Puzzling Math, Part 3: The Five Triangle Puzzle

by Jeremy Knight, Grants Pass High School, jeremy@knightmath.com
The Oregon Mathematics Teacher , May/June 2014

Grade Level: 5-12+ (to varying levels)
Objectives: Special right triangles, similar triangles, angle measurement, triangle-angle sum theorem, working with radicals, and triangle area.

Common Core Standards (by cluster):

Practice Standards: (1) Make sense of problems and persevere in solving them, (2) Reason abstractly and quantitatively, (3) Construct viable arguments and critique the reasoning of others, (4) Model with Mathematics, (5) Use appropriate tools strategically

6.G.A: Solve real-world problems involving area, surface and volume (stand. 1, 3).

7.G.B: Solve real-life and mathematical problems involving angle measure, area, surface area, and volume (stand.6).

8.EE.A: Work with radicals and integer exponents (stand.2).

*8.G.A:*Understand congruence and similarity using physical models, transparencies and geometric software (stand. 5).

8.G.B: Understand and Apply the Pythagorean Theorem (stand. 7).

HSG-SRT.C.8: Define trigonometric ratios and solve problems involving right triangles (stand. 8).

HSN-RN.B: Use properties of rational and irrational numbers (stand. 3).

Time: 45-60 minutes

Some triangles are always right, some triangles are just special, and then some triangles get the honor of being *special right triangles.* These special right triangles are a key concept that students develop in a high school geometry course and then apply regularly as they enter the world of analytic trigonometry in Algebra 2 courses and beyond. As useful as these triangles are, it is often difficult for students to remember the important relationships among the side lengths. The “Five Triangle Puzzle” is a fun and engaging way for students to work with these 30°-60°-90° triangles and gain a more firm grasp on the side lengths of triangles.

The Five Triangle Puzzle is another great puzzle, like the Five Square Puzzle from the first part of this series, that I learned in my days working as a puzzle maker for Dr. Richard Thiessen, an internationally recognized mathematical puzzle expert and president of the A.I.M.S. education foundation. This puzzle can be used for a wide range of math classes from grades 4 and up. Students from middle elementary and older can benefit from the problem solving and spatial visualization challenge of playing with the puzzle (Introductory Phase). Middle school classes can use this activity to hone their skills with finding possible angles in a triangle, and discuss similar triangles (Phase 1). Finally, high school Geometry and Algebra 2 classes can learn from all the following phases as they explore the rich mathematics involved in this puzzle (Phase 2).

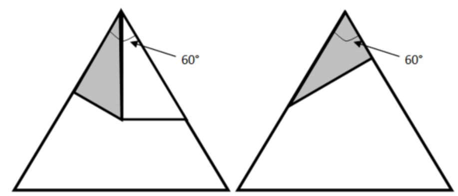
Introductory Phase: Playing with Triangles (grade 4 and up)
The Five Triangle puzzle is made of two small triangles and three large triangles. The student handout has puzzle pieces to cut out and questions to answer. The goal is to make one large triangle using all five pieces. In this introductory phase, students need to be given time to simply play with the puzzle while attempting to accomplish the objective through experimentation. It turns out that there is more than one solution (which you may want to withhold from students to start), and that is where the math gets involved. However, for younger students (4th or 6th grade), this is still a fun puzzle to use when students are introduced to right triangles.

For elementary students, a final goal of this activity would be to just have fun with a puzzle while working with right triangles and practicing spatial visualization. If students are not going to be doing the phase 1 questions, the teacher should tell the students there are two triangles that can be made: an equilateral triangle and an obtuse isosceles triangle. A handout with outlines of the final triangle shapes can also help clarify the student's destination.

Phase 1: What triangles are possible? (grade 6 and up)

After attempting to solve the puzzle with trial-and-error, some students will possibly come up with one of the solutions, but this is a challenging puzzle that will usually require some careful thought (or hints.) Now it is time to tell them that there is more than one solution. At this, students may begin to ask, “How many solutions are there?” and, "What types of triangles are they?" The teacher’s answer to these questions should be something along the lines of, “Those are great questions! Let’s do the math.”

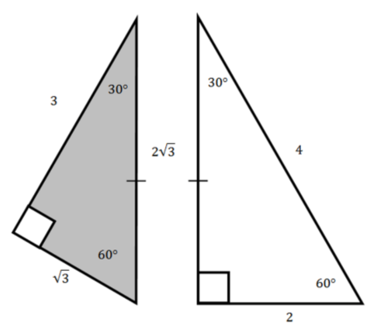
The first question that students need to address is what possible angles can be made by joining one or more of the puzzle pieces at their vertices with shared sides. The fact that the puzzle pieces are 30°-60°-90° triangles can be given to the students, or they could deduce this by comparing angles, or they could measure the angles with a protractor. This may also be a good time to point out that the triangle pieces are good examples of similar triangles, and it may be that the solution is another larger similar triangle.



As shown to the right (and on the handout), it is possible to have a 60° angle in the final triangle by putting one puzzle piece with its 60° angle corresponding to the vertex of the final triangle. On the other hand we could place two puzzle pieces with both of their 30° angles together to make a 60° angle in the final triangle. Using this logic in the Phase 1 section, students should conclude that the angles less than 180° which we can make with the pieces are 30°, 60°, 90°, 120°, and 150°.

Now that we know which angles can potentially be made in the final triangle, we can decide which type of triangles may be possible for our final solution. To do this, we need to find all the three-angle combinations of these five angles (allowing for repeated angles) which have a sum of 180°. The only combinations that total 180° are 30°-60°-90°, 60°-60°-60°, and 30°-120°-30°. Middle school or geometry prerequisite classes can work through Phase 1 of this activity to accomplish the goal of measuring angles, recognizing and using similar triangles, applying the triangle-angle sum theorem, and using the CCSS practice standards. Teachers of these classes may want to stop at Phase 1 on the handout and then challenge students to try to find one of the three possible triangle solutions. Geometry or post-geometry classes may continue onto Phase 2.

Phase 2: Digging deeper with radicals and area (high school geometry level and up)



We now want to decide if we can eliminate any of these three possible solution triangles so we do not waste time trying to build an impossible triangle. To do this we will consider the total area of the five pieces to see if we can make the necessary side lengths with our puzzle pieces. To begin this phase, students need to find all of the side lengths given that the hypotenuse of the large triangle is 4. To do this, they will need to apply the properties of 30°-60°-90° triangles. Then with these side lengths determined we find the area of the individual pieces, giving us a total area of $9\sqrt{3}$.

The first possible solution triangle investigated is a large 30°-60°-90°. If the short leg of this solution was *x*, the long leg would be $x\sqrt{3}$. Using the formula for the area of a triangle we get the equation

$$\frac{1}{2}(x)\left(x\sqrt{3}\right) = 9\sqrt{3}$$

Solving this equation we get $x = \sqrt{18} = 3\sqrt{2}$. However, if we look at all the side lengths of the puzzle pieces, there are no side lengths that are a multiple of $\sqrt{2}$. Therefore, we can conclude that a 30°-60°-90° solution triangle is not possible.

How about the 60°-60°-60° solution triangle? Letting half of the base equal y, we get the equation

$$\frac{1}{2}(2y)(y\sqrt{3}) = 9\sqrt{3} \quad \rightarrow \quad y^2\sqrt{3} = 9\sqrt{3}.$$

This gives us $y = 3$ which means the side length of the solution triangle would be 6. We can make a side length of 6 because the pieces have sides of 4 and 2. So, this suggests that a construction is possible and gives us a good clue how to actually make this large equilateral triangle.

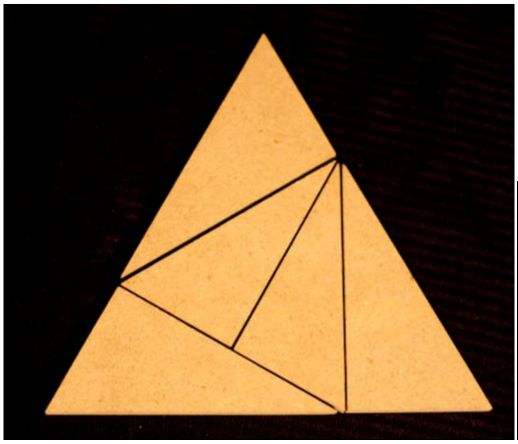
Finally, we consider the 30°-120°-30° triangle by letting the height be z and half of the base be $\sqrt{3}$. This gives us the same equation as last time

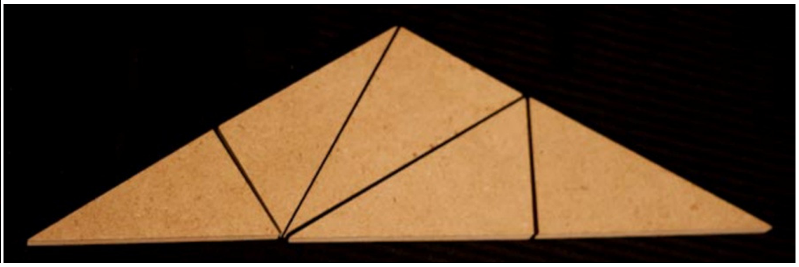
$$\frac{1}{2}(2z\sqrt{3})(z) = 9\sqrt{3} \quad \rightarrow \quad z^2 = 9\sqrt{3}.$$

Here the solution of $z = 3$ implies that the base length of the triangle must be $6\sqrt{3}$. It is possible to make a base length of $6\sqrt{3}$ with the puzzle pieces, so we have found the possible second solution triangle.

Conclusion: Using our discoveries to complete the puzzle
We now conclude that the two possible solutions are the 60°-60°-60° and 30°-120°-30° triangles. With this information and the base lengths that we found, the puzzle solutions can be found more easily (see [BACK PAGE OF JOURNAL] for solution). Although the reasoning and the math involved in Phase 2 can be challenging, the outcome is rewarding. Furthermore, the process of taking a problem and breaking it down to investigate possible solutions is an invaluable learning experience for all students.

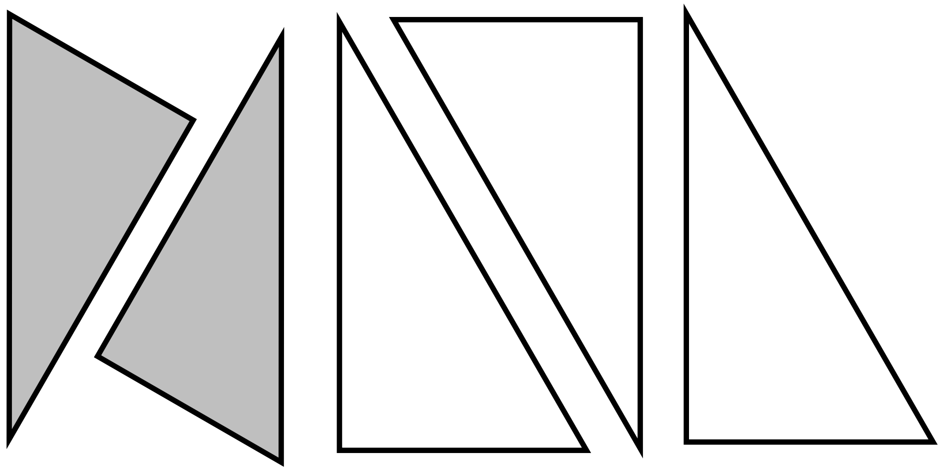
Solutions [Images to be placed in the back of the TOMT journal]:





The Five Triangle Puzzle

The object of this puzzle is to use all five right triangles below to form one large triangle.



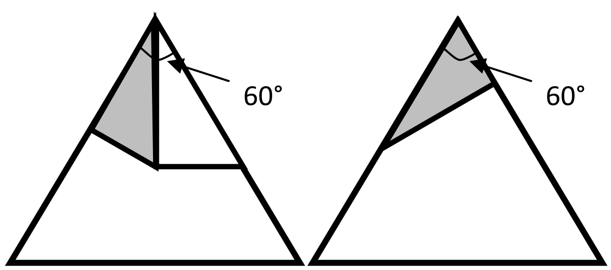
Begin by cutting out the triangle above. Notice that there are two smaller right triangles and three larger right triangles, and the hypotenuse of the smaller triangle is equal in measure to the longer leg of the larger triangle.

Questions to consider

There is one more twist to this puzzle… it has more than one solution! The answers to the following questions will help you to find all the solutions to the 5-triangle puzzle.

It is important to note that the puzzle piece are not just your everyday right triangles, these are *special right triangles.* They are special because their angle measures are 30°, 60°, and 90°, making it half of an equilateral triangle. We first need to think about the possible types of triangles that we can end up with in our final triangle.

Phase 1: Consider possible triangles



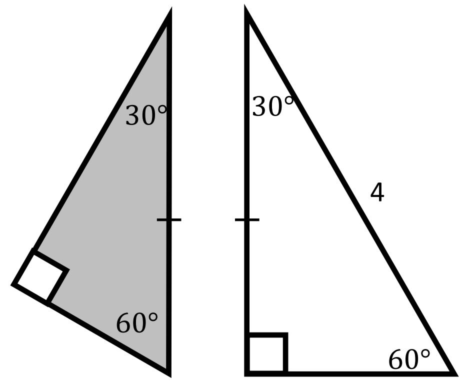
1. When we use the pieces to build the final triangle, one possible angle in the final triangle could be 60° by either placing two 30° angles in a corner or placing one 60° in the corner as shown in the example at the right.

   List all the possible angles (no greater than180°) that could be made by combining one or more of the puzzle piece angles (30°, 60°, or 90°). There are exactly 5 such possible angles.

   ___________ ___________ ___________ ___________ ___________

2. We know that the angles in a triangle must have a sum of 180°. Using the possible final angles you found in #1 above, what are all the possible combinations of angle measures that we could end up in our final triangle?

Phase 2: Digging deeper with radicals and areas
Leave all lengths and area measures in simplest radical form



3. If the hypotenuse of the large triangle is 4 units, find the length of each side of the two triangles and label them.

4. Find the area of the two triangles in simplest radical form:

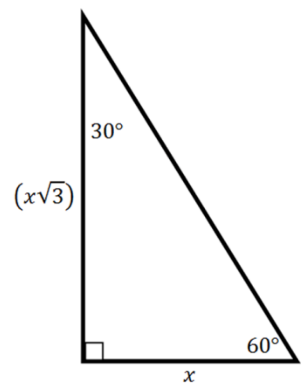
   A(Small Triangle)=

   A(Large Triangle)=

5. When we combine all five triangles, what will the area of the complete triangle be? (*Hint: The whole is the sum of its parts.*)

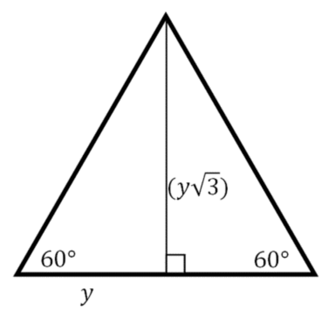
   A(Final Triangle)=

6. Let's find out if it is possible to make one large 30°-60°-90° triangle with the puzzle pieces. Use the area formula for a triangle and your answer from #5 to figure out what x would have to be in the triangle to the right.



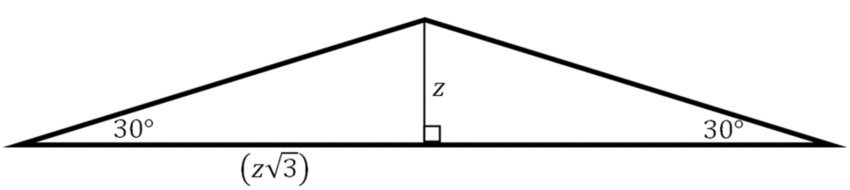
   Is it possible to use the puzzle pieces to make a length of this x?
   So, is it possible to have a final triangle with angles of 30°-60°-90°?

7. Now, let's find out if it is possible to make one large 60°-60°-60° triangle with the puzzle pieces. Use the area formula for a triangle and your answer from #5 to figure out what y would have to be in the triangle to the right.



   What would the length of the entire base be?
   Is it possible to use the puzzle pieces to make this base length?
   So, is it possible to have a final triangle with angles of 60°-60°-60°?

8. Finally, is it possible to make one large $30° - 150° - 30°$ triangle?
   If so, what would z have to be below? How long is the base?



Conclusion: After answering these questions, you should know what the two possible types of final triangles are. Now, use this information to help you find both solutions to the puzzle.