Jumping into the *Leap Frog* Puzzle

By Jeremy Knight
Math Teacher, Grants Pass High School, jeremy@knightmath.com

Activity Grade Level: *5th grade and up*
Materials: *Golf Tees and Peg board puzzles*
or 9x1 grids with colored marker chips,
and Leap Frog Handout

Our world is saturated with the beauty of mathematical patterns. From the constant daily rising and setting of the sun by which we set our clocks, to the complex and chaotic weather systems that envelop our globe, the patterns that have been set in motion in our universe are the basis for all of our mathematical systems. One of my goals as a math teacher is that students will leave our classrooms with a greater appetite for insight and understanding as they observe the patterns in the world and continue to ask, "Why does this happen?"

So, how do we accomplish such a lofty task as inspiring students to care about patterns? One good way is to use puzzles to engage their natural curiosity. The Leap Frog puzzle is a simple problem that can lead to some deep mathematical results when properly motivated with a good question, "What is the minimum number of moves?"

The Puzzle

The Leap Frog puzzle consists of a peg board with nine holes in a line, four brown pegs, and four white pegs – golf tees work nicely for pegs. The puzzle can be made with any two colors of pegs, or even flat marker chips on a 9-by-1 paper grid as shown in figure 1. We begin by placing the two groups of like-colored pegs in the four holes on either end of the board and leaving one hole in the middle. The objective of the puzzle is to move the pegs so that the groups of brown and white pegs switch sides. The permissible moves are a *slide* of one peg to an adjacent hole that is open, or a *jump* of <u>one</u> peg (of any color). You may ask, "Can we move 'backwards'?" Go right ahead. We will not place this restriction at this time.

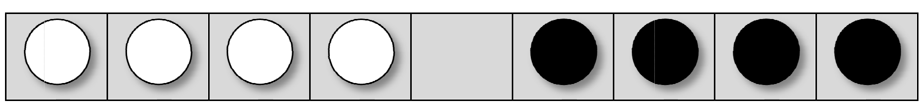


Figure 1 – Alternate Colored-Chip Game Board

Jumping into the Math

With these basic instructions, students can quickly begin solving the puzzle as they allow their curiosity to take over. It won't take long until students start to say, "I got it!" – and with a little bit of challenge, fun, and success, they are hooked. We can now prompt students to employ a crucial skill in mathematics – forming good questions. After commending students for completing the initial objective and discussing their process, we want to lead them to ask the question,

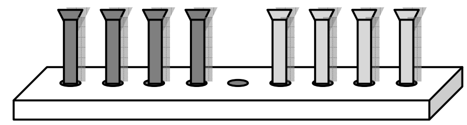
"What is the *minimum number* of moves needed to complete the puzzle?"

Now an entertaining puzzle has prompted a great math question! After allowing students to complete the puzzle while counting their moves, then writing some results on the board, we can have some great discussion about what is needed to acquire the, "minimum solution". This discussion should lead to the fact that if we want to get the minimum number of moves, we cannot move backwards because this wastes moves. Now we have formulated a new, but equivalent, question,

"How many moves are needed to complete the puzzle *without moving backwards*?"

The Minimum Number of Moves – An Inductive Approach

We now have a new goal to solve the four peg (per side) puzzle without going backwards. With a little more trial-and-error, students will soon find that this is a difficult task. It is now time to introduce the "Wishful Thinking Method" of inductive reasoning to answer the question. The process of the Wishful Thinking Method is to



- Wish for an easier problem,
- Solve the easier problem,
- Then look for a pattern as the problem gets harder.

So, without changing the rules of the game, we can make it "easier" by imagining a Leap Frog game with one peg per side, we will call this the "one-peg puzzle". Students can remove the outer three pegs to model this smaller board as in figure 2 (the outer three holes are now off limits!). Now we ask the question, "What is the minimum number of moves to solve the puzzle with *one* peg per side?" We quickly find the solution to the one-peg puzzle requires *three moves*. Now let's add a peg (figure 3).

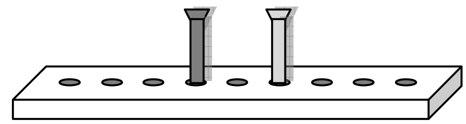


Figure 2

How many moves are needed for *two* pegs per side? Remember, we don't want to go backwards, so this puzzle will get more difficult as the number of pegs grows. After a little time, students should be able to see that the two-peg puzzle requires *eight moves*. Time to add another peg.

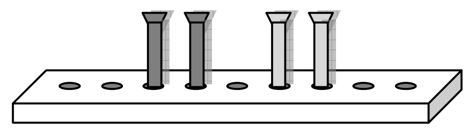


Figure 3

With one peg removed from the outer hole, we need to make the pegs switch places. At this point, it is challenging to see the algorithm that will allow us to complete the 3-peg puzzle. So, let's take a diversion from counting moves to find an algorithm for completing the puzzle.

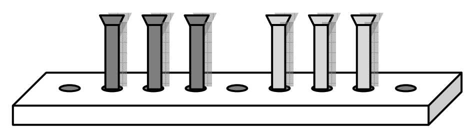


Figure 4

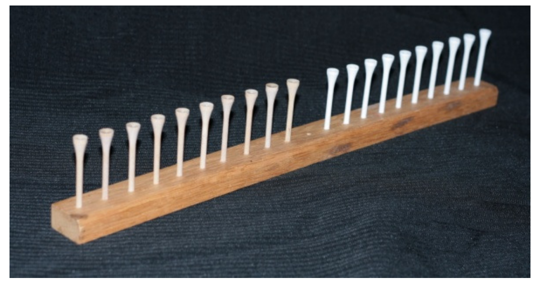
A Useful Diversion – Finding the Algorithm

How do we know which piece to move next? Well, let's analyze the types of moves that were made to complete the smaller puzzles. To do this, observe that there are two types of moves, a "slide" of one peg to an adjacent open hole and a "jump" of one peg over another. We will denote these as "s" and "j" respectively, ignoring the color that slides or jumps. Now, if we record the moves for the one-peg puzzle, we have the moves *s-j-s* to complete the puzzle. The two-peg puzzle requires the moves *s-j-s-j-j-s-j-s*. We notice that there is an interesting symmetry to this list, and observe that there is one jump, then two jumps, then one separated by a single *s*. It is also useful to observe that the groups of jumps alternate colors.

If we follow the pattern of the first two puzzles, we can conjecture that the three-peg puzzle would require the moves *s-j-s-j-j-s-j-j-j-s-j-j-s-j-s*. This suggests the exact steps needed to complete the four-peg puzzle without moving backwards. We can now quickly find an algorithm for any number of pegs. The following list shows the number of moves for the one-peg to the five-peg puzzles where groups of jumps have been replaced with the number of jumps (i.e. "*j-j-j*" has been replaced with "3"):

s-1-s
s-1-s-2-s-1-s
s-1-s-2-s-3-s-2-s-1-s
s-1-s-2-s-3-s-4-s-3-s-2-s-1-s
s-1-s-2-s-3-s-4-s-5-s-4-s-3-s-2-s-1-s

Back to the Minimum Question



We can now answer the question of what the minimum number of moves is for four pegs. Using the pattern above, we can see that there are 24 moves needed for the four-peg puzzle. Hooray! We answered our question… but what if we had more pegs? How many moves would a ten-peg puzzle require?

Pegs	*# of Moves*	*+1*
1	3	4
2	8	9
3	15	16
4	24	25
…	…	…
10	?	?
n	?	?

To answer this, we need to make a table with the numbers we have collected. However, the numbers 3, 8, 15, 24 do not look too familiar to most of us (unless you realize patterns that increase by odd numbers are related to square numbers). So, let's make a "helping column" by adding 1. Now we see a pattern, these are square numbers. Filling in this extra column with square numbers, we see that a ten-peg puzzle would require $(10+1)^2 - 1 = 120$ moves. Finally, if we have *n-pegs*, we would require $(n+1)^2 - 1$ moves. Another pattern you may recognize is that the number of moves are

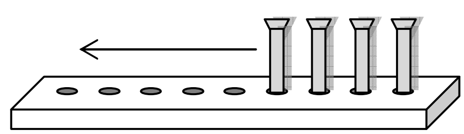
$$1 \cdot 3 = 3, \quad 2 \cdot 4 = 8, \quad 3 \cdot 5 = 15, \quad 4 \cdot 6 = 24, \ldots$$

This leads us to conclude that we require $n(n+2)$ moves. This confirms our previous observations since

$$(n+1)^2 - 1 = n^2 + 2n = n(n+2).$$

The Silver Bullet – A Deductive Approach

After discussing this puzzle with Irv Lubliner, he shared with me the following beautiful deductive proof to answer the question of minimum moves for a four-peg puzzle and an *n*-peg puzzle. This method will prove that it is a *necessary* condition for the minimum-move solution of the four-peg puzzle to be 24 moves, but it is not *sufficient* to prove that the solution does exist. However, our algorithm above showed that the minimum solution does, in fact, exist. So, we can now use deductive reasoning to prove the minimum number of moves.



Assume that there were only four white pegs on the board. To move the white pegs to the opposite side of the board, it would require five moves each. This would take $4 pegs \cdot 5 slides = 20 slides$. Likewise, if the brown pegs did not have to jump the white pegs, they would take up 20 slide moves for a grand total of 40 slides. However, jumping conserves "slide" moves because a jump goes further. So each jump will subtract one slide move from the total of 40 moves.

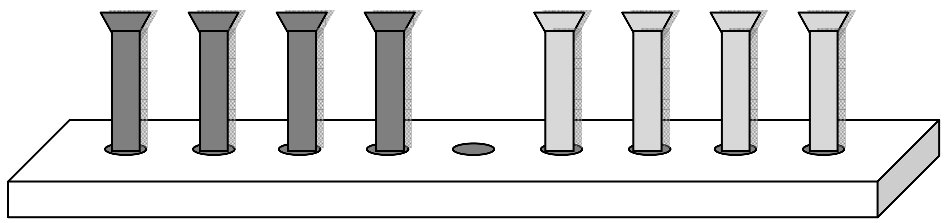
Consider the first white peg. For this peg to get to the opposite side, it must jump or be jumped four times. Likewise, every white peg on the board must jump over a brown or be jumped by a brown. Therefore, there must be $4 jumps \cdot 4 whites = 16 jumps$ to get the whites on the opposite side of the brown. Finally, we can put this all together to conclude that the minimum number of moves for four-peg puzzle requires

$$40\ slides - 16\ jumps = 24\ moves\ total.$$

Likewise, the n-peg puzzle requires $2n(n+1) - n^2$. With a little algebra, we see that the deductive solution is equivalent to the inductive solution of $(n+1)^2 - 1$ moves.

Leap Frog

Student Handout



Object: Move the pegs one at a time so the pegs end up on the opposite sides of the board. A peg may jump over only one other peg per move or slide to an open hole next to it.

Level 1 Game: Complete the puzzle with "backwards" moves allowed.
Level 2 Game: Complete the puzzle without moving "backwards"

The big question: What is the minimum number of moves that it will take to swap the pegs of a ten-peg puzzle?

Fill in the chart to find a pattern that will help you answer the big question.

Number of Pegs	Number of Moves
1	
2	
3	
4	
. . .	. . .
10	
$\boldsymbol{n}$	