Puzzling Math, Part 1: The Five-Square Puzzle

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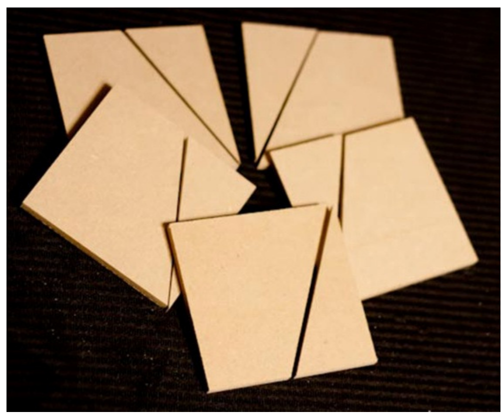
Grade Level: 8-12+
Objectives: To apply the Pythagorean theorem and square area to determine sides with irrational lengths, and to motivate the use of simplified radical notation instead of decimal approximation.
Common Core Standards (by cluster):
Practice Standards: (1) Make sense of problems and persevere in solving them, (2) Reason abstractly and quantitatively, (3) Construct viable arguments and critique the reasoning of others, (4) Model with Mathematics, (5) Use appropriate tools strategically
6.G.A: Solve real-world problems involving area, surface and volume (stand. 3).
7.G.B: Solve real-life and mathematical problems involving angle measure, area, surface area, and volume (stand.6).
8.EE.A.2: Work with radicals and integer exponents (stand.2).
8.G.B: Understand and Apply the Pythagorean Theorem (stand. 7).
HSG-SRT.C.8: Define trigonometric ratios and solve problems involving right triangles (stand. 8).
HSN-RN.B: Use properties of rational and irrational numbers (stand. 3).
Time: 45 minutes



"What is a radical anyway, and why do we need to simplify it?" As students move into the world of Algebra, these are crucial questions for them to firmly grasp. Irrational numbers can be quite difficult for students to work with due to a lack of a physical model. When it comes to counting numbers, children have built-in manipulatives with their fingers that can add, subtract, multiply, divide. However, if we try to represent the square root of 2 with fingers, blocks, or any other type of counter we are at a loss.

The Pythagoreans believed that all numbers could be expressed as rational numbers. Legend has it that the Pythagoreans were so convinced of this fact that when Hippasus presented a geometric proof of the irrationality of $\sqrt{2}$, he was thrown into the sea for proposing such a *radical* idea. Although your students today may not throw you overboard for presenting the concept of an irrational number, it is not unusual in a middle or high school classroom to hear a little groaning among the ranks when the radicals come out.

The Five-Square Puzzle is an excellent activity to give students a physical model of a square root and motivate the need to simplify a radical. This puzzle was first introduced to me by Dr. Richard Thiessen of the A.I.M.S. foundation, who has the wonderful gift of seeing the beauty of the mathematics in almost any puzzle. Puzzles can be powerful tools for engaging students in deep mathematical problem solving while not even realizing that they are doing math.

One square from five

In this puzzle, five squares are divided into two pieces by connecting the midpoint of one side to a vertex of the opposite side. The objective is to rearrange the pieces of the five smaller squares to make one large square. Students will need photocopies of the puzzle handout which includes puzzle pieces to cut out as well as questions to answer. If a more durable puzzle is desired, copies can be made on cardstock.

When students have cut out the puzzle pieces, allow them exploration time to try to solve the puzzle independently. The majority of students enjoy playing with puzzles, which motivates them to engage in the math investigation to follow. It is rare for a student to stumble upon the large square solution without using the math. However, many students will create a rectangle. This is a commendable accomplishment and can be considered a "level 1" solution, although it is not the ultimate goal.

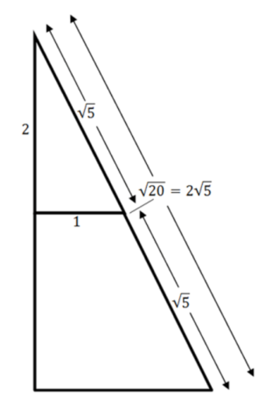
Mathematics to the rescue!

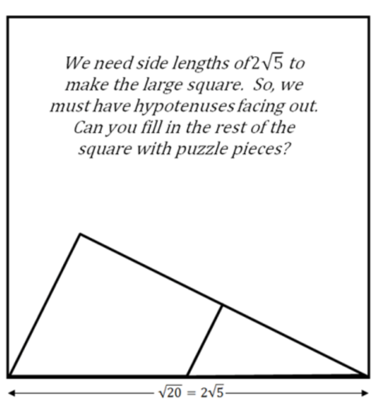
After attempting to solve the puzzle by trial and error, students will reach a point where they need more than just guessing. This is the time when students should be directed to the key questions on the handout to focus on the mathematics involved. We begin by defining the squares as 2×2 squares with an area of 4 un^2. This means that all five squares have a total area of 20 un^2.

This prompts a discussion about the desired length of the side of the final square. Applying the formula for the area of a square, we see that the side must be $\sqrt{20}$, which is not a whole number. However, this gives us a tangible model that shows us that

"$\sqrt{20}$ is the length of the side of a square with area of 20 square units."

Now, since the side is $\sqrt{20}$ units, we see that it cannot be made by adding puzzle piece sides of lengths 1 and 2. However, using algebraic techniques to simplify the radical, we see that $\sqrt{20} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$. Thus, we need a new length of $\sqrt{5}$ to make this puzzle possible. This motivates us to use the Pythagorean theorem to find the length of the triangle hypotenuse is $\sqrt{5}$. This is exactly what we need!





The math solves the puzzle

With our new discovery, we now see that the outside of the square needs to have the length of $2\sqrt{5}$. Since the hypotenuse of the right triangle, and its matching side on the trapezoid is $\sqrt{5}$, this tells us that we need 2 of these "slanted" lengths facing out. Using this fact, students can flip the pieces to make theses "slanted" sides face out. This will feel quite unnatural for students, so they will need to be reminded that we need a large side length of $\sqrt{20} \approx 4.47$, which cannot be made using sides of 1 or 2 units.

This process of using mathematics to solve a puzzle gives students a wonderful example of how taking time to think through the math involved in a situation can lead us to a solution when trial-and-error has us frustrated and running in circles. So many students have developed a math-phobia that has convinced them that math is an intangible and irrelevant. The physical model and direct application of radicals in this puzzle help to convince students that viewing problems through mathematics is a powerful way to lift the veil and help us to see clearly the simple solution to a beautiful problem.

5-Square Puzzle

Object: Make a large square out of the pieces of the five small squares

Begin by cutting out the pieces below and move them to try to make one large square

Answer the following questions to help you solve the puzzle;

1. What is the area of the small square?
2. What is the total area of the five squares?
3. What will the area of the final, large square be?
4. What will the length of one side of the final square be?
   Can this be made by putting 1's and 2's together?
5. What is the length of the hypotenuse of the triangle cut from the small square?

Now try using this new information to better attack the puzzle solution.