b(PQ) = y$

$\log_b \frac{P}{Q} = y$

$\log_b P^c = y$

<u>**Properties of Logarithms**</u>

Let b, R, and S be positive real numbers with $b \neq 1$, and c a

Product Rule: $\log_b(PQ) = \log_b P + \log_b Q$

Quotient Rule: $\log_b \frac{P}{Q} = \log_b P - \log_b Q$

Power Rule: $\log_b P^c = c \log_b P$

Where b, R, and S be positive real numbers with $b \neq 1$, and c is any real number.

Change of Base Formula for Logarithms

$$\log_b x = \frac{\log_a x}{\log_a b}$$

<u>Example</u> Use the change-of-base formula to evaluate the following to 3 decimal places. Check your answer by evaluating the power.

a) $\log_2 5 = \frac{\log 5}{\log 2} = \frac{\ln 5}{\ln 2} \approx 2.322$

b) $\log_5 130 = \frac{\log 130}{\log 5} = \frac{\ln 130}{\ln 5} \approx 3.024$

c) $\log_5 2 = \frac{\log 2}{\log 5} = \frac{\ln 2}{\ln 5} \approx 0.431$

d) $\log_{0.5} 4 = \frac{\log 4}{\log 0.5} = \frac{\ln 4}{\ln 0.5} = -2$

Exercises

Solve the following using Logarithmic Identities.

1. $10^x + 5 = 90$

   $\boldsymbol{x = \log(85)}$

   $\mathbf{1.9294189257143}$

2. $5(10^x) = 200$

   $\boldsymbol{x = log(40)}$

   $\mathbf{1.602059991328}$

3. $3(e^{x+1}) = 17$

   $\boldsymbol{x = ln\left(\frac{17}{3}\right) - 1}$

   $\mathbf{0.7346010553881}$

4. $4e^{2x} - 3 = 9$

   $\boldsymbol{x = \frac{ln(3)}{2}}$

   $\mathbf{0.5493061443341}$

Use the change-of-base formula to evaluate the logarithm.

5. $\log_3 30 =$

   3.0959032742894

6. $\log_7 30 =$

   1.7478696965085

7. $\log_{0.5} 15 =$

   $\mathbf{-3.9068905956085}$

8. $\log_{0.2} 20 =$

   $\mathbf{-1.8613531161468}$

Solve each equation algebraically. Get a numerical approximation for your solution and check it by substitution.

9. $5^x = 512$

   $\boldsymbol{x = 9\log_5 2}$

   $\mathbf{3.8760890226605}$

10. $3^{5x} = 100$

    $x = \frac{2\log_3 10}{5}$

    $\mathbf{0.8383613097158}$

11. $e^x = 217.5$

    $x = ln(217.5)$

    5.3821988505287

12. $2.5^x = 300$

    $\boldsymbol{x = log_{2.5} 300}$

    $\mathbf{6.2248610361799}$

13. $4(5^x) = 210$

    $\boldsymbol{x = log_5\left(\frac{105}{2}\right)}$

    $\mathbf{2.4609915915348}$

14. $4^{x+1} - 2 = 10$

    $\boldsymbol{x = \frac{log_2 12}{2} - 1}$

    $\mathbf{0.7924812503606}$

15. $4\left(1 + .25^{x/4}\right) = 40$

    $1 + .25^{x/4} = 10$

    $.25^{x/4} = 9$

    $\frac{x}{4} = \log_{.25} 9$

    $\boldsymbol{x = 4\,log_{.25} 9}$

    $\boldsymbol{x = -6.34}$

The formula for interest that is *compound continuously* is $A = Pe^{rt}$, where A=final amount, P=starting amount, r=interest rate(as a decimal), and t=time in years.

Find the missing variable.

16. $A = \$200, P = \$100, r = 2.3\%$

    $200 = 100e^{.023t}$

    $t = \frac{1000\,ln(2)}{23}$

    30.1368339373889

17. $A = \$3000, P = \$100, t = 30$

    $3000 = 100e^{30r}$

    $r = \frac{ln(30)}{30}$

    0.1133732460554