Puzzling Math, Part 2: The Tower of Hanoi & the End of the World!

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Grade Level: 6-12+
Objectives: Create and describe algorithms, recognize number patterns, model exponential growth, apply inductive reasoning, and break down a complex problem into manageable steps.
Common Core Standards (by cluster):
Practice Standards: (1) Make sense of problems and persevere in solving them, (2) Reason abstractly and quantitatively, (4) Model with Mathematics, (5) Use appropriate tools strategically, (7) Look for and make sense of structure, (8) Look for and express regularity in repeated reasoning
8.EE.A: Work with radicals and integer exponents (stand. 1).
HSF-LE.A: Construct and compare linear, quadratic, and exponential models (stand. 1, 2).
HSA-CED.A: Create equations that describe numbers or relationships (stand. 1).
HSA-SSE.B: Write expressions in equivalent forms to solve problems (stand. 3).
HSF-BF.A: Build a function that models a relationship between two quantities (stand. 1).
Time: 45 minutes

Solving complex multi-step problems can be a daunting task for math students of all ages. The great problem solver, George Polya, often encouraged students to, "wish" for a simpler problem when faced with a difficult question. This powerful *wishful thinking method* can be modeled well using pattern-based questions that often surface when playing with puzzles. The *Tower of Hanoi* puzzle is a classic puzzle that is an exceptional tool for engaging students in a fun activity which creates a perfect opportunity to apply the wishful thinking method to solve a challenging problem.

A puzzle to end the world!
The Tower of Hanoi puzzle was inspired by the legend of a temple in India which had three posts with 64 discs of different sizes stacked onto one of the posts. The priests in the temple were given the job of moving the discs from one peg to another with the requirements that only one disc can be moved at a time and a larger disc may never be placed on top of a smaller disc. Furthermore, the legend states that when the puzzle is completed by the priests, the earth would vanish! This motivates an important question: "If the monks moved exactly one disc correctly every second, how long would it take them to complete the puzzle?" In other words, according to the legend, when will the world end?

The original puzzle created by French mathematician Edouard Lucas in 1883 had eight discs. However a modified six-disc version of the puzzle on three pegs version is sufficient for this classroom activity. The goal of the puzzle is to start with all the discs on the left peg and move them to the right pegs. Only one disc may be moved at a time, and a larger disc may never be on top of a smaller disc.

To use this puzzle in the classroom, a teacher may choose to use the blackline master below to make cardstock puzzles (the handout has squares instead of the standard circles for ease of cutting), buy or make wooden versions of the puzzle, use an online applet such as the one at www.illuminations.nctm.org, or use an iPad app such as "Tap Towers".

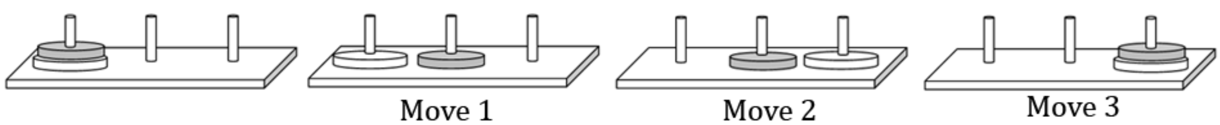
Breaking it down with "Wishful Thinking"

With any puzzle activity, it is important for students to begin by exploring and playing with the puzzle on their own to spark their interest. Given the goal of transferring the stack of discs from one peg to another *in the fewest number of moves*, students will first try to solve the puzzle by trial-and-error. Most students will begin to discover some general techniques for moving the pieces, and some may solve the puzzle given enough time.

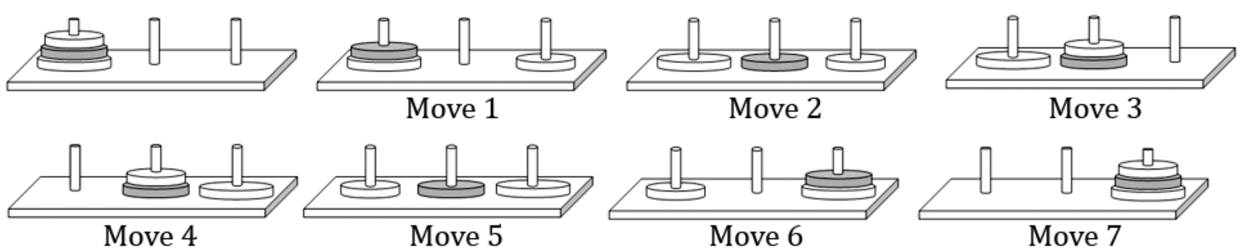
With a little class discussion, it should become clear that finding the fewest number of steps for the six-disc puzzle is not a simple question that can be easily answered by experimenting. So, we need to employ the wishful thinking method. Since we have a challenging question which we cannot answer, we need to "wish" for a simpler problem, solve this simpler problem, then look for a pattern to answer the original question.

To begin this discussion, ask students what would make this problem simpler without changing any of the rules. They should conclude that having fewer discs would simplify the problem. So, let's wish big and try the puzzle with not three, not two, but only one disc! If we remove all discs but one, the question becomes, "What is the minimum number of moves to solve the puzzle with 1 disc?" Clearly, it only requires one move.

"That was too easy," the students will say. Well, now we will make it twice as hard and try a 2-disc puzzle. Ask students to remove all but 2 discs, and then they should be able to complete this puzzle in three moves.



That is still not very challenging, so we can step it up a notch to solve the puzzle with three discs. Although there is only one more disc, the number of moves increases quickly (maybe it is an exponential pattern!). Students should find that the three-disc puzzle requires seven moves as shown below.



# of Discs	1	2	3	4	5	6
# of Moves (min.)	1	3	7	?	?	?

We now have some data for the first three puzzles. At this point, it is time to begin looking for a pattern in the number of moves. With the first three small puzzles, it is easy to agree that we have found the minimum number of moves. However, for four, five, or six discs, it is easier to take a careful look at the algorithm that we have developed for the first three cases to count moves for larger puzzles.

Finding an Algorithm

This puzzle provides an excellent opportunity for students to recognize and define an algorithm – a set of defined steps to produce a result. Consider the relationship between the moves for the three-disc puzzle and the two-disc puzzle. In the three-disc puzzle, we have three intermediate steps:

1. Move the two smaller discs to the middle peg (3 moves),
2. Move the largest disc to the right peg (1 move), and
3. Move the two smaller discs to the right peg (3 moves).

Notice that steps 1 and 3 are accomplished using the same steps from the two-disc puzzle. Using this same algorithm, we can solve the four-disc puzzle with steps:

1. Move the *three* smaller discs to the middle peg (7 moves),
2. Move the largest disc to the right peg (1 move), and
3. Move the *three* smaller discs to the right peg (7 moves).

This gives us a total of 15 moves. With the same logic, we can easily show that we need 31 moves and 63 moves for the five-disc and six-disc puzzle respectively.

So, when will the world end … according to the legend?

Our last step is to return to the legend to figure out how long it would take to solve a 64-disc puzzle. Our data table is now as follows:

# of Discs	1	2	3	4	5	6	…	10	…	64	…	*n*
# of Moves (min.)	1	3	7	15	31	63	…		…		…	

Students can now look carefully at the number of moves to find a pattern. If we observe the differences in the moves, we see that they are 2, 4, 8, 16, and 32! That's exciting because these are powers of two, but we still need to find a connection between the number of discs and number of moves. A good suggestion for students is to "check the neighbors," or more specifically to add one to the number of moves. Adding a new row to our table giving us:

# of Discs	1	2	3	4	5	6	…	10	…	64	…	*n*
# of Moves (min.)	1	3	7	15	31	63	…	2^{10}-1= 1023	…	2^{64}-1	…	2^n-1
Moves+1	2=2^1	4=2^2	8=2^3	16=2^4	32=2^5	64=2^6	…	2^{10}	…	2^{64}	…	2^n

When we add one to the moves, we get the powers of two. This can now be extended to 10, 64, and n discs. Subtracting one gives us our final answer.

So, how long will it take the priests to complete the 64 disc puzzle bringing about the end of the world if they move one disc every second continually?

$$2^{64}-1 \textit{ sec.}=18446744073709551615 \textit{ sec.}=14{,}038{,}618{,}016{,}522 \textit{ years}$$

Conclusions and Extensions

So, if you want to see if this prediction is true you will have to stick around a few trillion years! The Tower of Hanoi puzzle is a wonderful vehicle for students to practice breaking down larger problems into smaller problems and using patterns to answer difficult questions. The process of developing, analyzing, and articulating algorithms is also a valuable exercise that will equip students with the tools needed to solve more complex problems.

This puzzle has many rich and challenging extensions such as adding pegs and changing the rules, many of which are still unsolved problems [Gardner]. So, a puzzle that at first appears to be a child's toy, proves to be the source of some amazingly beautiful and endless mathematics. It is such a joy to find so much intriguing math in the simplest of places.

Other Resources

Gardner, Martin. *Hexaflexagons, Probability, Paradoxes, and the Tower of Hanoi*. Cambridge University Press. 2009.

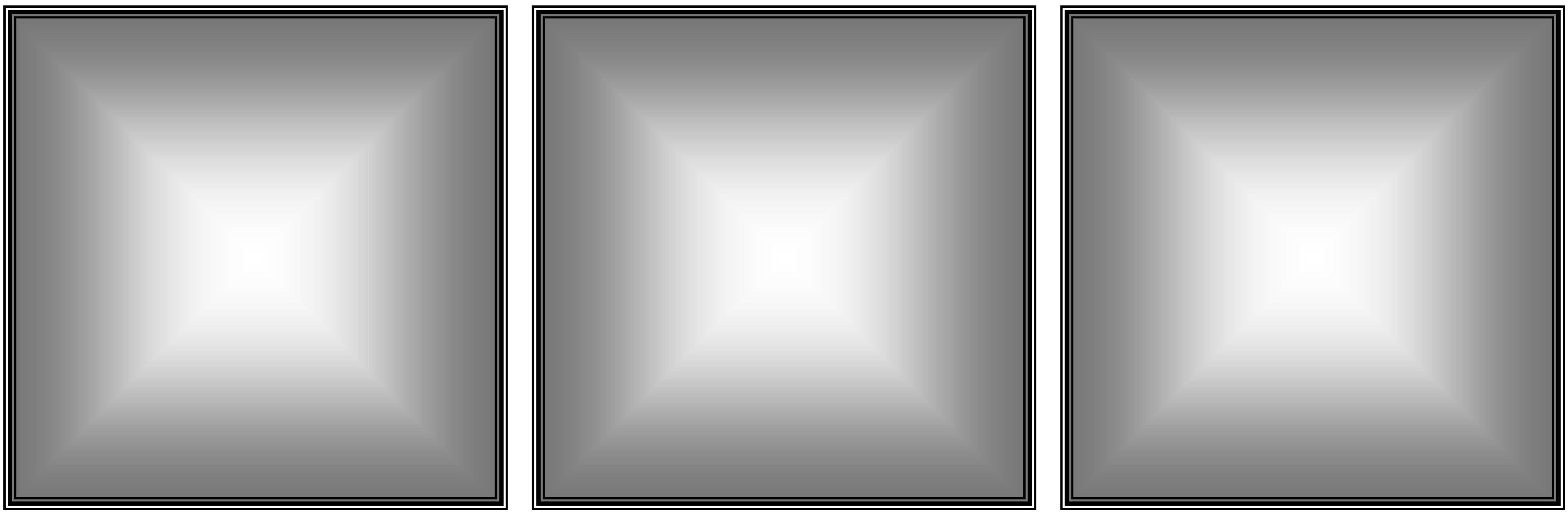
Reimer, Will. *What's Next? Using Patterns to Solve Problems, Vol 1;* AIMS Education. 2012;

Tower of Hanoi – *handout 1*

Object: Below are three "base squares". Begin by stacking all of the cut-out pieces in order on the left square with the largest piece on the bottom and smallest on top. The goal is to move the whole stack from the left base square to the right base square one at a time.

Rules: Larger squares may never be on top of smaller squares. You may only move one piece at a time.

The Big Question: What is the minimum number of moves needed to complete the six-piece puzzle? How about 10, 64, or n pieces?



# of Squares	1	2	3	4	5	6	…	10	…	64	…	n
Minimum. # of Moves							…		…		…	

Tower of Hanoi – *handout 2*

Cut out the squares below and use them on the Tower of Hanoi game board.