7.2 Applications of the Normal Distribution

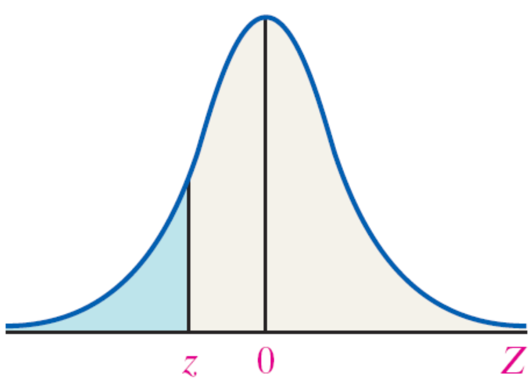
Standardizing a Normal Random Variable
Suppose that the random variable X is normally distributed with mean μ and standard deviation σ. Then the random variable

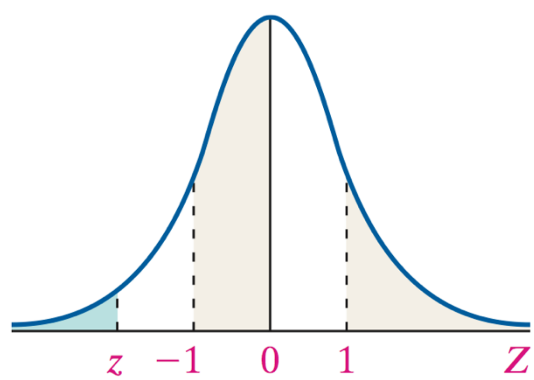
$$Z = \frac{X - \mu}{\sigma}$$

is normally distributed with mean $\mu = 0$ and standard deviation $\sigma = 1$.The random variable Z is said to have the standard normal distribution.

Standard Normal Curve and the Z-value

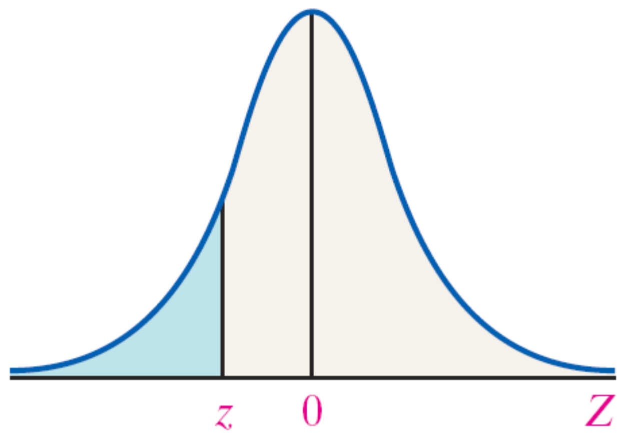
- The value of Z tells us the area *underneath the* curve to the left of a given value of X.
- Since the area of the standard normal curve is 1, the area under the curve to the left of z_0 is the probability that a random Z will be less than z_0. That is
$$Area\ to\ Left\ of\ z_0 = P(Z < z_0)$$
- The table gives the area under the standard normal curve for values to the left of a specified Z-score, z, as shown in the figure.





Example: IQ scores can be modeled by a normal distribution with $\mu = 100$ and $\sigma = 15$.
An individual whose IQ score is 120, is 1.33 standard deviations above the mean.

The table gives the area under the standard normal curve for values to the left of a specified *Z*-score, *z*, as shown in the figure.

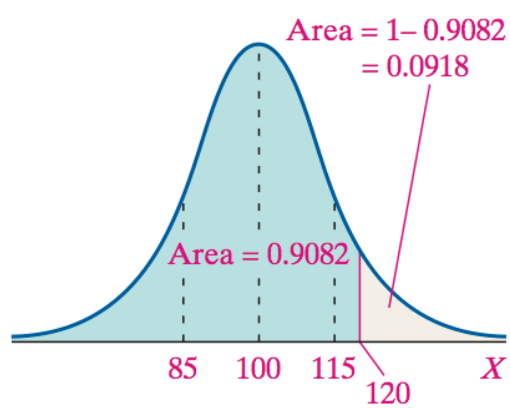


a) Find the area under the standard normal curve to the left of $z = 1.33$

Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.776
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.927
1.5	0.9332	[illegible]	0.9357	[illegible]	0.9382	0.9394	0.94

b) Use the Complement Rule to find the area to the right of $z = 1.33$.



Calculator: Type [2nd] [VARS] (to get Distributions)

- $P(Z < z_0)$ = normalcdf(-E99, z_0)

  For example,

- P(Z < -1.2)=normalcdf(-E99,-1.2)=.115

Note: Area under the normal curve to the right of z_o = 1 **–** Area to the left of z_o

Example

Find the area under the standard normal curve to the right of z = 1.25.

Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.535
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.575
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.614
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.651
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.687
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.901
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.917
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.931
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.94

Calculator $P(Z > z_0)$ =?

Key things to remember when using your Calculator

- $P(Z > z_0)$ represents the proportion (probability, or percentage) of the z-scores that are larger than a specific z-score z_0.
- The **Standard** Normal Curve has $\mu = 0, \sigma = 1$
- For example, $P(Z > 0) = 0.5$.
- What about $P(Z > 1.3)$ =?
- $P(Z > 1.3)$=normalcdf(1.3,E99)

Example

Find the area under the standard normal curve between $z = -1.02$ and $z = 2.94$.

Area between –1.02 and 2.94

= (Area left of z = 2.94) – (area left of z = –1.02)

Calculator $P(z_1 < Z < z_2)$ = normalcdf(z_1, z_2)

For example,

$P(-1.5 < Z < 1.5)$

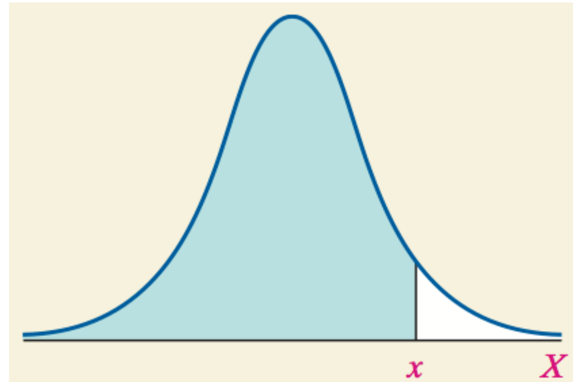
= normalcdf(-1.5, 1.5)

=.866

Problem: Find the area to the left of x.
Approach: Shade the area to the left of x.
Solution:

1. Convert the value of x to a z-score.
2. Use the Z-Table to find the row and column that correspond to z. The area to the left of x is the value where the row and column intersect.

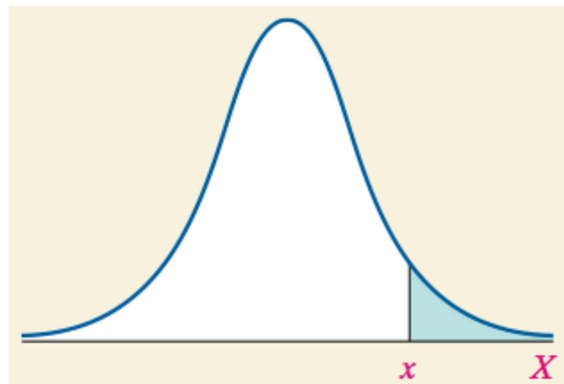
   Or… Use technology to find the area.
   normalcdf(E99, z, 0, 1)



Problem: Find the area to the right of x.
Approach: Shade the area to the right of x.
Solution:

1. Convert the value of x to a z-score.
2. Use the Z-Table to find the area to the left of z (also is the area to the left of x). The area to the right of z (also x) is 1 minus the area to the left of z.

   Or… Use technology to find the area.
   normalcdf(z, E99, 0, 1)



Problem: Find the area between x_1 and x_2.
Approach: Shade the area between x_1 and x_2.
Solution:

1. Convert the values of x to a z-scores.
2. Use Table V to find the area to the left of z_1 and to the left of z_2.
   The area between z_1 and z_2 is (area to the left of z_2) – (area to the left of z_1).

   Or… Use technology to find the area
   normalcdf(z1, z2, 0, 1)

Calculating the probabilities of a normal distribution through z-scores

- Method 1: $P(a < X < b) = normalcdf(a, b, \mu, \sigma)$
- Method 2: $P(a < X < b) = P(z1 < Z < z2) = normalcdf(z1, z2)$
  here, z1 and z2 are the z-scores of a and b, respectively.

Summary If the distribution is a normal distribution, a TI function *normalcdf* can be used.

$$P(X < b) = normalcdf(-E99, b, \mu, \sigma)$$
$$P(X > a) = normalcdf(a, E99, \mu, \sigma)$$
$$P(a < X < b) = normalcdf(a, b, \mu, \sigma)$$

Finding the Value of a Normal Random Variable

Procedure for Finding the Value of a Normal Random Variable

Step 1: Draw a normal curve and shade the area corresponding to the proportion, probability, or percentile.
Step 2: Use Table V to find the *z*-score that corresponds to the shaded area.
Step 3: Obtain the normal value from the formula $x = \mu + z\sigma$.

OR

Use technology: **invnorm (area left, μ,σ)**

Example:

Find a z-score that corresponds to an area of .66 to the left.

How to find the k^th^ percentile of a normal distribution with μ and σ

- P_k represents the k^{th} percentile.
- Here, k^{th} percentile is the number that separates bottom $k \cdot 100\%$ from top $(1-k) \cdot 100\%$

Example

The combined (verbal + quantitative reasoning) score on the GRE is normally distributed with mean 1049 and standard deviation 189.
(Source: http://www.ets.org/Media/Tests/GRE/pdf/994994.pdf.)

What is the score of a student whose percentile rank is at the 85^{th} percentile?

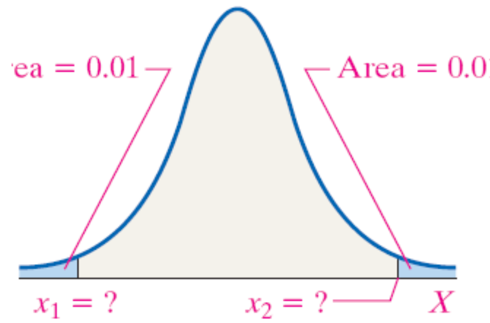
- First find the z-score that corresponds to the 85^{th} percentile
  (That is, the area under the standard normal curve to the left is .85.)

- Now find $x = \mu + z\sigma$

- Calculator: 2^{nd} VARS invnorm (.85, 1049, 189)

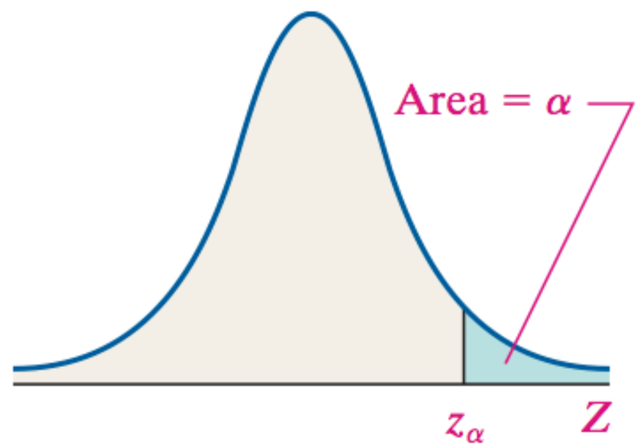
Example

It is known that the length of a certain steel rod is normally distributed with a mean of 100 cm and a standard deviation of 0.45 cm. Suppose the manufacturer wants to accept 90% of all rods manufactured. Determine the length of rods that make up the middle 90% of all steel rods manufactured. (Use your Calculator)



A useful note:

The notation z_α (pronounced "*z* sub alpha") is the *z*-score such that the area under the standard normal curve to the right of z_α is α.



- z_α = invNorm(1- α)
- For example, $z_{.10}$ = invNorm(1- .10)=1.28